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Advances in
Mathematical
Fluid Mechanics

# New Directions in Mathematical Fluid Mechanics

## The Alexander V. Kazhikhov Memorial Volume

Andrei V. Fursikov
Giovanni P. Galdi
Vladislav V. Pukhnachev
Editors

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## Advances in Mathematical Fluid Mechanics

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# Preface

On November 3, 2005, Alexander Vasil'evich Kazhikhov left this world, untimely and unexpectedly.

He was one of the most influential mathematicians in the mechanics of fluids, and will be remembered for his outstanding results that had, and still have, a considerably significant influence in the field. Among his many achievements, we recall that he was the founder of the modern mathematical theory of the Navier-Stokes equations describing one- and two-dimensional motions of a viscous, compressible and heat-conducting gas. A brief account of Professor Kazhikhov's contributions to science is provided in the following article "Scientific portrait of Alexander Vasil'evich Kazhikhov".

This volume is meant to be an expression of high regard to his memory, from most of his friends and his colleagues. In particular, it collects a selection of papers that represent the latest progress in a number of new important directions of Mathematical Physics, mainly of Mathematical Fluid Mechanics. These papers are written by world renowned specialists. Most of them were friends, students or colleagues of Professor Kazhikhov, who either worked with him directly, or met him many times in official scientific meetings, where they had the opportunity of discussing problems of common interest.

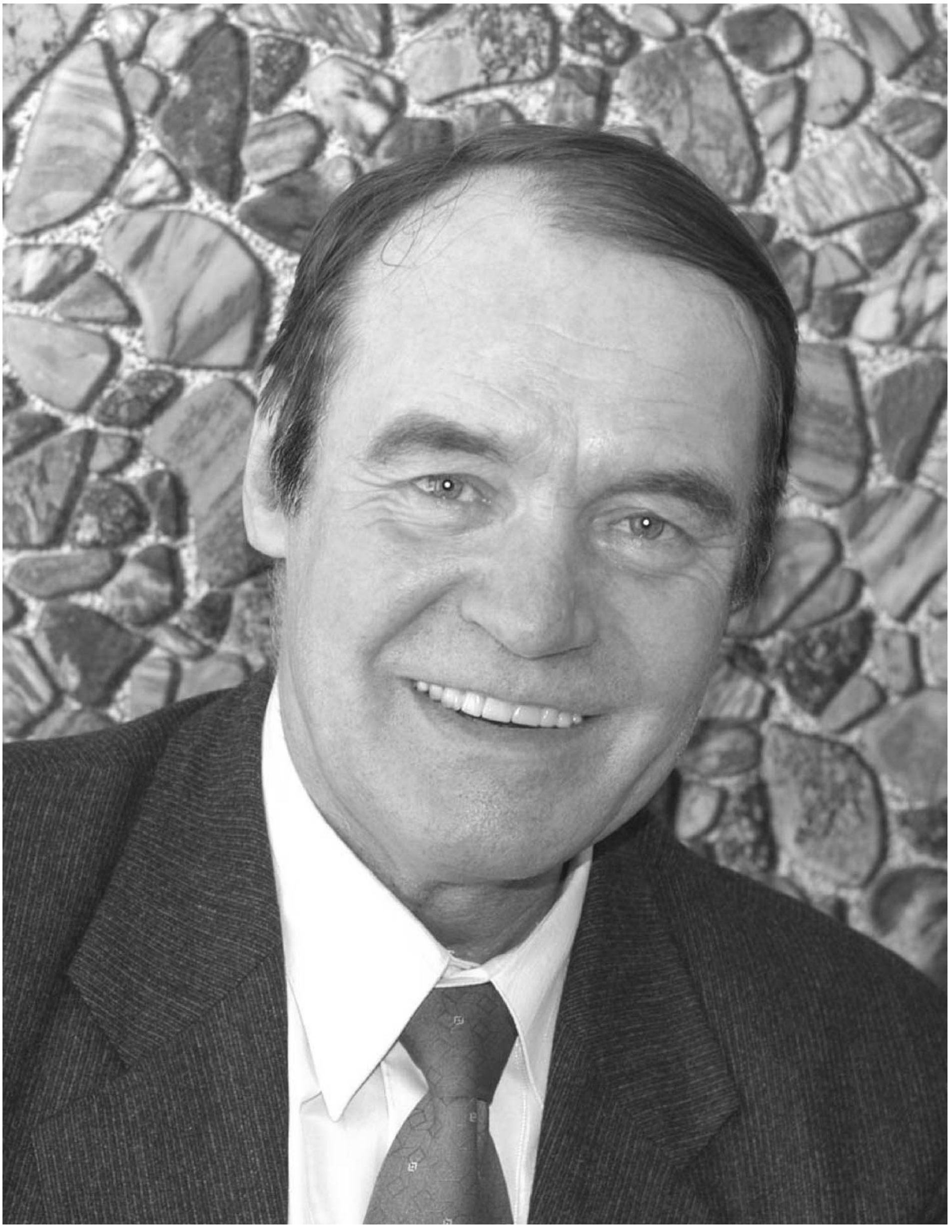
We shall not give the detailed description of the results presented in this volume, but, rather, we shall only give a short list of the main areas where these results have been obtained. These areas range from boundary value problems for different types of fluid dynamic equations, to certain models describing the properties of compressible flows, to limits of different kind with vanishing parameters. They also include control problems for fluid flow, stability problems for equilibrium figures of a liquid, problems connected with elastic bodies, Poisson-Boltzmann equation, and problems with insufficient information about initial and/or boundary data.

In many articles the reader will find an account of the state-of-the-art of the corresponding disciplines that may serve as a stimulating starting point for further research.

Also for this reason, we believe that this volume could be helpful to specialists as well as to researchers who would like to become acquainted with certain aspects of Mathematical Fluid Mechanics.

Moscow, Pittsburgh, and Novosibirsk, September 2009

Andrei V. Fursikov Giovanni P. Galdi Vladislav V. Pukhnachev



*Aleksander V. Kazhikov: August* 28, 1946–*November* 3, 2005

# Scientific Portrait of Alexander Vasilievich Kazhikhov

Alexander Vasilievich Kazhikhov was born on the 28th of August in 1946 in the village of Proskokovo, Kemerovo Region. In 1947 his family moved to the town of Kolyvan, Novosibirsk Region, where he graduated from a secondary school. In 1964 Alexander Kazhikhov entered the Mechanics and Mathematics Department at the Novosibirsk State University. Upon graduation in 1969, he continued his education as a post graduate student. In 1971 Alexander Kazhikhov defended his Candidate of Science thesis entitled "Global solvability of some boundary value problems in hydrodynamics", and occupied a position at the Theoretical Department of the Institute for Hydrodynamics of the Siberian Branch of the Soviet Union Academy of Sciences, where he had been working as a full-time scientific researcher for the whole of his academic career. The degree of Doctor of Science was awarded to him in 1982 on the basis of a successful defence of the thesis entitled "Boundary value problems for the viscous gas equations and equations of nonhomogeneous fluids".

Alexander Kazhikhov published about 80 scientific works, the monograph "Boundary value problems in mechanics of nonhomogeneous fluids" written in 1983 in collaboration with S.N. Antontsev and V.N. Monakhov being the most known among them. All of Alexander Kazhikhov's works belong to the field of mathematical hydrodynamics. This field descends from the works of L. Euler, who in 1750 derived his famous equations of an ideal liquid. Later C.L. Navier (1822) and G.G. Stokes (1845) generalized Euler's equations taking into account viscosity effects. Since then, great progress has been achieved in understanding of the classical models of fluids. However, to date, there still remains unanswered a set of important mathematical questions regarding solvability of these equations, as well as uniqueness and stability of their solutions. Differential equations of fluid mechanics are of the greatest interest in applied mathematics, due to numerous applications in meteorology, aerodynamics, thermodynamics, physics of plasma, and many other fields. The contribution of Alexander Kazhikhov is very significant and has been widely internationally recognized.

Three major directions can be distinguished in Kazhikhov's studies.

He constructed the theory of boundary value problems for the Navier-Stokes equations of one-dimensional motion of viscous heat-conducting gas. His pioneering results in this field were established in the 1970s, by means of *a priori* estimates techniques. Alexander Kazhikhov became a master of those techniques.

Further, one of the first ever results on global solvability for equations of multi-dimensional motions of viscous gas is due to him; it was published in 1995 in a joint article with V.A. Weigant. Also, Kazhikhov established the fundamentals of the contemporary theory of viscous nonhomogeneous incompressible fluids, which was solidly demonstrated in the world-famous monograph of P.-L. Lions entitled "Mathematical topics in fluid mechanics" (1996).

One more set of remarkable results of Alexander Kazhikhov relates to the classical Euler equations of an ideal incompressible liquid. It must be emphasized that the question about well-posedness of boundary value problems for Euler's equations is nontrivial, even within the local setting. The theory built by N.M. Gunter, L. Lichtenstein, W. Wolibner, and N.E. Kochin left open an important question on well-posedness of the boundary value problem on flow of liquid through a given domain. In particular, it was unclear whether it was legitimate to impose boundary conditions on the velocity vector at the entrance of the flow region. Kazhikhov justified the well-posedness of this boundary value problem and of some closely related formulations proposed for modeling of liquid flows through given domains.

As a matter of fact, the words "for the first time" characterize many of Alexander Kazhikhov's results. The notion of renormalized solutions to the differential mass conservation law was introduced in 1989 by P.-L. Lions and R.J. DiPerna and remains rooted in one of the earlier works of Kazhikhov on equations of viscous nonhomogeneous fluids (1974). This notion aims at improving convergence of approximate solutions. In his work, Kazhikhov found that the weakly convergent sequence of approximate densities converges, in fact, strongly. This was verified by an analysis of the equation whose solution is the weak limit of the sequence of squares of approximate densities. Further generalization of this result became a basis of contemporary theory. The generalization consists in replacing the quadratic expression with a proper convex function.

The scientific achievements of Alexander Kazhikhov were honored, in 1978 and 1984, by the Prize of the Siberian Branch of the Soviet Union Academy of Sciences, and, in 1989, by the Silver Medal of the All Soviet Industrial Exposition. In 2003 he became Laureate of the Lavrentiev Prize of the Russian Academy of Sciences. Within the community of specialists on the Navier-Stokes equations he was outstanding for his persistent tackling of the most complicated and fundamental problems.

Alexander Kazhikhov frequently lectured abroad. He participated in all significant international conferences on mathematical hydrodynamics. One of the conferences was organized by Japanese mathematicians in Fukuoka in honor of his 50th anniversary.

Alexander Kazhikhov devoted much time and effort to scientific, administrative, and pedagogical activities. He was a member of the editorial boards of "Journal of Mathematical Fluid Mechanics", "Siberian Mathematical Journal", and "Vestnik – Quarterly Journal of Novosibirsk State University". For many years he actively participated in functions of various dissertation expertise councils and of the United scientific board for mathematics and mechanics at the Siberian Branch of the Russian Academy of Sciences. Starting from 1971, Alexander Kazhikhov taught at the Novosibirsk State University. From 1986 till 1991 he was Dean of the Mathematics and Mechanics Department. His lectures were always lucid and interesting. He took an enthusiastic interest Mathematical and Physical Olympiads for secondary school students. Many current researchers chose the profession of scientist under the influence of Professor Kazhikhov's charisma. Among his students, there are 16 Candidates and 5 Doctors of Science.

The science community around the world recognizes that the scientific legacy of Professor Alexander Vasilievich Kazhikhov will influence the progress of mathematical hydrodynamics for many more years.

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# Boundary Control Problems for Stationary Equations of Heat Convection

G.V. Alekseev and D.A. Tereshko

**Abstract.** Boundary control problems for the stationary Boussinesq equations under nonhomogeneous Dirichlet boundary condition for the velocity and mixed boundary conditions for the temperature are considered. Velocity vector on the boundary and heat flux on a part of the boundary are used as controls. Quadratic tracking-type functionals for the velocity or vorticity fields play the role of cost functionals. Solvability and uniqueness theorem for the considered boundary value problem is formulated. Optimality systems describing first-order necessary optimality conditions are derived and analyzed. Sufficient conditions to the data ensuring the local uniqueness and stability of optimal solutions for concrete cost functionals and controls are established. Numerical algorithm based on Newton's method for the optimality system and finite element method for linearized boundary value problems is proposed. Some computational results connected with the vortex reduction in the backward-facing-step channel by means of the heat flux on a part of the boundary are given and analyzed.

**Mathematics Subject Classification (2000).** 35B37, 76D55.

**Keywords.** Heat convection, flow control, uniqueness and stability estimates.

## 1. Introduction. Statement of extremum problem

Much attention has recently been given to statement and investigation of new problems for the models of hydrodynamics and heat convection. The control problems for Navier-Stokes and Boussinesq equations are examples of this kind of problems. A significant number of papers is devoted their study (see for example [1–6]).

Along with control problems, the inverse problems for models of heat and mass transfer play an important role. In these problems the unknown densities of boundary or distributed sources or the coefficients of model differential equations or boundary conditions are recovered from additional information on the solution to the original boundary value problem [7–12]. We note that inverse problems can be reduced to corresponding extremum problems by choosing a suitable minimized cost functional that adequately describes the inverse problem under consideration. As a result, both control and inverse problems can be analyzed by applying a unified approach based on the constrained optimization theory in Hilbert spaces [13].

The goal of this paper is a theoretical analysis of inverse extremum problems for the following model of heat transfer in a viscous incompressible heat-conducting fluid:

$$-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} - \tilde{\beta}T\mathbf{G},\ \operatorname{div}\mathbf{u} = 0\ \Omega\ \ \mathbf{u} = \mathbf{g}\ \Gamma, \tag{1.1}$$

$$\lambda\Delta T + \mathbf{u}\cdot\operatorname{grad}T = f\ \Omega,\ T = \psi\ \Gamma_D,\ \lambda(\partial T/\partial n + \alpha T) = \chi\ \Gamma_N. \tag{1.2}$$

Here $\Omega$ is a bounded domain in $\mathbb{R}^d$ $(d = 2,3)$ with a Lipschitz boundary $\Gamma$ consisting of two parts $\Gamma_D$ and $\Gamma_N$; $\mathbf{u}$ and $T$ are the velocity and temperature; $p = P/\rho$, where $P$ is the pressure and $\rho = \text{const} > 0$ is the density of the medium; $\nu = \text{const} > 0$, $\lambda = \text{const} > 0$ are the viscosity and thermal conductivity, respectively; $\mathbf{f}$ and $f$ are the volume densities of the external body forces and heat sources; $\mathbf{G} = (0,0,-G)$ is the acceleration of gravity, and $\tilde{\beta}$, $\mathbf{g}$, $\psi$, $\alpha$ and $\chi$ are given functions. In what follows, problem (1.1), (1.2) with given $\mathbf{f}$, $\mathbf{g}$, $\tilde{\beta}$, $f$, $\psi$, $\alpha$ and $\chi$ are referred to as problem 1. We note that all the quantities in (1.1), (1.2) are dimensional and their dimensions are defined in terms of SI units.

In this paper an inverse extremum problem will be formulated for the system (1.1), (1.2). It consists of finding a triple $(\mathbf{u}, p, T)$ and boundary functions $(\mathbf{g}, \chi)$ from (1.1), (1.2) and conditions on the minimum of the specific cost functional which depends on $\mathbf{u}$, $p$, $T$, $\mathbf{g}$ and $\chi$. Our main attention is focused on analysis of the local uniqueness and stability of solutions to the inverse extremum problems in question. We also discuss results of numerical experiments for concrete extremum problems.

As in [8] we shall use the Sobolev spaces $H^s(D)$ with $s \in \mathbb{R}$ and $L^r(D)$ with $r \geq 2$ where $D$ denotes $\Omega$, its subset $Q$, $\Gamma$ or its part $\Gamma_0$ with positive measure. The corresponding spaces of vector functions are denoted by $\mathbf{H}^s(D)$ and $\mathbf{L}^r(D)$. The norms and inner products in $H^s(Q)$, $H^s(\Gamma)$ and their vector analogies are denoted by $\|\cdot\|_{s,Q}$, $\|\cdot\|_{s,\Gamma}$ and $(\cdot,\cdot)_{s,Q}$, $(\cdot,\cdot)_{s,\Gamma}$. The inner products and norms in $L^2(Q)$ or $\mathbf{L}^2(Q)$ are denoted by $(\cdot,\cdot)_Q$ and $\|\cdot\|_Q$. If $Q = \Omega$ then we set $\|\cdot\|_\Omega = \|\cdot\|$, $(\cdot,\cdot)_\Omega = (\cdot,\cdot)$. The inner product and norm in $L^2(\Gamma_N)$ are denoted by $(\cdot,\cdot)_{\Gamma_N}$ and $\|\cdot\|_{\Gamma_N}$. The norm and seminorm in $H^1(\Omega)$ and $\mathbf{H}^1(\Omega)$ are denoted by $\|\cdot\|_1$ and $|\cdot|_1$. The duality relation for the pair of dual spaces $X$ and $X^*$ is denoted by $\langle\cdot,\cdot\rangle_{X^*\times X}$ or simply $\langle\cdot,\cdot\rangle$. Let the following assumptions hold.

(i) $\Omega$ is a bounded domain in $\mathbb{R}^d$, $d = 2,3$ with a boundary $\Gamma \in C^{0,1}$ consisting of $N$ connected components $\Gamma_i$, $i = 1,2,\ldots,N$. The open segments $\Gamma_D$ and $\Gamma_N$ of $\Gamma$ obey the conditions $\Gamma_D \in C^{0,1}$, $\Gamma_N \in C^{0,1}$, $\Gamma_D \neq \emptyset$, $\Gamma_D \cap \Gamma_N = \emptyset$, $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N$.

Let $\mathcal{D}(\Omega)$ be the space of infinitely differentiable finite in $\Omega$ functions, $H_0^1(\Omega)$ be a closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$, $\mathbf{H}_0^1(\Omega) = H_0^1(\Omega)^d$, $\mathbf{V} = \{\mathbf{v} \in \mathbf{H}_0^1(\Omega) : \operatorname{div}\mathbf{v} = 0\}$, $\mathbf{H}^{-1}(\Omega) = \mathbf{H}_0^1(\Omega)^*$, $L_0^2(\Omega) = \{p \in L^2(\Omega) : (p,1) = 0\}$, $\mathcal{T} = H^1(\Omega,\Gamma_D) \equiv \{S \in H^1(\Omega) : S|_{\Gamma_D} = 0\}$. We shall use the following inequalities which are implied by the embedding theorems and the continuity of the trace operator

$$\|\mathrm{rot}\,\mathbf{u}\| \le C_1\|\mathbf{u}\|_1, \quad \|\mathrm{div}\,\mathbf{u}\| \le C_2\|\mathbf{u}\|_1,$$
$$\|\mathbf{u}\|_Q \le C_3\|\mathbf{u}\|_1, \quad \|\mathbf{u}\|_{1/2,\Gamma} \le C_\Gamma\|\mathbf{u}\|_1 \ \forall \mathbf{u} \in \mathbf{H}^1(\Omega). \tag{1.3}$$

Here $C_1, C_2, C_3, C_\Gamma$ are constants depending on $\Omega$.

Together with $\mathbf{H}^1(\Omega)$ and $\mathbf{H}^{1/2}(\Gamma)$ we shall consider their closed subspaces $\tilde{\mathbf{H}}^1(\Omega) = \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \mathbf{u}\cdot\mathbf{n}|_{\Gamma_N} = 0, (\mathbf{u},\mathbf{n})_{\Gamma_i} = 0,\ i = \overline{1,N}\}$, $\tilde{\mathbf{H}}^1_{\mathrm{div}}(\Omega) = \{\mathbf{v} \in \tilde{\mathbf{H}}^1(\Omega) : \mathrm{div}\mathbf{v} = 0\}$, $\tilde{\mathbf{H}}^{1/2}(\Gamma) = \{\mathbf{u}|_\Gamma : \mathbf{u} \in \tilde{\mathbf{H}}^1(\Omega)\}$, and also duals $\tilde{\mathbf{H}}^1(\Omega)^*$, $\tilde{\mathbf{H}}^{1/2}(\Gamma)^*$ of the spaces $\tilde{\mathbf{H}}^1(\Omega)$, $\tilde{\mathbf{H}}^{1/2}(\Gamma)$. Let us introduce the following bilinear and trilinear forms: $a_0 : \mathbf{H}^1(\Omega)^2 \to \mathbb{R}$, $b : \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \to \mathbb{R}$, $a_1 : H^1(\Omega)^2 \to \mathbb{R}$, $b_1 : H^1(\Omega) \times \mathbf{H}^1_0(\Omega) \to \mathbb{R}$, $c : \mathbf{H}^1(\Omega)^3 \to \mathbb{R}$, $c_1 : \mathbf{H}^1(\Omega) \times H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by

$$a_0(\mathbf{u},\mathbf{v}) = (\nabla\mathbf{u},\nabla\mathbf{v}),\ b(\mathbf{v},q) = -(\mathrm{div}\mathbf{v},q),\ c(\mathbf{u},\mathbf{v},\mathbf{w}) = ((\mathbf{u}\cdot\mathrm{grad})\mathbf{v},\mathbf{w}),$$

$$a_1(T,S) = (\nabla T,\nabla S),\ b_1(S,\mathbf{v}) = (\mathbf{b}S,\mathbf{v}),\ c_1(\mathbf{u},T,S) = (\mathbf{u}\cdot\nabla T,S),\ \ \mathbf{b} = \tilde{\beta}\mathbf{G}.$$

We note that forms $c$ and $c_1$ possess the next properties [14, 15]

$$c(\mathbf{u},\mathbf{v},\mathbf{w}) = -c(\mathbf{u},\mathbf{w},\mathbf{v}),\ \ c(\mathbf{u},\mathbf{v},\mathbf{v}) = 0\ \forall\mathbf{u} \in \tilde{\mathbf{H}}^1_{\mathrm{div}}(\Omega),\ \ (\mathbf{v},\mathbf{w}) \in \mathbf{H}^1_0(\Omega)\times\mathbf{H}^1(\Omega), \tag{1.4}$$

$$c_1(\mathbf{u},T,S) = -c_1(\mathbf{u},S,T), c_1(\mathbf{u},T,T) = 0\ \ \forall\mathbf{u} \in \tilde{\mathbf{H}}^1_{\mathrm{div}}(\Omega), (T,S) \in \mathcal{T}\times H^1(\Omega). \tag{1.5}$$

Besides all the forms are continuous and the following technical lemma holds [15].

**Lemma 1.** *Under conditions* (i) *there exist constants $\delta_0$, $\gamma_0$, $\gamma_1$, $\gamma_2$ and $\beta_1$ depending on $\Omega$ such that*

$$|a_0(\mathbf{u},\mathbf{v})| \le \|\mathbf{u}\|_1\|\mathbf{v}\|_1\ \forall(\mathbf{u},\mathbf{v}) \in \mathbf{H}^1(\Omega)^2,\ a_0(\mathbf{v},\mathbf{v}) \ge \delta_0\|\mathbf{v}\|_1^2\ \ \forall\mathbf{v} \in \mathbf{H}^1_0(\Omega), \tag{1.6}$$

$$|a_1(T,S)| \le \|T\|_1\|S\|_1\ \ \forall(T,S) \in H^1(\Omega)\times H^1(\Omega),\ \ a_1(T,T) \ge \delta_1\|T\|_1^2\ \ \forall T \in \mathcal{T}, \tag{1.7}$$

$$|c(\mathbf{u},\mathbf{v},\mathbf{w})| \le \gamma_0\|\mathbf{u}\|_1\|\mathbf{v}\|_1\|\mathbf{w}\|_1\ \ \forall(\mathbf{u},\mathbf{v},\mathbf{w}) \in \mathbf{H}^1(\Omega)^3, \tag{1.8}$$

$$|c_1(\mathbf{u},T,S)| \le \gamma_1\|\mathbf{u}\|_1\|T\|_1\|S\|_1\ \ \forall\mathbf{u} \in \mathbf{H}^1(\Omega),\ (T,S) \in H^1(\Omega)\times H^1(\Omega), \tag{1.9}$$

$$|b_1(T,\mathbf{v})| \le \beta_1\|T\|_1\|\mathbf{v}\|_1\ \ \forall T \in H^1(\Omega), \mathbf{v} \in \mathbf{H}^1_0(\Omega), \tag{1.10}$$

$$|(\chi,T)_{\Gamma_N}| \le \|\chi\|_{\Gamma_N}\|T\|_{\Gamma_N} \le \gamma_2\|\chi\|_{\Gamma_N}\|T\|_1\ \forall T \in H^1(\Omega). \tag{1.11}$$

*Bilinear form $b(\cdot,\cdot)$ satisfies the* inf-sup *condition*

$$\inf_{q\in L^2_0(\Omega),q\ne0}\ \sup_{\mathbf{v}\in\mathbf{H}^1_0(\Omega),\mathbf{v}\ne0}\frac{b(\mathbf{v},q)}{\|\mathbf{v}\|_1\|q\|} \ge \beta = \mathrm{const} > 0. \tag{1.12}$$

Let in addition to (i) the following conditions hold:

(ii) $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$, $f \in L^2(\Omega)$, $\mathbf{b} \in \mathbf{L}^2(\Omega)$, $\psi \in H^{1/2}(\Gamma_D)$, $\alpha \in L^2(\Gamma_N)$;

(iii) $\mathbf{g} \in \tilde{\mathbf{H}}^{1/2}(\Gamma)$, $\chi \in L^2(\Gamma_N)$.

Now we divide the set of all input data in problem (1.1), (1.2) into two groups. One consists of the control functions $\mathbf{g}$ and $\chi$, and the other consists of fixed data, namely $\mathbf{f}$, $f$, $\mathbf{b}$, $\psi$ and $\alpha$. Assume that controls $\mathbf{g}$, $\chi$ vary over some sets $K_1$ and $K_2$ such that

(j) $K_1 \subset \tilde{\mathbf{H}}^{1/2}(\Gamma)$, $K_2 \in L^2(\Gamma_N)$ are nonempty convex closed subsets.

Let $X = \tilde{\mathbf{H}}^1(\Omega) \times L_0^2(\Omega) \times H^1(\Omega)$, $Y = \mathbf{H}^{-1}(\Omega) \times L_0^2(\Omega) \times \tilde{\mathbf{H}}^{1/2}(\Gamma) \times \mathcal{T}^* \times H^{1/2}(\Gamma_D)$, $K = K_1 \times K_2$, $\mathbf{x} = (\mathbf{u}, p, T) \in X$. Introduce an operator $F \equiv (F_1, F_2, F_3, F_4, F_5) : X \times K \to Y$, defined by

$$\langle F_1(\mathbf{x}, u), \mathbf{v}\rangle = \nu a_0(\mathbf{u}, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) + b_1(T, \mathbf{v}) - \langle \mathbf{f}, \mathbf{v}\rangle,$$

$$\langle F_2(\mathbf{x}, u), s\rangle = b(\mathbf{u}, s) \equiv -(\operatorname{div} \mathbf{u}, s),\ F_3(\mathbf{x}, u) = \mathbf{u}|_\Gamma - \mathbf{g},\ F_5(\mathbf{x}, u) = T|_{\Gamma_D} - \psi,$$

$$\langle F_4(\mathbf{x}, u), S\rangle = \lambda a_1(T, S) + \lambda(\alpha T, S)_{\Gamma_N} + c_1(\mathbf{u}, T, S) - (f, S) - (\chi, S)_{\Gamma_N}.$$

We multiply the first equation in (1.1) by $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$, the equation in (1.2) by $S \in \mathcal{T}$, integrate the results over $\Omega$ with use of Green formulas, and use boundary conditions in (1.1), (1.2) to obtain a week formulation of problem 1. It consists of finding a triple $\mathbf{x} = (\mathbf{u}, p, T) \in X$ satisfying the relations

$$\nu a_0(\mathbf{u}, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) + b_1(T, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v}\rangle \ \ \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \tag{1.13}$$

$$\lambda a_1(T, S) + \lambda(\alpha T, S)_{\Gamma_N} + c_1(\mathbf{u}, T, S) = \langle l, S\rangle \ \equiv (f, S) + (\chi, S)_{\Gamma_N} \ \forall S \in \mathcal{T}, \tag{1.14}$$

$$\operatorname{div} \mathbf{u} = 0 \text{ in } \Omega,\ \mathbf{u}|_\Gamma = \mathbf{g},\ \ T|_{\Gamma_D} = \psi, \tag{1.15}$$

which one can rewrite in equivalent form

$$F(\mathbf{x}, u) \equiv F(\mathbf{u}, p, T, \mathbf{g}, \chi) = 0. \tag{1.16}$$

This triple $(\mathbf{u}, p, T) \in X$ will be called the weak solution to problem 1.

Let $I : X \to \mathbb{R}$ be a weakly lower semicontinuous functional. Setting $K = K_1 \times K_2$, $u = (\mathbf{g}, \chi)$, $u_0 = (\mathbf{f}, f, \mathbf{b}, \psi, \alpha)$ we formulate the following constrained minimization problem:

$$\begin{gathered} J(\mathbf{x}, u) = (\mu_0/2)I(\mathbf{x}) + (\mu_1/2)\|\mathbf{g}\|^2_{1/2,\Gamma} + (\mu_2/2)\|\chi\|^2_{\Gamma_N} \to \inf, \\ F(\mathbf{x}, u) = 0,\ (\mathbf{x}, u) \in X \times K. \end{gathered} \tag{1.17}$$

Here $\mu_0 > 0$ and $\mu_1 \geq 0, \mu_2 \geq 0$ are positive dimensional parameters which serve to regulate the relative importance of each of the terms in (1.17). Another purpose of introducing $\mu_l$ is to ensure the uniqueness and stability of solutions to the control problems under consideration (see below).

The possible cost functionals are defined as

$$I_1(\mathbf{u}) = \|\mathbf{u} - \mathbf{v}_d\|^2_Q,\ \ I_2(\mathbf{u}) = \|\mathbf{u} - \mathbf{v}_d\|^2_{1,Q},\ \ I_3(\mathbf{u}) = \|\operatorname{rot} \mathbf{u} - \zeta_d\|^2_Q. \tag{1.18}$$

Here $Q$ is a subset of $\Omega$, $\mathbf{v}_d \in \mathbf{L}^2(Q)$ (or $\mathbf{v}_d \in \mathbf{H}^1(Q)$) and $\zeta_d \in \mathbf{L}^2(Q)$ are functions which are interpreted as measured velocity and vorticity fields. Define $Z_{ad} = \{(\mathbf{x},u) \in X \times K : F(\mathbf{x},u) = 0, J(\mathbf{x},u) < \infty\}$. Let us assume in addition to (j) that

(jj) $\mu_0 > 0$, $\mu_1 \geq 0$, $\mu_2 \geq 0$ and $K$ is a bounded subset or $\mu_0 > 0$, $\mu_1 > 0$, $\mu_2 > 0$ and $I$ is bounded from below.

According to the general theory of extremum problems (see [13]) we introduce an element $\mathbf{y}^* = (\xi, \sigma, \zeta, \theta, \zeta^c) \in Y^* = \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \tilde{\mathbf{H}}^{1/2}(\Gamma)^* \times \mathcal{T} \times H^{1/2}(\Gamma_D)^*$

which is referred to as the adjoint state, and we define the Lagrangian $\mathcal{L} : X \times K \times \mathbb{R}^+ \times Y^* \to \mathbb{R}$, where $\mathbb{R}^+ = \{\lambda \in \mathbb{R} : \lambda \geq 0\}$, by the formula

$$\mathcal{L}(\mathbf{x}, u, \lambda_0, \mathbf{y}^*) = \lambda_0 J(\mathbf{x}, u) + \langle F_1(\mathbf{x}, u), \xi \rangle + (F_2(\mathbf{x}, u), q) + \langle \zeta, F_3(\mathbf{x}, u) \rangle_\Gamma + \kappa \langle F_4(\mathbf{x}, u), \theta \rangle + \kappa \langle \zeta^c, F_5(\mathbf{x}, u) \rangle_{\Gamma_D}.$$

Here

$$\langle \zeta, \mathbf{g} \rangle_\Gamma = \langle \zeta, \mathbf{g} \rangle_{\tilde{\mathbf{H}}^{1/2}(\Gamma)^* \times \tilde{\mathbf{H}}^{1/2}(\Gamma)} \quad \text{for} \quad \zeta \in \tilde{\mathbf{H}}^{1/2}(\Gamma)^*,$$
$$\langle \zeta^c, \psi \rangle_{\Gamma_D} = \langle \zeta^c, \psi \rangle_{H^{1/2}(\Gamma_D)^* \times H^{1/2}(\Gamma_D)} \quad \text{for} \quad \zeta^c \in H^{1/2}(\Gamma_D),$$

$\kappa$ is a dimensional parameter. Let the dimension $[\kappa]$ be chosen so that the dimensions of $\xi, s, \theta$ at the adjoint state $\mathbf{y}^*$ coincide with those at the basic state, i.e.,

$$[\xi] = [\mathbf{u}] = L_0/T_0, \ [\theta] = [T] = K_0, \ [s] = [p] = L_0^2/T_0^2. \tag{1.19}$$

Here $L_0$, $T_0$, $M_0$, $K_0$ denote the SI dimensions of the length, time, mass and temperature units expressed in meters, seconds, kilograms and degrees Kelvin respectively. A simple analysis of (1.19) shows that a necessary condition for fulfillment of (1.19) is $[\kappa] = L_0^2/T_0^2 K_0^2$.

Below we shall use some results concerning problem 1 and extremum problem (1.17). The proofs of the theorems are similar to those in [15].

**Theorem 1.** *Let conditions* (i), (ii) *be satisfied. Then for any $u \in K$ problem* 1 *has a weak solution $(\mathbf{u}, p, T) \in X$ that satisfies the estimates $\|\mathbf{u}\|_1 \leq M_\mathbf{u}(u_0, u)$, $\|p\| \leq M_p(u_0, u)$, $\|T\|_1 \leq M_T(u_0, u)$. Here $M_\mathbf{u}(u_0, u)$, $M_p(u_0, u)$ and $M_T(u_0, u)$ are nondecreasing continuous functions of the norms $\|\mathbf{f}\|_{-1}$, $\|f\|$, $\|\mathbf{b}\|$, $\|\alpha\|_{\Gamma_N}$, $\|\mathbf{g}\|_{1/2,\Gamma}$, $\|\psi\|_{1/2,\Gamma_D}$, $\|\chi\|_{\Gamma_N}$. If the functions $\mathbf{f}, f, \mathbf{g}, \psi, \chi$ are small* (*or the viscosity $\nu$ is high*) *in the sense that*

$$\frac{\gamma_0 M_\mathbf{u}(u_0, u)}{\delta_0 \nu} + \frac{1}{\delta_0 \nu} \frac{\beta_1 \gamma_1 M_T^0(u_0, u)}{\delta_1 \lambda} < 1, \tag{1.20}$$

*then the weak solution to problem 1 is unique. Here $\delta_0, \delta_1, \gamma_0, \gamma_1, \beta_1$ are the constants from* (1.6)–(1.10).

**Theorem 2.** *Under conditions* (i), (ii), (j) *and* (jj) *let $I : X \to \mathbb{R}$ be a weakly lower semicontinuous functional and $Z_{ad} \neq \emptyset$. Then, control problem* (1.17) *has at least one solution.*

**Theorem 3.** *Under conditions* (i), (ii), (j) *let $\mu_0 > 0$, $\mu_l > 0$ or $\mu_0 > 0$, $\mu_l \geq 0$ and $K_l$ be the bounded sets, $l = 1, 2$. Then, control problem* (1.17) *has at least one solution for $I = I_k$, $k = 1, 2, 3$.*

**Theorem 4.** *Under conditions* (i), (ii), (j), (jj) *let $(\hat{\mathbf{x}}, \hat{u}) \equiv (\hat{\mathbf{u}}, \hat{p}, \hat{T}, \hat{\mathbf{g}}, \hat{\chi}) \in X \times K$ be a local minimizer in problem* (1.17) *and let the functional $I$, not depending on pressure $p$, be continuously differentiable at the point $\hat{\mathbf{x}}$. Then, there exists a nonzero Lagrange multiplier $(\lambda_0, \mathbf{y}^*) = (\lambda_0, \xi, \sigma, \zeta, \theta, \zeta^c) \in \mathbb{R}^+ \times \mathbf{V} \times L_0^2(\Omega) \times$*

$\tilde{\mathbf{H}}^{1/2}(\Gamma)^*\times\mathcal{T}\times H^{1/2}(\Gamma_D)^*$ *that satisfies the Euler-Lagrange equation* $F'_{\mathbf{x}}(\hat{\mathbf{x}},\hat{u})^*\mathbf{y}^* = -\lambda_0 J'_{\mathbf{x}}(\hat{\mathbf{x}},\hat{u})$, *which is equivalent to the identities*

$$\nu a_0(\mathbf{w},\xi)+c(\hat{\mathbf{u}},\mathbf{w},\xi)+c(\mathbf{w},\hat{\mathbf{u}},\xi)+\kappa c_1(\mathbf{w},\hat{T},\theta)+b(\mathbf{w},\sigma)+\langle\zeta,\mathbf{w}\rangle_\Gamma$$
$$= -\lambda_0(\mu_0/2)\langle I'_{\mathbf{u}}(\hat{\mathbf{x}}),\mathbf{w}\rangle \ \ \forall\mathbf{w}\in\tilde{\mathbf{H}}^1(\Omega), \tag{1.21}$$

$$\kappa[\lambda a_1(\tau,\theta)+\lambda(\alpha\tau,\theta)_{\Gamma_N}+c_1(\hat{\mathbf{u}},\tau,\theta)+\langle\zeta^c,\tau\rangle_{\Gamma_D}]$$
$$+b_1(\tau,\xi) = -\lambda_0(\mu_0/2)\langle I'_T(\hat{\mathbf{x}}),\tau\rangle \ \ \forall\tau\in H^1(\Omega), \tag{1.22}$$

*and satisfies the minimum principle* $\mathcal{L}(\hat{\mathbf{x}},\hat{u},\lambda_0,\mathbf{y}^*)\le\mathcal{L}(\hat{\mathbf{x}},u,\lambda_0,\mathbf{y}^*)$ *for all* $u\in K$, *which is equivalent to the variational inequality*

$$\begin{aligned}\lambda_0\mu_1(\hat{\mathbf{g}},\mathbf{g}-\hat{\mathbf{g}})_{1/2,\Gamma}-\langle\zeta,\mathbf{g}-\hat{\mathbf{g}}\rangle_\Gamma+\lambda_0\mu_2(\hat{\chi},\chi-\hat{\chi})_{\Gamma_N}\\ -\kappa(\theta,\chi-\hat{\chi})_{\Gamma_N}\ge 0\ \forall u=(\mathbf{g},\chi).\end{aligned}\tag{1.23}$$

**Theorem 5.** *Let the assumptions of Theorem* 4 *be satisfied and inequality* (1.20) *hold for all* $u\in K$. *Then:* 1) *homogeneous problem* (1.21), (1.22) (*under* $\lambda_0=0$) *has only a trivial solution* $\mathbf{y}^*\equiv(\xi,\sigma,\zeta,\theta,\zeta^c)=0$; 2) *any nontrivial Lagrange multiplier satisfying* (1.21), (1.22) *is regular, i.e., it has the form* $(1,\mathbf{y}^*)$.

Relations (1.21), (1.22), together with variational inequality (1.23) and operator constraint (1.16), constitute an optimality system. It consists of three parts. The first part has the form of weak statement (1.13)–(1.15) of problem 1, which is equivalent to operator equation (1.16). The second part consists of identities (1.21), (1.22) for the Lagrange multipliers $\xi$, $\sigma$, $\zeta$, $\theta$ and $\zeta^c$. Finally, the last part represents the minimum principle, which is equivalent to inequality (1.23) with respect to the controls $\mathbf{g}$ and $\chi$.

## 2. General property of the optimality system solutions

Let us consider control problem (1.17). Denote by $(\mathbf{x}_1,u_1)\equiv(\mathbf{u}_1,p_1,T_1,\mathbf{g}_1,\chi_1)\in X\times K$ its solution. By $(\mathbf{x}_2,u_2)\equiv(\mathbf{u}_2,p_2,T_2,\mathbf{g}_2,\chi_2)\in X\times K$ we denote a solution of problem

$$\tilde{J}(\mathbf{x},u)=\frac{\mu_0}{2}\tilde{I}(\mathbf{x})+\frac{\mu_1}{2}\|\mathbf{g}\|^2_{1/2,\Gamma}+\frac{\mu_2}{2}\|\chi\|^2_{\Gamma_N}\to\inf,\ F(\mathbf{x},u)=0,\ (\mathbf{x},u)\in X\times K, \tag{2.1}$$

which is obtained from (1.17) by replacing a functional $I$ in (1.17) with another one $\tilde{I}$. In view of Theorem 1 the following estimates for pairs $(\mathbf{u}_i,p_i)$ hold:

$$\begin{aligned}\|\mathbf{u}_i\|_1 &\le M^0_{\mathbf{u}}=\sup_{u\in K}M_{\mathbf{u}}(u_0,u),\\ \|p_i\| &\le M^0_p=\sup_{u\in K}M_p(u_0,u),\\ \|T_i\|_1 &\le M^0_T=\sup_{u\in K}M_T(u_0,u).\end{aligned}\tag{2.2}$$

Let us introduce "model" Reynolds number $\mathcal{R}e$, Rayleigh number $\mathcal{R}a$ and Prandtl number $\mathcal{P}$ by

$$\mathcal{R}e = \frac{\gamma_0 M^0_{\mathbf{u}}}{\delta_0 \nu},\ \mathcal{R}a = \frac{\gamma_1}{\delta_0 \nu}\frac{\beta_1 M^0_T}{\delta_1 \lambda},\ \mathcal{P} = \frac{\delta_0 \nu}{\delta_1 \lambda} \tag{2.3}$$

and assume that

$$\mathcal{R}e + \mathcal{R}a \equiv \frac{\gamma_0 M^0_{\mathbf{u}}}{\delta_0 \nu} + \frac{\gamma_1}{\delta_0 \nu}\frac{\beta_1 M^0_T}{\delta_1 \lambda} < 1/2. \tag{2.4}$$

Denote by $(1, \mathbf{y}_i^*) \equiv (1, \xi_i, \sigma_i, \zeta_i, \theta_i, \zeta_i^c)$, $i = 1, 2$, the Lagrange multipliers corresponding to the solutions $(\mathbf{x}_i, u_i)$ (these multipliers are uniquely determined under condition (2.4)). By definition elements $(\xi_i, \sigma_i, \zeta_i, \theta_i, \zeta_i^c)$ satisfy relations

$$\nu a_0(\mathbf{w}, \xi_i) + c(\mathbf{u}_i, \mathbf{w}, \xi_i) + c(\mathbf{w}, \mathbf{u}_i, \xi_i) + \kappa c_1(\mathbf{w}, T_i, \theta_i) + b(\mathbf{w}, \sigma_i) + \langle \zeta_i, \mathbf{w}\rangle_\Gamma$$

$$= -(\mu_0/2)\langle (I^i)'_{\mathbf{u}}(\mathbf{x}_i), \mathbf{w}\rangle\ \forall \mathbf{w} \in \tilde{\mathbf{H}}^1(\Omega), \tag{2.5}$$

$$\kappa[\lambda a_1(\tau, \theta_i) + \lambda(\alpha\tau, \theta_i) + c_1(\mathbf{u}_i, \tau, \theta_i) + \langle \zeta_i^c, \tau\rangle_{\Gamma_D}] + b_1(\tau, \xi_i)$$

$$= -(\mu_0/2)\langle (I^i)'_T(\mathbf{x}_i), \tau\rangle\ \ \forall \tau \in H^1(\Omega). \tag{2.6}$$

Here we renamed $I = I^1$, $\tilde{I} = I^2$. Let $\mathbf{g} = \mathbf{g}_1 - \mathbf{g}_2$, $\chi = \chi_1 - \chi_2$, $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$, $p = p_1 - p_2$, $T = T_1 - T_2$, $\xi = \xi_1 - \xi_2$, $\sigma = \sigma_1 - \sigma_2$, $\zeta = \zeta_1 - \zeta_2$, $\theta = \theta_1 - \theta_2$, $\zeta^c = \zeta_1^c - \zeta_2^c$. Subtracting equations (1.13)–(1.15), written for $\mathbf{u}_2, p_2, T_2, u_2$ from the corresponding equations (1.13)–(1.15) for $\mathbf{u}_1, p_1, T_1, u_1$ gives

$$\nu a_0(\mathbf{u}, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}_1, \mathbf{v}) + c(\mathbf{u}_2, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) + b_1(T, \mathbf{v}) = 0\ \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega), \tag{2.7}$$

$$\lambda a_1(T, S) + \lambda(\alpha T, S) + c_1(\mathbf{u}, T_1, S) + c_1(\mathbf{u}_2, T, S) = \ (\chi, S)_{\Gamma_N}\ \forall S \in \mathcal{T}, \tag{2.8}$$

$$\operatorname{div} \mathbf{u} = 0 \text{ in } \Omega,\ \mathbf{u}|_\Gamma = \mathbf{g},\ T|_{\Gamma_D} = 0. \tag{2.9}$$

Setting $\mathbf{g} = \mathbf{g}_1$, $\chi = \chi_1$ in (1.23) under $\lambda_0 = 1$, written for $\hat{\mathbf{g}} = \mathbf{g}_2$, $\hat{\chi} = \chi_2$, $\zeta = \zeta_2$ and setting $\mathbf{g} = \mathbf{g}_2$, $\chi = \chi_2$ in (1.23) written for $\hat{\mathbf{g}} = \mathbf{g}_1$, $\hat{\chi} = \chi_1$, $\zeta = \zeta_1$, we obtain $\mu_1(\mathbf{g}_2, \mathbf{g})_{1/2,\Gamma} - \langle \zeta_2, \mathbf{g}\rangle_\Gamma + \mu_2(\chi_2, \chi)_{\Gamma_N} - \kappa(\theta_2, \chi)_{\Gamma_N} \geq 0$, $-\mu_1(\mathbf{g}_1, \mathbf{g})_{1/2,\Gamma} + \langle \zeta_1, \mathbf{g}\rangle_\Gamma - \mu_2(\chi_1, \chi)_{\Gamma_N} + \kappa(\theta_1, \chi)_{\Gamma_N} \geq 0$. Adding up these inequalities yields the relation

$$-\langle \zeta, \mathbf{g}\rangle_\Gamma - \kappa(\theta, \chi)_{\Gamma_N} \leq -\mu_1\|\mathbf{g}\|^2_{1/2,\Gamma} - \mu_2\|\chi\|^2_{\Gamma_N}. \tag{2.10}$$

Subtract equations (2.5), (2.6) written for $(\mathbf{x}_2, u_2, \mathbf{y}_2^*)$ from corresponding equations for $(\mathbf{x}_1, u_1, \mathbf{y}_1^*)$. We obtain

$$\nu a_0(\mathbf{w}, \xi) + c(\mathbf{u}_1, \mathbf{w}, \xi) + c(\mathbf{u}, \mathbf{w}, \xi_2) + c(\mathbf{w}, \mathbf{u}_1, \xi)$$

$$+c(\mathbf{w}, \mathbf{u}, \xi_2) + \kappa c_1(\mathbf{w}, T_1, \theta) + \kappa c_1(\mathbf{w}, T, \theta_2) + b(\mathbf{w}, \sigma)$$

$$= -\langle \zeta, \mathbf{w}\rangle_\Gamma - (\mu_0/2)\langle I'_{\mathbf{u}}(\mathbf{x}_1) - \tilde{I}'_{\mathbf{u}}(\mathbf{x}_2), \mathbf{w}\rangle\ \forall \mathbf{w} \in \tilde{\mathbf{H}}^1(\Omega), \tag{2.11}$$

$$\kappa[\lambda a_1(\tau, \theta) + \lambda(\alpha\tau, \theta)_{\Gamma_N} + c_1(\mathbf{u}_1, \tau, \theta) + c_1(\mathbf{u}, \tau, \theta_2) + \langle \zeta^c, \tau\rangle_{\Gamma_D}]$$

$$+b_1(\tau, \xi) = -(\mu_0/2)\langle I'_T(\mathbf{x}_1) - I'_T(\mathbf{x}_2), \tau\rangle\ \forall \tau \in H^1(\Omega). \tag{2.12}$$

Set $\mathbf{w} = \mathbf{u}$, $\tau = T$ in (2.11), (2.12) and add up the results. Taking into account (2.9) we obtain

$$\nu a_0(\mathbf{u},\xi)+c(\mathbf{u}_1,\mathbf{u},\xi)+2c(\mathbf{u},\mathbf{u},\xi_2)+c(\mathbf{u},\mathbf{u}_1,\xi)+\kappa c_1(\mathbf{u},T_1,\theta)+\kappa c_1(\mathbf{u},T,\theta_2)+\langle\zeta,\mathbf{g}\rangle_\Gamma$$
$$+\kappa[\lambda a_1(T,\theta)+\lambda(\alpha T,\theta)_{\Gamma_N}+c_1(\mathbf{u}_1,T,\theta)+c_1(\mathbf{u},T,\theta_2)]$$
$$+b_1(T,\xi)+b(\xi,p) = -(\mu_0/2)\langle I'_{\mathbf{u}}(\mathbf{x}_1)-\tilde{I}'_{\mathbf{u}}(\mathbf{x}_2),\mathbf{u}\rangle-(\mu_0/2)\langle I'_T(\mathbf{x}_1)-\tilde{I}'_T(\mathbf{x}_2),T\rangle. \tag{2.13}$$

Set $\mathbf{v} = \xi$ in (2.7), $S = \kappa\theta$ in (2.8) and subtract the results from (2.13). Using (2.10) and identities

$$2c(\mathbf{u},\mathbf{u},\xi_2)+c(\mathbf{u}_1,\mathbf{u},\xi)-c(\mathbf{u}_2,\mathbf{u},\xi) = 2c(\mathbf{u},\mathbf{u},\xi_2)+c(\mathbf{u},\mathbf{u},\xi) = c(\mathbf{u},\mathbf{u},\xi_1+\xi_2),$$
$$2c_1(\mathbf{u},T,\theta_2)+c_1(\mathbf{u},T_1,\theta)-c_1(\mathbf{u},T_2,\theta) = c_1(\mathbf{u},T,\theta_1+\theta_2),$$

we obtain

$$c(\mathbf{u},\mathbf{u},\xi_1+\xi_2)+\kappa c_1(\mathbf{u},T,\theta_1+\theta_2)+(\mu_0/2)\langle I'_{\mathbf{u}}(\mathbf{x}_1)-\tilde{I}'_{\mathbf{u}}(\mathbf{x}_2),\mathbf{u}\rangle$$
$$+(\mu_0/2)\langle I'_T(\mathbf{x}_1)-\tilde{I}'_T(\mathbf{x}_2),T\rangle \le -\mu_1\|\mathbf{g}\|^2_{1/2,\Gamma}-\mu_2\|\chi\|^2_{\Gamma_N}. \tag{2.14}$$

Thus the following result holds.

**Theorem 6.** *Let, under conditions of Theorem 4 for $I^1 = I$ and $I^2 = \tilde{I}$, pairs $(\mathbf{x}_1,u_1) = (\mathbf{u}_1,p_1,T_1,\mathbf{g}_1,\chi_1)$ and $(\mathbf{x}_2,u_2) = (\mathbf{u}_2,p_2,T_2,\mathbf{g}_2,\chi_2)$ be solutions to problems (1.17) and (2.1) respectively; $\mathbf{y}_i^* = (\xi_i,\sigma_i,\zeta_i,\theta_i,\zeta_i^c)$, $i = 1,2$, are the Lagrange multipliers corresponding to these solutions $(\mathbf{x}_i,u_i)$. Then the relation (2.14) for differences $\mathbf{u} = \mathbf{u}_1-\mathbf{u}_2$, $p = p_1-p_2$, $T = T_1-T_2$, $\mathbf{g} = \mathbf{g}_1-\mathbf{g}_2$, $\chi = \chi_1-\chi_2$ holds.*

Denote by $\mathbf{u}_0 \in \mathbf{H}^1(\Omega)$ a function such that $\operatorname{div}\mathbf{u}_0 = 0$, $\mathbf{u}_0|_\Gamma = \mathbf{g}$, $\|\mathbf{u}_0\|_1 \le C_0\|\mathbf{g}\|_{1/2,\Gamma}$. Here a constant $C_0$ depends on $\Omega$. The existence of $\mathbf{u}_0$ follows from [14, p. 24]. Set $\mathbf{u} \equiv \mathbf{u}_1-\mathbf{u}_2 = \mathbf{u}_0+\tilde{\mathbf{u}}$ where $\tilde{\mathbf{u}} \in \mathbf{V}$ is a specific function. Set $\mathbf{u} = \mathbf{u}_0+\tilde{\mathbf{u}}$, $\mathbf{v} = \tilde{\mathbf{u}}$ in (2.7). Taking into account (1.4) we obtain

$$\nu a_0(\tilde{\mathbf{u}},\tilde{\mathbf{u}}) = -\nu a_0(\mathbf{u}_0,\tilde{\mathbf{u}})-c(\mathbf{u}_0,\mathbf{u}_1,\tilde{\mathbf{u}})-c(\tilde{\mathbf{u}},\mathbf{u}_1,\tilde{\mathbf{u}})-c(\mathbf{u}_2,\mathbf{u}_0,\tilde{\mathbf{u}})-b_1(T,\tilde{\mathbf{u}}).$$

Using estimates (1.6), (1.8), (1.10), (2.2) and this relation we deduce, that

$$\delta_0\nu\|\tilde{\mathbf{u}}\|_1^2 \le \nu\|\mathbf{u}_0\|_1\|\tilde{\mathbf{u}}\|_1+\gamma_0 M^0_{\mathbf{u}}\|\tilde{\mathbf{u}}\|_1^2+2\gamma_0 M^0_{\mathbf{u}}\|\mathbf{u}_0\|_1\|\tilde{\mathbf{u}}\|_1+\beta_1\|T\|_1\|\tilde{\mathbf{u}}\|_1. \tag{2.15}$$

It follows from (2.4) that

$$(\delta_0\nu/2) < \delta_0\nu-\gamma_0 M^0_{\mathbf{u}}-\frac{\beta_1\gamma_1}{\delta_1\lambda}M^0_T \le \delta_0\nu-\gamma_0 M^0_{\mathbf{u}}. \tag{2.16}$$

Rewriting (2.15) in view of (2.16) as

$$(\delta_0\nu/2)\|\tilde{\mathbf{u}}\|_1^2 \le (\delta_0\nu-\gamma_0 M^0_{\mathbf{u}})\|\tilde{\mathbf{u}}\|_1^2 \le (\nu+2\gamma_0 M^0_{\mathbf{u}})\|\mathbf{u}_0\|_1\|\tilde{\mathbf{u}}\|_1+\beta_1\|T\|_1\|\tilde{\mathbf{u}}\|_1,$$

we obtain that

$$\|\tilde{\mathbf{u}}\|_1 \le (2/\delta_0\nu)(\nu+2\gamma_0 M^0_{\mathbf{u}})\|\mathbf{u}_0\|_1+(2\beta_1/\delta_0\nu)\|T\|_1 \le (2\delta_0^{-1}+4\mathcal{R}e)\|\mathbf{u}_0\|_1$$
$$+(2\beta_1/\delta_0\nu)\|T\|_1 \le 2M\|\mathbf{u}_0\|_1+(2\beta_1/\delta_0\nu)\|T\|_1,\quad M \equiv \delta_0^{-1}+2\mathcal{R}e.$$

Taking into account that $\mathbf{u} = \mathbf{u}_0 + \tilde{\mathbf{u}}$, we deduce the estimate

$$\begin{aligned}\|\mathbf{u}\|_1 &\le \|\mathbf{u}_0\|_1 + \|\tilde{\mathbf{u}}\|_1 \le (2M+1)\|\mathbf{u}_0\|_1 + (2\beta_1/\delta_0\nu)\|T\|_1\\ &\le C_0(2M+1)\|\mathbf{g}\|_{1/2,\Gamma} + (2\beta_1/\delta_0\nu)\|T\|_1.\end{aligned} \tag{2.17}$$

An analogous estimate holds for the pressure difference $p = p_1 - p_2$. We make use of inf-sup condition (1.12) to obtain this estimate. By (1.12) for the function $p = p_1 - p_2$ and any (small) number $\delta > 0$ there exists a function $\mathbf{v}_0 \in \mathbf{H}_0^1(\Omega)$, $\mathbf{v}_0 \ne 0$ such that $b(\mathbf{v}_0, p) \ge \beta_0\|\mathbf{v}_0\|_1\|p\|$, $\beta_0 = (\beta - \delta) > 0$. Set $\mathbf{v} = \mathbf{v}_0$ in the identity for $\mathbf{u}$ in (2.7). Using this estimate and (1.6), (1.8) we have

$$\beta_0\|\mathbf{v}_0\|_1\|p\| \le b(\mathbf{v}_0, p) \le (\nu + 2\gamma_0 M_{\mathbf{u}}^0)\|\mathbf{v}_0\|_1\|\mathbf{u}\|_1 + \beta_1\|T\|_1\|\mathbf{v}_0\|_1.$$

As $\|\mathbf{v}_0\|_1 \ne 0$ we deduce from this relation that

$$\|p\| \le \frac{\nu + 2\gamma_0 M_{\mathbf{u}}^0}{\beta_0}\|\mathbf{u}\|_1 + \frac{\beta_1}{\beta_0}\|T\|_1 = \frac{\delta_0\nu}{\beta_0}M\|\mathbf{u}\|_1 + \frac{\beta_1}{\beta_0}\|T\|_1. \tag{2.18}$$

Using (2.17) we obtain the following estimate for $\|p\|$:

$$\|p\| \le \frac{\delta_0\nu}{\beta_0}C_0 M(2M+1)\|\mathbf{g}\|_{1/2,\Gamma} + \frac{\beta_1}{\beta_0}(2M+1)\|T\|_1. \tag{2.19}$$

Based on Theorem 6 and estimates (2.17)–(2.19) we establish in the next section sufficient conditions to input data which provide uniqueness and stability of the solution $(\hat{\mathbf{x}}, \hat{u})$ to problem (1.17) for a number of concrete cost functionals and controls.

**Remark 2.1.** Let us note that if $u = \mathbf{g}$ (or $u = \chi$) then this extremum problem can be considered as a particular case of the general extremum problem (1.17) corresponding to the situation when $K_2$ (or $K_1$) is singleton: $K_2 = \{\chi\}$ (or $K_1 = \{\mathbf{g}\}$).

## 3. Uniqueness and stability of solutions of extremum problems

In this section we firstly consider the problem (1.17) in the case when $I = I_1$ and $u = \mathbf{q} \in K_1$, i.e., we consider the extremum problem

$$\begin{gathered} J(\mathbf{v}, \mathbf{q}) \equiv \frac{\mu_0}{2}\|\mathbf{v} - \mathbf{v}_d\|_Q^2 + \frac{\mu_1}{2}\|\mathbf{q}\|_{1/2,\Gamma}^2 \to \inf,\\ F(\mathbf{x}, \mathbf{q}) = 0,\ \mathbf{x} = (\mathbf{v}, q, S) \in X, \mathbf{q} \in K_1. \end{gathered} \tag{3.1}$$

Let $(\mathbf{x}_1, u_1) \equiv (\mathbf{u}_1, p_1, T_1, \mathbf{g}_1)$ be a solution to problem (3.1) which corresponds to a function $\mathbf{v}_d \equiv \mathbf{u}_d^{(1)} \in \mathbf{L}^2(Q)$, $(\mathbf{x}_2, u_2) \equiv (\mathbf{u}_2, p_2, T_2, \mathbf{g}_2)$ that is a solution to (3.1) which corresponds to another function $\tilde{\mathbf{v}}_d \equiv \mathbf{u}_d^{(2)} \in \mathbf{L}^2(Q)$. Setting $\mathbf{u}_d = \mathbf{u}_d^{(1)} - \mathbf{u}_d^{(2)}$, we note that

$$\begin{aligned} \langle (I_1)'_{\mathbf{u}}(\mathbf{u}_i), \mathbf{w}\rangle &= (\mathbf{u}_i - \mathbf{u}_d^{(i)}, \mathbf{w})_Q,\\ \langle (I_1)'_{\mathbf{u}}(\mathbf{u}_1) - (I_1)'_{\mathbf{u}}(\mathbf{u}_2), \mathbf{u}\rangle &= (\mathbf{u} - \mathbf{u}_d, \mathbf{u})_Q\\ &= \|\mathbf{u}\|_Q^2 - (\mathbf{u}, \mathbf{u}_d)_Q,\ \ (I_1)'_T = 0. \end{aligned} \tag{3.2}$$

The relations (2.7), (2.9) for problem (3.1) do not change while relations (2.8), (2.5), (2.6) and the main inequality (2.14) take subject to (3.2) form

$$\lambda a_1(T,S)+\lambda(\alpha T,S)_{\Gamma_N}+c_1(\mathbf{u},T_1,S)+c_1(\mathbf{u}_2,T,S)=0\ \ \forall S\in\mathcal{T}, \tag{3.3}$$

$$\nu a_0(\mathbf{w},\xi_i)+c(\mathbf{u}_i,\mathbf{w},\xi_i)+c(\mathbf{w},\mathbf{u}_i,\xi_i)+\kappa c_1(\mathbf{w},T_i,\theta_i)+b(\mathbf{w},\sigma_i)+\langle\zeta_i,\mathbf{w}\rangle_\Gamma$$
$$=-\mu_0(\mathbf{u}_i-\mathbf{u}_d^{(i)},\mathbf{w})_Q\ \forall\mathbf{w}\in\tilde{\mathbf{H}}^1(\Omega), \tag{3.4}$$

$$\kappa[\lambda\tilde{a}(\tau,\theta_i)+\lambda(\alpha\tau,\theta_i)+c_1(\mathbf{u}_i,\tau,\theta_i)+\langle\zeta_i^c,\tau\rangle_{\Gamma_D}]+b_1(\tau,\xi_i)=0\ \ \forall\tau\in H^1(\Omega), \tag{3.5}$$

$$c(\mathbf{u},\mathbf{u},\xi_1+\xi_2)+\kappa c_1(\mathbf{u},T,\theta_1+\theta_2)+\mu_0(\|\mathbf{u}\|_Q^2-(\mathbf{u},\mathbf{u}_d)_Q)\le-\mu_1\|\mathbf{g}\|_{1/2,\Gamma}^2. \tag{3.6}$$

It follows from (2.9) that $T\in\mathcal{T}$. Set $S=T$ in (3.3). Using (1.5) we obtain that

$$\lambda a_1(T,T)+\lambda(\alpha T,T)_{\Gamma_N}=-c_1(\mathbf{u},T_1,T). \tag{3.7}$$

It follows from (1.9), (1.10), (2.2) that

$$|c_1(\mathbf{u},T_1,T)|\le\gamma_1 M_T^0\|\mathbf{u}\|_1\|T\|_1,\ \ |b_1(T,\mathbf{u})|\le\beta_1\|\mathbf{u}\|_1\|T\|_1. \tag{3.8}$$

Taking into account (1.7), (3.8), we obtain from (3.7) that

$$\delta_1\lambda\|T\|^2\le\gamma_1 M_T^0\|\mathbf{u}\|_1\|T\|_1.$$

From this inequality we deduce the following estimate for $\|T\|_1$:

$$\|T\|_1\le\frac{\gamma_1 M_T^0}{\delta_1\lambda}\|\mathbf{u}\|_1. \tag{3.9}$$

Using (2.17) and (3.9) we have

$$\|\mathbf{u}\|_1\le C_0(2M+1)\|\mathbf{g}\|_{1/2,\Gamma}+\frac{2\beta_1}{\delta_0\nu}\frac{\gamma_1 M_T^0}{\delta_1\lambda}\|\mathbf{u}\|_1. \tag{3.10}$$

It follows from (3.10) and (2.3) that $(1-2\mathcal{R}a)\|\mathbf{u}\|_1\le C_0(2M+1)\|\mathbf{g}\|_{1/2,\Gamma}$. Taking into account that $2\mathcal{R}a<1$ by (2.4) we obtain from this estimate, (3.9) and (2.19) that

$$\|\mathbf{u}\|_1\le\frac{C_0(2M+1)}{1-2\mathcal{R}a}\|\mathbf{g}\|_{1/2,\Gamma}, \tag{3.11}$$

$$\|T\|_1\le\frac{\gamma_1 C_0 M_T^0(2M+1)}{\delta_1\lambda(1-2\mathcal{R}a)}\|\mathbf{g}\|_{1/2,\Gamma},\ \ \|p\|\le\frac{C_0\delta_0\nu(2M+1)(M+\mathcal{R}a)}{\beta_0(1-2\mathcal{R}a)}\|\mathbf{g}\|_{1/2,\Gamma}. \tag{3.12}$$

Set $\mathbf{w}=\xi_i$, $\tau=\theta_i$ in (3.4), (3.5). Using (1.4), (1.5) and conditions $\xi_i\in\mathbf{V}$, $\theta_i\in\mathcal{T}$ we deduce that

$$\nu a_0(\xi_i,\xi_i)=-c(\xi_i,\mathbf{u}_i,\xi_i)-\kappa c_1(\xi_i,T_i,\theta_i)-\mu_0(\mathbf{u}_i-\mathbf{u}_d^{(i)},\xi_i)_Q, \tag{3.13}$$

$$\kappa[\lambda a_1(\theta_i,\theta_i)+\lambda(\alpha\theta_i,\theta_i)_{\Gamma_N}]=-b_1(\theta_i,\xi_i),\ i=1,2. \tag{3.14}$$

It follows from (1.6)–(1.10), (1.3), (2.2) that

$$a_0(\xi_i,\xi_i)\ge\delta_0\|\xi_i\|_1^2,\ |c(\xi_i,\mathbf{u}_i,\xi_i)|\le\gamma_0\|\mathbf{u}_i\|_1\|\xi_i\|_1^2\le\gamma_0 M_{\mathbf{u}}^0\|\xi_i\|_1^2, \tag{3.15}$$

$$a_1(\theta_i,\theta_i)\ge\delta_1\|\theta_i\|_1^2,\ \ |b_1(\theta_i,\xi_i)|\le\beta_1\|\theta_i\|_1\|\xi_i\|_1,\ |c_1(\xi_i,T_i,\theta_i)|\le\gamma_1 M_T^0\|\xi_i\|_1\|\theta_i\|_1, \tag{3.16}$$

$$|(\mathbf{u}_i-\mathbf{u}_d^{(i)},\xi_i)_Q|\le\|\mathbf{u}_i-\mathbf{u}_d^{(i)}\|_Q\|\xi_i\|_Q\le C_3(C_3M_{\mathbf{u}}^0+\|\mathbf{u}_d^{(i)}\|_Q)\|\xi_i\|_1. \tag{3.17}$$

Taking into account (3.15)–(3.17) we deduce from (3.13) and (3.14) that

$$\|\theta_i\|_1 \le \frac{\beta_1}{\delta_1\lambda\kappa}\|\xi_i\|_1,$$

$$\left(\delta_0\nu - \gamma_0 M^0_{\mathbf{u}} - \frac{\beta_1\gamma_1}{\delta_1\lambda}M^0_T\right)\|\xi_i\|_1^2 \le \mu_0 C_3(C_3 M^0_{\mathbf{u}} + \|\mathbf{u}_d^{(i)}\|_Q)\|\xi_i\|_1.$$

Combining these inequalities with (2.16) and (2.3) gives

$$\|\xi_i\|_1 \le \frac{2\mu_0 C}{\gamma_0}(\mathcal{R}e + \mathcal{R}e^0), \quad \|\theta_i\|_1 \le \frac{\beta_1}{\delta_1\lambda\kappa}\frac{2\mu_0 C}{\gamma_0}(\mathcal{R}e + \mathcal{R}e^0), \tag{3.18}$$

where

$$C = C_3^2, \quad \mathcal{R}e^0 = \frac{\gamma_0}{\delta_0\nu C_3}\max(\|\mathbf{u}_d^{(1)}\|_Q, \|\mathbf{u}_d^{(2)}\|_Q). \tag{3.19}$$

Taking into account (1.8), (1.9), (3.9), (3.18) and (2.3) we have

$$|c(\mathbf{u},\mathbf{u},\xi_1+\xi_2)| \le \gamma_0\|\mathbf{u}\|_1^2(\|\xi_1\|_1 + \|\xi_2\|_1) \le 4\mu_0 C(\mathcal{R}e + \mathcal{R}e^0)\|\mathbf{u}\|_1^2,$$

$$\kappa|c_1(\mathbf{u},T,\theta_1+\theta_2)| \le \kappa\gamma_1\|\mathbf{u}\|_1\|T\|_1(\|\theta_1\|_1 + \|\theta_2\|_1)$$

$$\le \frac{\gamma_1 M_T^0}{\delta_1\lambda}\frac{\gamma_1\beta_1}{\delta_1\lambda}\frac{4\mu_0 C(\mathcal{R}e + \mathcal{R}e^0)}{\gamma_0}\|\mathbf{u}\|_1^2 = 4\mu_0 C(\mathcal{R}e + \mathcal{R}e^0)\frac{\gamma_1}{\gamma_0}\mathcal{P}\mathcal{R}a\|\mathbf{u}\|_1^2$$

which yields

$$|c(\mathbf{u},\mathbf{u},\xi_1+\xi_2) + \kappa c_1(\mathbf{u},T,\theta_1+\theta_2)| \le 4\mu_0 C(\mathcal{R}e + \mathcal{R}e^0)\left[1 + (\gamma_1/\gamma_0)\mathcal{P}\mathcal{R}a\right]\|\mathbf{u}\|_1^2. \tag{3.20}$$

Using (3.11) we deduce from (3.20) that

$$|c(\mathbf{u},\mathbf{u},\xi_1+\xi_2) + \kappa c_1(\mathbf{u},T,\theta_1+\theta_2)|$$

$$\le \frac{4\mu_0 C_0^2 C(2M+1)^2(\mathcal{R}e + \mathcal{R}e^0)[1 + (\gamma_1/\gamma_0)\mathcal{P}\mathcal{R}a]}{(1-2\mathcal{R}a)^2}\|\mathbf{g}\|^2_{1/2,\Gamma}. \tag{3.21}$$

Let input data for problem (3.1) and parameters $\mu_0, \mu_1$ be such that

$$(1-\varepsilon)\mu_1 \ge \frac{4\mu_0 C_0^2 C(2M+1)^2(\mathcal{R}e + \mathcal{R}e^0)[1 + (\gamma_1/\gamma_0)\mathcal{P}\mathcal{R}a]}{(1-2\mathcal{R}a)^2}, \ \varepsilon = \text{const} > 0. \tag{3.22}$$

Here and further $\varepsilon > 0$ is a constant. In view of (3.22) we find from (3.21) that

$$|c(\mathbf{u},\mathbf{u},\xi_1+\xi_2) + \kappa c_1(\mathbf{u},T,\theta_1+\theta_2)| \le (1-\varepsilon)\mu_1\|\mathbf{g}\|^2_{1/2,\Gamma}. \tag{3.23}$$

Taking into account (3.23) we come from (3.6) to the inequality

$$\mu_0(\|\mathbf{u}\|_Q^2 - (\mathbf{u},\mathbf{u}_d)_Q) \le -c(\mathbf{u},\mathbf{u},\xi_1+\xi_2) - \kappa c_1(\mathbf{u},T,\theta_1+\theta_2) - \mu_1\|\mathbf{g}\|^2_{1/2,\Gamma}$$

$$\le -\varepsilon\mu_1\|\mathbf{g}\|^2_{1/2,\Gamma}. \tag{3.24}$$

It follows from (3.24) that $\|\mathbf{u}\|_Q^2 \le (\mathbf{u},\mathbf{u}_d)_Q \le \|\mathbf{u}\|_Q\|\mathbf{u}_d\|_Q$, which yields $\|\mathbf{u}\|_Q \le \|\mathbf{u}_d\|_Q$. As $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$, $\mathbf{u}_d = \mathbf{u}_d^{(1)} - \mathbf{u}_d^{(2)}$ we deduce the estimate

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_Q \le \|\mathbf{u}_d^{(1)} - \mathbf{u}_d^{(2)}\|_Q. \tag{3.25}$$

The estimate (3.25) under condition $Q = \Omega$ has the sense of the stability estimate of the component $\hat{\mathbf{u}}$ of the solution $(\hat{\mathbf{u}}, \hat{p}, \hat{T}, \hat{\mathbf{g}})$ to problem (3.1) with respect to small disturbances in the $\mathbf{L}^2(\Omega)$ norm of the function $\mathbf{v}_d \in \mathbf{L}^2(\Omega)$ which enters into the expression for the functional $I_1$ in (1.18). In the case where $\mathbf{u}_d^{(1)} = \mathbf{u}_d^{(2)}$ it follows from (3.25) that $\mathbf{u}_1 = \mathbf{u}_2$. This yields together with (3.9), (2.18) and condition $\mathbf{u}|_\Gamma = \mathbf{g} \equiv \mathbf{g}_1 - \mathbf{g}_2$ in (2.7), that $T_1 = T_2$, $p_1 = p_2$ and $\mathbf{g}_1 = \mathbf{g}_2$. The latter means the uniqueness of the solution to problem (3.1) when $Q = \Omega$ and (3.22) holds.

We note that the uniqueness and stability of the solution to problem (3.1) under condition (3.22) holds, as it also does in the case where $Q \subset \Omega$, i.e., $Q$ is a part of $\Omega$. In order to prove this fact let us consider the inequality (3.24). Using (3.25) rewrite it in the form

$$\begin{aligned}\varepsilon\mu_1\|\mathbf{g}\|^2_{1/2,\Gamma} &\le c(\mathbf{u},\mathbf{u},\xi_1+\xi_2) + \kappa c_1(\mathbf{u},T,\theta_1+\theta_2) + \mu_1\|\mathbf{g}\|^2_{1/2,\Gamma}\\ &\le -\mu_0\|\mathbf{u}\|^2_Q + \mu_0\|\mathbf{u}\|_Q\|\mathbf{u}_d\|_Q \le \mu_0\|\mathbf{u}_d\|^2_Q.\end{aligned}$$

From this relation, (3.11) and (3.12) we deduce the following stability estimates:

$$\begin{aligned}\|\mathbf{g}_1-\mathbf{g}_2\|_{1/2,\Gamma} &\le \sqrt{\frac{\mu_0}{\varepsilon\mu_1}}\|\mathbf{u}_d^{(1)}-\mathbf{u}_d^{(2)}\|_{\mathbf{H}^s(Q)},\\ \|\mathbf{u}_1-\mathbf{u}_2\|_1 &\le \frac{C_0(2M+1)}{1-2\mathcal{R}a}\sqrt{\frac{\mu_0}{\varepsilon\mu_1}}\|\mathbf{u}_d^{(1)}-\mathbf{u}_d^{(2)}\|_{\mathbf{H}^s(Q)},\\ \|T_1-T_2\|_1 &\le \frac{\gamma_1 C_0 M_T^0(2M+1)}{\delta_1\lambda(1-2\mathcal{R}a)}\sqrt{\frac{\mu_0}{\varepsilon\mu_1}}\|\mathbf{u}_d^{(1)}-\mathbf{u}_d^{(2)}\|_{\mathbf{H}^s(Q)},\\ \|p_1-p_2\| &\le \frac{C_0\delta_0\nu(2M+1)(M+\mathcal{R}a)}{\beta_0(1-2\mathcal{R}a)}\sqrt{\frac{\mu_0}{\varepsilon\mu_1}}\|\mathbf{u}_d^{(1)}-\mathbf{u}_d^{(2)}\|_{\mathbf{H}^s(Q)}\ (M\equiv\delta_0^{-1}+2\mathcal{R}e)\end{aligned} \tag{3.26}$$

where $s = 0$. Thus we have proved the following theorem.

**Theorem 7.** *Let, under conditions* (i), (ii), (j) *and* (2.4)*, the quadruple* $(\mathbf{u}_i, p_i, T_i, \mathbf{g}_i)$ *be the solution to problem* (3.1) *corresponding to a given function* $\mathbf{u}_d^{(i)} \in \mathbf{L}^2(Q)$, $i = 1, 2$*, and let the condition* (3.22) *hold where* $C$ *and* $\mathcal{R}e^0$ *are defined in* (3.19)*. Then stability estimates* (3.25) *and* (3.26) *under* $s = 0$ *hold true.*

We emphasize that the uniqueness and stability of the solution to problem (3.1) both under $Q = \Omega$, and under $Q \subset \Omega$ is proved only if parameter $\mu_1$ in (3.1) is positive and satisfies (3.22). This means that term $(\mu_1/2)\|\mathbf{g}\|^2_{1/2,\Gamma}$ in the expression for the minimized functional $J$ in (3.1) has a regularizing effect on the extremum problem (3.1).

In the same manner one can study uniqueness and stability of solutions to extremum problems for other cost functionals depending on the velocity $\mathbf{u}$. Let us consider for example the extremum problem

$$\begin{gathered}J(\mathbf{v},\mathbf{q}) \equiv \frac{\mu_0}{2}\|\mathbf{v}-\mathbf{v}_d\|^2_{1,Q} + \frac{\mu_1}{2}\|\mathbf{q}\|^2_{1/2,\Gamma} \to \inf,\\ F(\mathbf{x},\mathbf{q}) = 0,\ \mathbf{x} = (\mathbf{v},q,S) \in X,\ \mathbf{q} \in K_1,\end{gathered} \tag{3.27}$$

which corresponds to the cost functional $I_2(\mathbf{v}) = \|\mathbf{v}-\mathbf{v}_d\|^2_{1,Q}$.

Denoting by $(\mathbf{x}_i, u_i) \equiv (\mathbf{u}_i, p_i, T_i, \mathbf{g}_i)$ the solution to problem (3.27) corresponding to a function $\mathbf{v}_d \equiv \mathbf{u}_d^{(i)} \in \mathbf{H}^1(Q)$, $i = 1, 2$, and setting $\mathbf{u}_d = \mathbf{u}_d^{(1)} - \mathbf{u}_d^{(2)}$ we note that

$$\begin{aligned}\langle (I_2)'_{\mathbf{u}}(\mathbf{u}_i), \mathbf{w}\rangle &= (\mathbf{u}_i - \mathbf{u}_d^{(i)}, \mathbf{w})_{1,Q},\\ \langle (I_2)'_{\mathbf{u}}(\mathbf{u}_1) - (I_2)'_{\mathbf{u}}(\mathbf{u}_2), \mathbf{u}\rangle &= (\mathbf{u} - \mathbf{u}_d, \mathbf{u})_{1,Q}\\ &= \|\mathbf{u}\|^2_{1,Q} - (\mathbf{u}, \mathbf{u}_d)_{1,Q}, \quad (I_2)'_T = 0.\end{aligned} \tag{3.28}$$

In view of (3.28), relations (2.9), (3.3), (3.5), (3.14) and estimates (3.9), (3.11), (3.12) do not change, while (2.14) and (2.5) under $\mathbf{w} = \xi_i$ take instead of (3.6), (3.13) the form

$$c(\mathbf{u}, \mathbf{u}, \xi_1 + \xi_2) + \kappa c_1(\mathbf{u}, T, \theta_1 + \theta_2) + \mu_0(\|\mathbf{u}\|_{1,Q} - (\mathbf{u}, \mathbf{u}_d)_{1,Q}) \le -\mu_1\|\mathbf{g}\|^2_{1/2,\Gamma}, \tag{3.29}$$

$$\nu a(\xi_i, \xi_i) = -c(\xi_i, \mathbf{u}_i, \xi_i) - \kappa c_1(\xi_i, T_i, \theta_i) - \mu_0(\mathbf{u}_i - \mathbf{u}_d^{(i)}, \xi_i)_{1,Q}. \tag{3.30}$$

Using the estimates (3.30) we deduce (instead of (3.17)) that

$$|(\mathbf{u}_i - \mathbf{u}_d^{(i)}, \xi_i)_{1,Q}| \le \|\mathbf{u}_i - \mathbf{u}_d^{(i)}\|_{1,Q}\|\xi_i\|_{1,Q} \le (M^0_{\mathbf{u}} + \|\mathbf{u}_d^{(i)}\|_{1,Q})\|\xi_i\|_1.$$

Proceeding as above we obtain (3.18) for $\|\xi_i\|_1$, $\|\theta_i\|_1$ and inequality (3.21) where

$$C = 1, \quad \mathcal{R}e^0 = (\gamma_0/\delta_0\nu)\max(\|\mathbf{u}_d^{(1)}\|_{1,Q}, \|\mathbf{u}_d^{(2)}\|_{1,Q}). \tag{3.31}$$

Let us assume that the condition (3.22) takes place where $C$ and $\mathcal{R}e^0$ are defined in (3.31). Using (3.22) we deduce (3.23). Taking into account (3.23) we obtain from (3.29) that

$$\begin{aligned}\mu_0(\|\mathbf{u}\|^2_{1,Q} - (\mathbf{u}, \mathbf{u}_d)_{1,Q}) &\le -c(\mathbf{u}, \mathbf{u}, \xi_1 + \xi_2) - \kappa c_1(\mathbf{u}, T, \theta_1 + \theta_2) - \mu_1\|\mathbf{g}\|^2_{1/2,\Gamma}\\ &\le -\varepsilon\mu_1\|\mathbf{g}\|^2_{1/2,\Gamma}.\end{aligned} \tag{3.32}$$

It follows from (3.32) that $\|\mathbf{u}\|^2_{1,Q} \le (\mathbf{u}, \mathbf{u}_d)_{1,Q}$ which yields $\|\mathbf{u}\|_{1,Q} \le \|\mathbf{u}_d\|_{1,Q}$ or

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_{1,Q} \le \|\mathbf{u}_d^{(1)} - \mathbf{u}_d^{(2)}\|_{1,Q}. \tag{3.33}$$

In the case where $Q = \Omega$ we deduce, from (3.33), relation $\mathbf{u}|_\Gamma = \mathbf{g} = \mathbf{g}_1 - \mathbf{g}_2$, (1.3), (3.9) and (2.18), the following estimates:

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_1 \le \|\mathbf{u}_d^{(1)} - \mathbf{u}_d^{(2)}\|_1, \ \|\mathbf{g}_1 - \mathbf{g}_2\|_{1/2,\Gamma} \le C_\Gamma\|\mathbf{u}_d^{(1)} - \mathbf{u}_d^{(2)}\|_1,$$

$$\|T_1 - T_2\|_1 \le \frac{\gamma_1 M^0_T}{\delta_1\lambda}\|\mathbf{u}_d^{(1)} - \mathbf{u}_d^{(2)}\|_1, \ \|p_1 - p_2\| \le \frac{\delta_0\nu(M + \mathcal{R}a)}{\beta_0}\|\mathbf{u}_d^{(1)} - \mathbf{u}_d^{(2)}\|_1. \tag{3.34}$$

The estimates (3.34) have the sense of stability estimates for the solution $(\hat{\mathbf{u}}, \hat{p}, \hat{T}, \hat{\mathbf{g}})$ to problem (3.27) under $Q = \Omega$ with respect to small disturbances in the $\mathbf{H}^1(\Omega)$ norm of the function $\mathbf{v}_d$ which enters into the expression for the functional $I_2$. In the case where $\mathbf{u}_d^{(1)} = \mathbf{u}_d^{(2)}$ we deduce from (3.34) that $\mathbf{u}_1 = \mathbf{u}_2$, $\mathbf{g}_1 = \mathbf{g}_2$, $T_1 = T_2$, $p_1 = p_2$ which means the uniqueness of the solution to problem (3.27) under $Q = \Omega$. If $Q \subset \Omega$, the estimates (3.34) do not hold true but using

(3.32) one can obtain instead of them rougher estimates such as (3.26). In fact rewriting (3.32) in view of (3.33) in the form

$$\varepsilon\mu_1\|\mathbf{g}\|^2_{1/2,\Gamma} \le -\mu_0\|\mathbf{u}\|^2_{1,Q} + \mu_0\|\mathbf{u}\|_{1,Q}\|\mathbf{u}_d\|_{1,Q} \le \mu_0\|\mathbf{u}_d\|^2_{1,Q}$$

and using (3.11), (3.12) we come to the estimates (3.26) under $s = 1$. Thus we have proved the following result.

**Theorem 8.** *Let, under conditions* (i), (ii), (j) *and* (2.4)*, the quadruple* $(\mathbf{u}_i, p_i, T_i, \mathbf{g}_i)$ *be a solution to problem* (3.27) *corresponding to the given function* $\mathbf{u}_d^{(i)} \in \mathbf{H}^1(Q)$*,* $i = 1, 2$*, and let the condition* (3.22) *hold where* $C$ *and* $\mathcal{R}e^0$ *are defined in* (3.31)*. Then stability estimates* (3.33) *and* (3.26) *under* $s = 1$ *hold true. Furthermore* (3.34) *holds if* $Q = \Omega$*.*

We again note that the uniqueness and stability of the solution to problem (3.27) both under $Q = \Omega$ and under $Q \subset \Omega$ is proved above under the condition that the parameter $\mu_1$ in (3.27) satisfies (3.22). We can not prove stability of the solution to problem (3.27) as we did for problem (3.1) in the case where $\mu_1 = 0$. But we can establish the local uniqueness of the solution to problem (3.27) under $\mu_1 = 0$ in the case where $Q = \Omega$. In fact setting $\mu_1 = 0$, $Q = \Omega$, $\mathbf{u}_d^{(1)} = \mathbf{u}_d^{(2)}$ in (3.29) we obtain the inequality

$$c(\mathbf{u}, \mathbf{u}, \xi_1 + \xi_2) + \kappa c_1(\mathbf{u}, T, \theta_1 + \theta_2) \le -\mu_0\|\mathbf{u}\|^2_1. \tag{3.35}$$

Let input data for problem (3.27) be such that

$$4(\mathcal{R}e + \mathcal{R}e^0)[1 + (\gamma_1/\gamma_0)\mathcal{P}\mathcal{R}a] < 1. \tag{3.36}$$

It follows from (3.20) under $C = 1$ and (3.35) that $\mathbf{u} = \mathbf{0}$, and from (3.9), (2.18) and relation $\mathbf{u}|_\Gamma = \mathbf{g}$ we deduce that $T_1 = T_2$, $p_1 = p_2$, $\mathbf{g}_1 = \mathbf{g}_2$. So the next theorem holds.

**Theorem 9.** *Let, under conditions* (i), (ii), (j) *and* (2.4)*,* $\mathbf{v}_d \in \mathbf{H}^1(\Omega)$ *be a given function,* $\mu_0 > 0$*,* $\mu_1 \ge 0$ *and let the condition* (3.36) *hold where*

$$\mathcal{R}e^0 = (\gamma_0/\delta_0\nu)\|\mathbf{v}_d\|_1.$$

*Then the solution* $(\hat{\mathbf{u}}, \hat{p}, \hat{T}, \hat{\mathbf{g}})$ *to problem* (3.27) *under* $Q = \Omega$ *is unique.*

Let us consider the extremum problem

$$\begin{gathered} J(\mathbf{v}, \mathbf{q}) \equiv \frac{\mu_0}{2}\|\mathrm{rot}\,\mathbf{v} - \eta_d\|^2_Q + \frac{\mu_1}{2}\|\mathbf{q}\|^2_{1/2,\Gamma} \to \inf, \\ F(\mathbf{x}, \mathbf{q}) = 0,\ \mathbf{x} = (\mathbf{v}, q, S) \in X,\ \mathbf{q} \in K_1, \end{gathered} \tag{3.37}$$

corresponding to the cost functional $I_3(\mathbf{v}) = \|\mathrm{rot}\,\mathbf{v} - \eta_d\|^2_Q$. Denoting by $(\mathbf{x}_i, u_i) = (\mathbf{u}_i, p_i, T_i, \mathbf{g}_i)$, $i = 1, 2$, the solution to problem (3.37) corresponding to the function $\eta_d = \zeta_d^{(i)} \in \mathbf{L}^2(Q)$, $i = 1, 2$, and setting $\zeta_d = \zeta_d^{(1)} - \zeta_d^{(2)}$, we note that

$$\begin{gathered} \langle (I_3)'_{\mathbf{u}}(\mathbf{u}_i), \mathbf{w}\rangle = (\mathrm{rot}\,\mathbf{u}_i - \zeta_d^{(i)}, \mathrm{rot}\,\mathbf{w})_Q, \\ \langle (I_3)'_{\mathbf{u}}(\mathbf{u}_1) - (I_3)'_{\mathbf{u}}(\mathbf{u}_2), \mathbf{u}\rangle = (\mathrm{rot}\,\mathbf{u} - \zeta_d, \mathrm{rot}\,\mathbf{u})_Q, \quad (I_3)'_T = 0. \end{gathered} \tag{3.38}$$

In view of (3.38), relations (2.9), (3.3), (3.5), (3.14) and estimates (3.9), (3.11), (3.12) do not change, while (2.14) and (2.5) under $\mathbf{w} = \xi_i$ transform to

$$c(\mathbf{u}, \mathbf{u}, \xi_1 + \xi_2) + \kappa Q_1(\mathbf{u}, T, \theta_1 + \theta_2) + \mu_0(\|\mathrm{rot}\,\mathbf{u}\|_Q^2 - (\zeta_d, \mathrm{rot}\,\mathbf{u})_Q) \le -\mu_1 \|\mathbf{g}\|_{1/2,\Gamma}^2, \tag{3.39}$$

$$\nu a_0(\xi_i, \xi_i) = -c(\xi_i, \mathbf{u}_i, \xi_i) - \kappa c_1(\xi_i, T_i, \theta_i) - \mu_0(\mathrm{rot}\,\mathbf{u}_i - \zeta_d^{(i)}, \mathrm{rot}\,\xi_i)_Q. \tag{3.40}$$

Using (1.3) and (2.2) we have

$$\begin{aligned} |(\mathrm{rot}\,\mathbf{u}_i - \zeta_d^{(i)}, \mathrm{rot}\,\xi_i)_Q| &\le (\|\mathrm{rot}\,\mathbf{u}_i\|_Q + \|\zeta_d^{(i)}\|_Q)\|\mathrm{rot}\,\xi_i\|_Q \\ &\le C_1(C_1 M_{\mathbf{u}}^0 + \|\zeta_d^{(i)}\|_Q)\|\xi_i\|_1. \end{aligned} \tag{3.41}$$

Taking into account (3.15), (3.16), (3.41) we deduce from (3.40) and (3.14) that

$$\|\theta_i\|_1 \le \frac{\beta_1 \|\xi_i\|_1}{\delta_1 \lambda \kappa},\ (\delta_0 \nu - \gamma_0 M_{\mathbf{u}}^0 - \frac{\beta_1 \gamma_1}{\delta_1 \lambda} M_T^0)\|\xi_i\|_1^2 \le \mu_0 C_1 (C_1 M_{\mathbf{u}}^0 + \|\zeta_d^{(i)}\|_Q)\|\xi_i\|_1.$$

In view of (2.16), from this inequality we obtain $\|\xi_i\|_1 \le (2\mu_0/\delta_0\nu) C_1^2 (M_{\mathbf{u}}^0 + C_1^{-1}\|\zeta_d^{(i)}\|_Q)$ which yields (3.18), (3.20) and (3.21) where

$$C = C_1^2,\ \ \mathcal{R}e^0 = \frac{\gamma_0}{\delta_0 \nu C_1} \max(\|\zeta_d^{(1)}\|_Q, \|\zeta_d^{(2)}\|_Q). \tag{3.42}$$

Let us assume that the condition (3.22) holds where $C$ and $\mathcal{R}e^0$ are defined in (3.42). Using (3.22) we deduce (3.23). Taking into account (3.23) we obtain from (3.39) that

$$\mu_0(\|\mathrm{rot}\,\mathbf{u}\|_Q^2 - (\mathrm{rot}\,\mathbf{u}, \zeta_d)_Q) \le -\varepsilon \mu_1 \|\mathbf{g}\|_{1/2,\Gamma}^2. \tag{3.43}$$

It follows from (3.43) that $\|\mathrm{rot}\,\mathbf{u}\|_Q^2 \le (\mathrm{rot}\,\mathbf{u}, \zeta_d)_Q$ which yields $\|\mathrm{rot}\,\mathbf{u}\|_Q \le \|\zeta_d\|_Q$ or

$$\|\mathrm{rot}\,\mathbf{u}_1 - \mathrm{rot}\,\mathbf{u}_2\|_Q \le \|\zeta_d^{(1)} - \zeta_d^{(2)}\|_Q. \tag{3.44}$$

Rewriting (3.43) in the form $\varepsilon \mu_1 \|\mathbf{g}\|_{1/2,\Gamma}^2 \le \mu_0 \|\zeta_d\|_Q^2$ and using (3.11), (3.12) we obtain the following stability estimates:

$$\begin{aligned} \|\mathbf{g}_1 - \mathbf{g}_2\|_{1/2,\Gamma} &\le \sqrt{\frac{\mu_0}{\varepsilon \mu_1}} \|\zeta_d^{(1)} - \zeta_d^{(2)}\|_Q, \\ \|\mathbf{u}_1 - \mathbf{u}_2\|_1 &\le \frac{C_0(2M+1)}{1 - 2\mathcal{R}a} \sqrt{\frac{\mu_0}{\varepsilon \mu_1}} \|\zeta_d^{(1)} - \zeta_d^{(2)}\|_Q, \\ \|T_1 - T_2\|_1 &\le \frac{\gamma_1 M_T^0}{\delta_1 \lambda} \frac{C_0(2M+1)}{1 - 2\mathcal{R}a} \sqrt{\frac{\mu_0}{\varepsilon \mu_1}} \|\zeta_d^{(1)} - \zeta_d^{(2)}\|_Q, \\ \|p_1 - p_2\| &\le \frac{C_0 \delta_0 \nu (2M+1)(M + \mathcal{R}a)}{\beta_0 (1 - 2\mathcal{R}a)} \sqrt{\frac{\mu_0}{\varepsilon \mu_1}} \|\zeta_d^{(1)} - \zeta_d^{(2)}\|_Q. \end{aligned} \tag{3.45}$$

Thus we have proved the following theorem.

**Theorem 10.** *Let, under conditions* (i), (ii), (j) *and* (2.4), *the quadruple* ($\mathbf{u}_i$, $p_i$, $T_i, \mathbf{g}_i$) *be the solution to problem* (3.37) *corresponding to a given function* $\zeta_d^{(i)} \in \mathbf{L}^2(Q)$, $i = 1, 2$, *and let the condition* (3.22) *hold where* $C$ *and* $\mathcal{R}e^0$ *are defined in* (3.42). *Then the stability estimates* (3.44), (3.45) *hold true.*

We can not prove the stability of the solution to problem (3.37) in the case where $\mu_1 = 0$. But we can establish the local uniqueness of the solution to problem (3.37) under more strict conditions on $\Omega$ and boundary vector $\mathbf{g}$ if we replace condition (j) by

(j′) $\Omega$ is a simply connected domain with the boundary $\Gamma \in C^{1,1}$; $K \subset \tilde{\mathbf{H}}^{1/2}(\Gamma)$ is a convex closed set consisting of functions $\mathbf{g}$ which satisfy the condition $\mathbf{g}\cdot\mathbf{n}|_\Gamma = q$ where $q \in H^{1/2}(\Gamma)$ is a given function.

Indeed let us note that (3.39) takes, under $\mu_1 = 0$, $\zeta_d^{(1)} = \zeta_d^{(2)}$, $Q = \Omega$, the form

$$c(\mathbf{u},\mathbf{u},\xi_1+\xi_2) + \kappa c_1(\mathbf{u},T,\theta_1+\theta_2) \le -\mu_0\|\mathrm{rot}\,\mathbf{u}\|^2. \tag{3.46}$$

Under the first condition in (j′) the difference $\mathbf{g} = \mathbf{g}_1 - \mathbf{g}_2$ has a zero normal component on $\Gamma$. Therefore taking into account the simple connectedness of the domain $\Omega$ we have the estimate $\|\mathbf{u}\|_1 \le C_4\|\mathrm{rot}\,\mathbf{u}\|$ with the constant $C_4$ depending on $\Omega$ [14]. Using this estimate we deduce from (3.20), (3.42) that

$$|c(\mathbf{u},\mathbf{u},\xi_1+\xi_2)+\kappa c_1(\mathbf{u},T,\theta_1+\theta_2)| \le 4\mu_0 C_1^2 C_4^2(\mathcal{R}e+\mathcal{R}e^0)[1+(\gamma_1/\gamma_0)\mathcal{P}\mathcal{R}a]\|\mathrm{rot}\,\mathbf{u}\|^2. \tag{3.47}$$

Let input data for problem (3.37) be such that

$$4C_1^2C_4^2(\mathcal{R}e+\mathcal{R}e^0)[1+(\gamma_1/\gamma_0)\mathcal{P}\mathcal{R}a] < 1. \tag{3.48}$$

It follows from (3.46) and (3.47) that $\mathrm{rot}\,\mathbf{u} = 0$ which yields $\mathbf{u} = \mathbf{0}$ or $\mathbf{u}_1 = \mathbf{u}_2$. From (3.9), (2.18) and the condition $\mathbf{u}|_\Gamma = \mathbf{g}$ we deduce that $T_1 = T_2$, $p_1 = p_2$, $\mathbf{g}_1 = \mathbf{g}_2$. Thus we have proved the following theorem.

**Theorem 11.** *Let, under conditions* (i), (ii), (j′) *and* (2.4), $\eta_d \in \mathbf{L}^2(\Omega)$ *be a given function,* $\mu_0 > 0$, $\mu_1 \ge 0$ *and let the condition* (3.48) *hold where* $\mathcal{R}e^0 = (\gamma_0/\delta_0\nu C_1)\|\eta_d\|$. *Then the solution* $(\hat{\mathbf{u}},\hat{p},\hat{T},\hat{\mathbf{g}})$ *to problem* (3.37) *under* $Q = \Omega$ *is unique.*

In conclusion let us consider the case where $I = I_1$ in (1.17) and $u = \chi \in K_2 \subset L^2(\Gamma_N)$, i.e., we consider the extremum problem

$$J(\mathbf{v},\chi) \equiv \frac{\mu_0}{2}\|\mathbf{v}-\mathbf{v}_d\|_Q^2 + \frac{\mu_2}{2}\|\chi\|_{\Gamma_N}^2 \to \inf,\ F(\mathbf{x},\chi) = 0,\ \mathbf{x} = (\mathbf{v},q,S) \in X, \chi \in K_2. \tag{3.49}$$

Let $(\mathbf{x}_1,u_1) \equiv (\mathbf{u}_1,p_1,T_1,\chi_1)$ be a solution to problem (3.49) which corresponds to a function $\mathbf{v}_d \equiv \mathbf{u}_d^{(1)} \in \mathbf{L}^2(Q)$, and let $(\mathbf{x}_2,u_2) \equiv (\mathbf{u}_2,p_2,T_2,\chi_2)$ be a solution to problem (3.49) which corresponds to another function $\tilde{\mathbf{v}}_d \equiv \mathbf{u}_d^{(2)} \in \mathbf{L}^2(Q)$. Setting $\mathbf{u}_d = \mathbf{u}_d^{(1)} - \mathbf{u}_d^{(2)}$, we note that for problem (3.49) the relations (3.2) hold true. In view of (3.2), relations (3.4), (3.5), (3.13), (3.14) and estimates (3.18), (3.19) do not change, while (2.14) and (2.9) take the form

$$c(\mathbf{u},\mathbf{u},\xi_1+\xi_2) + \kappa c_1(\mathbf{u},T,\theta_1+\theta_2) + \mu_0(\|\mathbf{u}\|_Q^2 - (\mathbf{u},\mathbf{u}_d)_Q) \le -\mu_2\|\chi\|_{\Gamma_N}^2, \tag{3.50}$$

$$\mathrm{div}\,\mathbf{u} = 0 \text{ in } \Omega,\ \ \mathbf{u}|_\Gamma = \mathbf{g} = \mathbf{0},\ \ T|_{\Gamma_D} = 0, \tag{3.51}$$

and instead of (3.3) we have to use the original identity (2.8). It follows from (3.51) that $T \in \mathcal{T}$. Setting $S = T$ in (2.8) we obtain from (1.5) that

$$\lambda a_1(T,T) + \lambda(\alpha T, T)_{\Gamma_N} = -c_1(\mathbf{u}, T_1, T) + (\chi, T)_{\Gamma_N}. \tag{3.52}$$

Using (1.7), (1.11) and the first estimate in (3.8) we deduce from (3.52) that $\delta_1\lambda\|T\|^2 \leq \gamma_1 M_T^0 \|\mathbf{u}\|_1 \|T\|_1 + \gamma_2\|\chi\|_{\Gamma_N}\|T\|_1$. This yields the following estimate for $\|T\|_1$:

$$\|T\|_1 \leq \frac{\gamma_1 M_T^0}{\delta_1\lambda}\|\mathbf{u}\|_1 + \frac{\gamma_2}{\delta_1\lambda}\|\chi\|_{\Gamma_N}. \tag{3.53}$$

As $\mathbf{g} = \mathbf{0}$ in view of (3.51) the estimates (2.17) and (2.19) take the form

$$\|\mathbf{u}\|_1 \leq \frac{2\beta_1}{\delta_0\nu}\|T\|_1, \;\; \|p\| \leq \frac{\beta_1(2M+1)}{\beta_0}\|T\|_1. \tag{3.54}$$

Taking into account the first estimate in (3.54) we obtain from (3.53) that

$$\|T\|_1 \leq \frac{2\beta_1}{\delta_0\nu}\frac{\gamma_1 M_T}{\delta_1\lambda}\|T\|_1 + \frac{\gamma_2}{\delta_1\lambda}\|\chi\|_{\Gamma_N}.$$

In view of (2.3) and (3.54) we deduce the following estimates for $T$, $\mathbf{u}$ and $p$:

$$\|T\|_1 \leq \frac{\gamma_2\|\chi\|_{\Gamma_N}}{\delta_1\lambda(1-2\mathcal{R}a)}, \;\; \|\mathbf{u}_1 \leq \frac{2\beta_1\gamma_2\|\chi\|_{\Gamma_N}}{\delta_0\nu\delta_1\lambda(1-2\mathcal{R}a)}, \;\; \|p\| \leq \frac{\beta_1\gamma_2(2M+1)\|\chi\|_{\Gamma_N}}{\beta_0\delta_1\lambda(1-2\mathcal{R}a)}. \tag{3.55}$$

It follows from (1.8), (1.9), (3.18) and (3.55) that

$$|c(\mathbf{u},\mathbf{u},\xi_1+\xi_2)| \leq \gamma_0\|\mathbf{u}\|_1^2(\|\xi_1\|_1 + \|\xi_2\|_1) \leq 4\mu_0 C\left(\frac{2\beta_1}{\delta_0\nu}\frac{\gamma_2}{\delta_1\lambda}\right)^2\frac{(\mathcal{R}e+\mathcal{R}e^0)}{(1-2\mathcal{R}a)^2}\|\chi\|^2_{\Gamma_N},$$

$$\kappa|c_1(\mathbf{u},T,\theta_1+\theta_2)| \leq \kappa\gamma_1\|\mathbf{u}\|_1\|T\|_1(\|\theta_1\|_1 + \|\theta_2\|_1)$$
$$\leq 4\mu_0 C\frac{2\beta_1}{\delta_0\nu}\left(\frac{\gamma_2}{\delta_1\lambda}\right)^2\frac{\gamma_1\beta_1}{\delta_1\lambda}\frac{(\mathcal{R}e+\mathcal{R}e^0)}{\gamma_0(1-2\mathcal{R}a)^2}\|\chi\|^2_{\Gamma_N}.$$

Here $C$ and $\mathcal{R}e^0$ are defined in (3.19). From these inequalities and (2.3) we deduce that

$$|c(\mathbf{u},\mathbf{u},\xi_1+\xi_2) + \kappa c_1(\mathbf{u},T,\theta_1+\theta_2)|$$
$$\leq 2\mu_0 C(\frac{2\beta_1}{\delta_0\nu}\frac{\gamma_2}{\delta_1\lambda})^2\frac{(\mathcal{R}e+\mathcal{R}e^0)}{(1-2\mathcal{R}a)^2}(\frac{2\mathcal{P}\gamma_0+\gamma_1}{\mathcal{P}\gamma_0})\|\chi\|^2_{\Gamma_N}. \tag{3.56}$$

Let the input data for problem (3.49) be such that

$$(1-\varepsilon)\mu_2 \geq 2\mu_0 C(\frac{2\beta_1}{\delta_0\nu}\frac{\gamma_2}{\delta_1\lambda})^2\frac{(\mathcal{R}e+\mathcal{R}e^0)}{(1-2\mathcal{R}a)^2}\left(\frac{2\mathcal{P}\gamma_0+\gamma_1}{\mathcal{P}\gamma_0}\right)\|\chi\|^2_{\Gamma_N}, \;\; \varepsilon = \text{const} > 0. \tag{3.57}$$

In view of (3.57) we find from (3.56) that

$$|c(\mathbf{u},\mathbf{u},\xi_1+\xi_2) + \kappa c_1(\mathbf{u},T,\theta_1+\theta_2)| \leq (1-\varepsilon)\mu_2\|\chi\|^2_{\Gamma_N}. \tag{3.58}$$

Combining (3.50) and (3.58) we obtain the inequality

$$\mu_0(\|\mathbf{u}\|_Q^2 - (\mathbf{u}, \mathbf{u}_d)_Q) \leq -c(\mathbf{u}, \mathbf{u}, \xi_1 + \xi_2) - \kappa c_1(\mathbf{u}, T, \theta_1 + \theta_2) - \mu_2\|\chi\|_{\Gamma_N}^2$$
$$\leq -\varepsilon\mu_2\|\chi\|_{\Gamma_N}^2. \tag{3.59}$$

Using (3.59) we deduce (3.25) which holds under condition (3.57). The uniqueness of the solution to problem (3.49) follows from this estimate and (3.57) when $Q = \Omega$.

Rewriting (3.59) in view of (3.25) in the form

$$\varepsilon\mu_2\|\chi\|_{\Gamma_N}^2 \leq c(\mathbf{u}, \mathbf{u}, \xi_1 + \xi_2) + \kappa c_1(\mathbf{u}, T, \theta_1 + \theta_2) + \mu_2\|\chi\|_{\Gamma_N}^2$$
$$\leq -\mu_0\|\mathbf{u}\|_Q^2 + \mu_0\|\mathbf{u}\|_Q\|\mathbf{u}_d\|_Q \leq \mu_0\|\mathbf{u}_d\|_Q^2,$$

and using (3.55) we obtain the following stability estimates:

$$\|\chi_1 - \chi_2\|_{\Gamma_N} \leq \sqrt{\frac{\mu_0}{\varepsilon\mu_2}}\|\mathbf{u}_d^{(1)} - \mathbf{u}_d^{(2)}\|_Q,$$
$$\|\mathbf{u}_1 - \mathbf{u}_2\|_1 \leq \frac{2\beta_1\gamma_2}{\delta_0\nu\delta_1\lambda(1-2\mathcal{R}a)}\sqrt{\frac{\mu_0}{\varepsilon\mu_2}}\|\mathbf{u}_d^{(1)} - \mathbf{u}_d^{(2)}\|_Q,$$
$$\|T_1 - T_2\|_1 \leq \frac{\gamma_2}{\delta_1\lambda(1-2\mathcal{R}a)}\sqrt{\frac{\mu_0}{\varepsilon\mu_2}}\|\mathbf{u}_d^{(1)} - \mathbf{u}_d^{(2)}\|_Q,$$
$$\|p_1 - p_2\| \leq \frac{\beta_1\gamma_2(2M+1)}{\beta_0\delta_1\lambda(1-2\mathcal{R}a)}\sqrt{\frac{\mu_0}{\varepsilon\mu_2}}\|\mathbf{u}_d^{(1)} - \mathbf{u}_d^{(2)}\|_Q \ (M \equiv \delta_0^{-1} + 2\mathcal{R}e). \tag{3.60}$$

So the next theorem holds.

**Theorem 12.** *Let under conditions* (i), (ii), (j) *and* (2.4)*, the quadruple* $(\mathbf{u}_i, p_i, T_i, \chi_i)$ *be the solution to problem* (3.49) *corresponding to a given function* $\mathbf{u}_d^{(i)} \in \mathbf{L}^2(Q)$*,* $i = 1, 2$*, and let the condition* (3.57) *hold where* $C$ *and* $\mathcal{R}e^0$ *are defined in* (3.19)*. Then the stability estimates* (3.25) *and* (3.60) *hold true.*

In the same manner one can study the extremum problem

$$\begin{gathered} J(\mathbf{v}, \chi) \equiv \frac{\mu_0}{2}\|\mathrm{rot}\,\mathbf{v} - \zeta_d\|_Q^2 + \frac{\mu_2}{2}\|\chi\|_{\Gamma_N}^2 \to \inf, \\ F(\mathbf{x}, \chi) = 0,\ \mathbf{x} = (\mathbf{v}, q, S) \in X, \chi \in K_2. \end{gathered} \tag{3.61}$$

The following theorem holds.

**Theorem 13.** *Let, under conditions* (i), (ii), (j) *and* (2.4)*, the quadruple* $(\mathbf{u}_i,\ p_i,\ T_i,\ \chi_i)$ *be the solution to problem* (3.61) *corresponding to a given function* $\zeta_d^{(i)} \in \mathbf{L}^2(Q)$*,* $i = 1, 2$*, and let the condition* (3.57) *hold where* $C$ *and* $\mathcal{R}e^0$ *are defined in* (3.42)*. Then the estimates* (3.44) *and the stability estimates*

$$\|\chi_1 - \chi_2\|_{\Gamma_N} \leq \sqrt{\frac{\mu_0}{\varepsilon\mu_2}}\|\zeta_d^{(1)} - \zeta_d^{(2)}\|_Q,$$
$$\|\mathbf{u}_1 - \mathbf{u}_2\|_1 \leq \frac{2\beta_1\gamma_2}{\delta_0\nu\delta_1\lambda(1-2\mathcal{R}a)}\sqrt{\frac{\mu_0}{\varepsilon\mu_2}}\|\zeta_d^{(1)} - \zeta_d^{(2)}\|_Q,$$

$$\|T_1 - T_2\|_1 \leq \frac{\gamma_2}{\delta_1\lambda(1-2\mathcal{R}a)}\sqrt{\frac{\mu_0}{\varepsilon\mu_2}}\|\zeta_d^{(1)} - \zeta_d^{(2)}\|_Q,$$

$$\|p_1 - p_2\| \leq \frac{\beta_1\gamma_2(2M+1)}{\beta_0\delta_1\lambda(1-2\mathcal{R}a)}\sqrt{\frac{\mu_0}{\varepsilon\mu_2}}\|\zeta_d^{(1)} - \zeta_d^{(2)}\|_Q \ \ (M \equiv \delta_0^{-1} + 2\mathcal{R}e) \qquad (3.62)$$

*hold true.*

We emphasize that the uniqueness and stability of solutions to problem (3.49) or (3.61) both under $Q = \Omega$ and under $Q \subset \Omega$ is proved only if the parameter $\mu_2$ in (3.49) or (3.61) is positive and satisfies (3.57). This means that term $(\mu_2/2)\|\chi\|^2_{\Gamma_N}$ in the expression for the minimized functional in (3.49) or (3.61) has a regularizing effect on the extremum problem (3.49) or (3.61).

## 4. Numerical analysis

The authors have developed algorithms for numerical solution of extremum problems studied in Section 2. These algorithms are based on using the Newton method. A separate paper by the authors will be devoted to the convergence analysis of these algorithms. Below we limit ourselves to discussion of the numerical results for extremum problem (3.49) with $I = I_1$ considered in a back-facing-step channel.

We consider the case where the function $\mathbf{g}$ is fixed and the set $K_2$ coincides with the entire space $L^2(\Gamma_N)$. Then the minimum will be reached in an internal point of set $K_2$ and it is possible to replace the minimum principle (1.23) with the identity $(\mu\chi - \theta, \phi)_{\Gamma_N} = 0$ for all $\phi \in L^2(\Gamma_N)$. Having expressed $\chi$ from this relation by the formula $\chi = \theta/\mu$, we can eliminate the control $\chi$ from the optimality system. The received relations we shall write down in the form of the operator equation $\Phi(\mathbf{u}, p, T, \xi, \sigma, \theta) = 0$. For its numerical solution the iterative algorithm based on the Newton method is proposed. This algorithm consists of the following steps:

1. For given $(\mathbf{u}_0, p_0, T_0, \xi_0, \sigma_0, \theta_0)$ and supposing $\mathbf{u}_n$, $p_n$, $T_n$, $\xi_n$, $\sigma_n$ and $\theta_n$ are known, we define $\tilde{\mathbf{u}}$, $\tilde{p}$, $\tilde{T}$, $\tilde{\xi}$, $\tilde{\sigma}$, $\tilde{\theta}$ by solving the following problem:

$$\Phi'(\mathbf{u}_n, p_n, T_n, \xi_n, \sigma_n, \theta_n)(\tilde{\mathbf{u}}, \tilde{p}, \tilde{T}, \tilde{\xi}, \tilde{\sigma}, \tilde{\theta}) = -\Phi(\mathbf{u}_n, p_n, T_n, \xi_n, \sigma_n, \theta_n).$$

2. Then we calculate new approximations $\mathbf{u}_{n+1}$, $p_{n+1}$, $T_{n+1}$, $\xi_{n+1}$, $\sigma_{n+1}$, $\theta_{n+1}$ for $\mathbf{u}$, $p$, $T$, $\xi$, $\sigma$, $\theta$ as

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \tilde{\mathbf{u}},\ p_{n+1} = p_n + \tilde{p},\ T_{n+1} = T_n + \tilde{T},$$

$$\xi_{n+1} = \xi_n + \tilde{\xi},\ \sigma_{n+1} = \sigma_n + \tilde{\sigma},\ \theta_{n+1} = \theta_n + \tilde{\theta}.$$

3. If the condition $\|T_{n+1} - T_n\| < \varepsilon$ for some sufficiently small number $\varepsilon$ is not satisfied, then we go to step 1.

We used free software freeFEM++ (www.freefem.org) for the discretization of direct boundary-value problems by the finite element method.

The computational experiments showed that if the initial guess is selected sufficiently close to the exact solution, then the algorithm converges for several iterations. The regularization parameter $\mu_2$ plays an important role. If its values are relatively large then we can not obtain small values of the functional $I_1$. But, on the other hand, very small values of the regularization parameter can lead to instability and oscillations in the numerical solution.

The following example is connected with the vortex reduction in the backward-facing-step channel by means of the “temperature” boundary control $\chi$. The initial flow without controls is the solution of the nonlinear problem (1.1) with $\mathbf{f} = 0$, $\tilde{\beta} = 0$ and Reynolds number Re $= 200$. The streamlines for this case are shown in Figure 1.

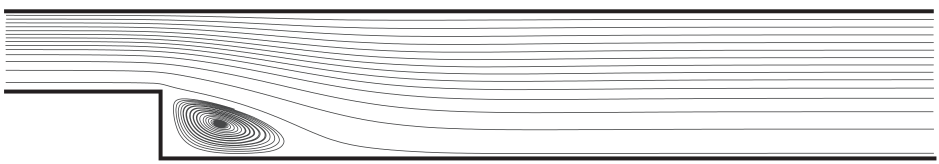

Figure 1. Streamlines for uncontrolled flow (Re=200)

One can see that it is a complicated flow with a vortex in the corner. The desired flow $\mathbf{v}_d$ is the solution of the linear Stokes equations.

We want to find the solution $\mathbf{u}$ of the nonlinear problem (1.1), (1.2) with Reynolds number Re $= 200$, Rayleigh number Ra $= 10^5$ closed to the desired velocity field $\mathbf{v}_d$. For this purpose we solve the extremum problem (3.49) with the functional $I_1$ and the boundary control $\chi$. The received flow is shown in Figure 2.

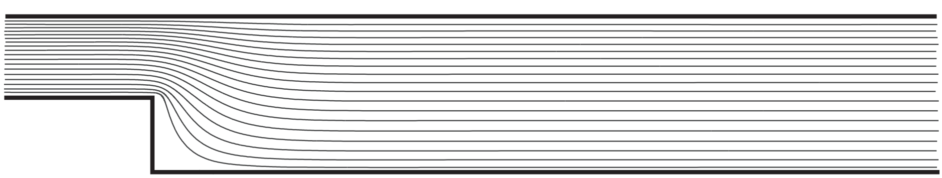

Figure 2. Streamlines for controlled flow (Re=200)

Similar results have been received for the functional $I_3$ in the case of the vorticity minimization ($\zeta_d = \mathbf{0}$). Looking at this flow we can see that the “temperature” control $\chi$ allows us to create a velocity field with desired properties.

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# Homogenization of the Poisson–Boltzmann Equation

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**Abstract.** By the homogenization approach we justify a two-scale model of ion equilibrium between solid layers. By up-scaling, the electric potential equation in nanoslits separated by thin solid layers is approximated by a homogenized macroscale equation in the form of the Poisson equation for an induced vertical electrical field.

**Mathematics Subject Classification (2000).** 78A35, 35B27.

**Keywords.** Nonlocal Poisson-Boltzmann equation, existence, homogenization

## 1. Introduction

The spontaneous separation of charge at solid-liquid interfaces is ubiquitous in microfluidic devices, and is central to electrokinetic actuation of flow [5]. Several chemical mechanisms can give rise to the spontaneous separation of charge between two phases. The most relevant to microfluidics are ionization of surface groups and preferential adsorption of ions of one charge or the other. The surface charge generates an electric field, which pulls oppositely charged ions (counterions) toward the surface, and pushes like charges (co-ions) away from it. Counterions preferentially concentrate near the surface, effectively shielding the bulk solution from the surface charge. The shielding layer is often referred to as the double electric layer.

Detailed descriptions of the internal structure of the double electric layer are often based on the Gouy-Chapman-Stern model [4], where the double electric layer is comprised of a Stern layer and a diffuse layer. The Stern layer consists of counterions which are immobilized on the surface, and its thickness is dictated by the size of the ions. The diffuse layer lies just beyond the Stern layer, and is responsible for the electrokinetic phenomena relevant to microfluidic devices.

For description of electroosmosis, it is sufficient to treat the diffuse layer ion distribution in the Boltzmann limit in which ions are treated as point particles in a mean field. The Nernst ionic flux equations coupled with the Poisson equation for electrical field give rise to the Poisson–Boltzmann equation for the electric potential $\varphi$. We perform derivation of this equation for a layered structure to explain why it is a non-local equation.

We develop an asymptotic approach for the qualitative analysis of the Poisson–Boltzmann equation. To this end we consider a vertical membrane, and we treat this membrane as a number of thin horizontal liquid layers of the same thickness $h_f$ separated by thin solid layers of the same thickness $h_s$. If $N$ is the total number of liquid layers, the total membrane thickness is equal to $L = N(h_f + h_s)$. In our study the total thickness $L$ is fixed and the ratio $\delta = (h_f + h_s)/L$ is a small parameter.

Our asymptotic analysis is the well-known homogenization procedure based on two-scale asymptotic expansions [1, 2, 7, 10] to up-scale the micromechanical picture of ion distribution near solid surfaces. We derive both microscale and macroscale equations. The first equations serve to identify constant coefficients in the second equations.

Proofs of the mathematical results below are strongly based on a priori estimates, independent of $\delta$, of the norm $\|\varphi\|_{H^1}$ for solutions of the nonlinear nonlocal Poisson–Boltzmann equation

$$(\varepsilon(z)\varphi_z)_z = -f(\varphi), \quad 0 < z < L, \tag{1.1}$$

where $\varepsilon(z)$ is a discontinuous stepwise periodic function with the periodicity cell $a_n < z < a_{n+1}$, $a_{n+1} - a_n = O(\delta)$; given a liquid interval $a_n < z < b_n$ ($b_n < a_{n+1}$), the nonlocal term $f(\varphi)$ is defined as

$$f(\varphi)|_{a_n<z<a_{n+1}} = 4\pi \mathbf{1}|_{a_n<z<b_n} \sum_{\pm} c_i^- q_i e^{\frac{q_i}{kT}\left(\varphi(d_n)-\varphi(z)\right)}, \quad d_n = \frac{a_n + b_n}{2},$$

where $\mathbf{1}(z)|_\omega$ is the characteristic function of the set $\omega$. The theory that we develop for equation (1.1) reveals that there are three types of electrolytes depending on the sign of the number $E = \sum_\pm c_i^- q_i$. Particularly, in the case of the "convex" electrolyte when $E < 0$, any solution of (1.1) satisfies the alternative property: either $\varphi$ is monotone or there is a unique point $z_c$ such that $\varphi_z(z_c) = 0$ and $(z - z_c)\varphi_z(z) > 0$ for $z \neq z_c$. To study the potential of the convex electrolyte in the alternative case when $\varphi$ is not monotone, we pass to a rearrangement function $\varphi'(z')$, which is a shift transformation of the function $\varphi(z)$, such that

$$\varepsilon_f \varphi'_{z'z'} = -f'(\varphi'), \quad 0 < z' < L' = \Phi L, \tag{1.2}$$

$$f'(\varphi')|_{a'_n<z'<a'_{n+1}} = 4\pi \sum_{\pm} c_i^- q_i e^{\frac{q_i}{kT}\left(\varphi'(d'_n)-\varphi'(z')\right)}, \quad d'_n = \frac{a'_n + a'_{n+1}}{2},$$

where $\varepsilon_f = \text{const}$ is the value of $\varepsilon(z)$ on the liquid domain and the points $a'_n$ are chosen in such a way that $a'_{n+1} - a'_n = b_n - a_n$ and $a_0 = a'_0$. It is essential that $\varphi'_{z'}(z') = \varphi_z(z)$ at the corresponding points $z'$ and $z$, and the function $\varphi'_{z'}(z')$ is continuous everywhere, whereas $\varphi_z(z)$ has jumps at the points $a_n$ and $b_n$. Next, we introduce the local function $w'^{(n)}(z') = \varphi'(z') - \varphi'(d'_n)$ which solves on each interval $a'_n < z' < a'_{n+1}$ the "local" equation

$$\varepsilon_f w_{z'z'} = -4\pi \sum_{\pm} c_i^- q_i e^{-\frac{q_i}{kT} w}, \quad w(d'_n) = 0. \tag{1.3}$$

Taking into account the fact that the right-hand side of the equation in (1.3) is a convex function of $w$, we establish a comparison inequality for any two solutions of (1.3). We find that the local function $w'^{(m)}(z')$ is given by an explicit formula at the interval $(a'_m, a'_{m+1})$ containing the point $z_c$, that enables us to estimate the norm of $w'^{(m)}(z')$ in $H^1(a'_m < z' < a'_{m+1})$. Applying the comparison inequality we verify that there is an extension $W'(z')$ of local solution $w'^{(m)}(z')$ onto the entire interval $0 < z' < L'$ such that $W'(z')$ serves as a majorant for any local function $w'^{(n)}(z')$. In this way, we estimate the norm of $\varphi'(z')$ in $H^1(0, L')$ using the local equality $w'^{(n)}_{z'}(z') = \varphi'_{z'}(z')$.

## 2. Basic equations

When in equilibrium, ion components in a binary electrolyte solution satisfy the Nernst equation [11]

$$0 = -c_i q_i d\psi - kT dc_i, \quad \mathbf{E} = -\nabla\psi, \quad i = +, -, \tag{2.1}$$

where $c_i$ is the ion molar concentration, $\psi$ is the potential of the electric field $\mathbf{E}$, $k$ is the Boltzmann constant, $q_i$ is the ion charge, $e > 0$ is the elementary charge, $q_i = z_i e$, $z_i$ is the valency of each ionic species, and $T$ is the temperature. Equality (2.1) implies that the chemical potentials are constant when all the contact components are in equilibrium.

The charge conservation law is the Poisson equation

$$\operatorname{div} \mathbf{D} = 4\pi \sum_{\pm} c_i q_i, \quad \mathbf{D} = \varepsilon_f \mathbf{E}, \quad \mathbf{E} = -\nabla\varphi, \tag{2.2}$$

where $\mathbf{D}$ is the electric induction vector and $\varepsilon_f$ is the dielectric permittivity of the electrolyte. Inside the solid dielectric, the electrical field obeys the equations

$$\operatorname{div} \mathbf{D} = 0, \quad \mathbf{D} = \varepsilon_s \mathbf{E}, \quad \mathbf{E} = -\nabla\varphi, \tag{2.3}$$

where $\varepsilon_s$ is the dielectric permittivity of the solid dielectric.

We consider the potential field in the layer of thickness $L$ consisting of $N$ thin fluid layers $a_n < z < b_n$ of the same thickness $h_f$ separated by slits $b_n < z < a_{n+1}$ of a solid dielectric of the same thickness $h_s$. The central points $d_n$ of the liquid intervals $a_n < z < b_n$ are the points of reference where the ion concentrations $c_i$ take the prescribed values $c_i^-$.

Let $Q_f$ and $Q_s$ stand for fluid and solid domain

$$Q_f = \{x, z : -\infty < x < +\infty, z \in \Omega_f\}, \ Q_s = \{x, z : -\infty < x < +\infty, z \in \Omega_s\},$$

$$\Omega_f = \bigcup_{n=0}^{N-1} \{a_n < z < b_n\}, \quad \Omega_s = \bigcup_{n=0}^{N-1} \{b_n < z < a_{n+1}\},$$

$$a_n = n(h_f + h_s), \quad b_n = a_n + h_f, \quad d_n = a_n + h_f/2.$$

We look for a solution $(\varphi, c_+, c_-)$ which depends on the variable $z \in [0, L]$. In the fluid domain $\Omega_f$, such a solution solves the system

$$-c_i q_i \varphi_z - kT c_{iz} = 0, \tag{2.4}$$

$$\varepsilon_f \varphi_{zz} = -4\pi \sum_{\pm} c_i q_i. \tag{2.5}$$

In the solid domain $\Omega_s$ the potential $\varphi$ solves the equation

$$\varepsilon_s \varphi_{zz} = 0. \tag{2.6}$$

Conditions of continuity of the potential $\varphi$ and the induction field $D$ are

$$\text{at} \quad z = a_n \quad \text{and} \quad z = b_n : [\varphi] = [\varepsilon \varphi_z] = 0, \quad \text{at} \quad z = d_n : c_i = c_i^-, \tag{2.7}$$

where $n = 1, \ldots, N-1$ and $[\varphi]|_{z=z_0}$ stands for the jump of a discontinuous function $\varphi$ at the point $z_0$:

$$[\varphi]|_{z=z_0} = \lim_{\sigma \to 0} (\varphi(z_0 + \sigma) - \varphi(z_0 - \sigma)).$$

We assume that $\varphi$ satisfies the external boundary conditions

$$\varphi|_{z=0} = \zeta_0, \quad \varphi|_{z=L} = \zeta_L, \tag{2.8}$$

Let us derive some consequence of the above formulation. In what follows we assume that the dielectric permittivity function

$$\varepsilon = \begin{cases} \varepsilon_f, & z \in \Omega_f, \\ \varepsilon_s, & z \in \Omega_s, \end{cases} \tag{2.9}$$

is extended periodically on $\mathbb{R}$.

Let us exclude the concentrations $c_i$. One can write equation (2.4) as

$$\frac{d}{dz}(q_i \varphi + kT \ln c_i) = 0.$$

Integrating between $d_n$ and $z \in (a_n, b_n)$, we obtain

$$c_i = c_i^- \exp\Big[\frac{q_i}{kT}(\varphi(d_n) - \varphi(z))\Big]. \tag{2.10}$$

Hence, the potential $\varphi$ solves in each liquid domain $(a_n, b_n)$ the Poisson–Boltzmann equation [11]

$$\varepsilon_f \varphi_{zz} = -4\pi \sum_{\pm} c_i^- q_i \exp\Big[\frac{q_i}{kT}(\varphi(d_n) - \varphi(z))\Big]. \tag{2.11}$$

With $[z]_e$ standing for the entire part of a number $z$, the functions

$$H_a(z) = h\left[\frac{z}{h}\right]_e, \quad H_d(z) = \frac{h_f}{2} + h\left[\frac{z}{h}\right]_e, \quad H_b(z) = h_f + h\left[\frac{z}{h}\right]_e, \tag{2.12}$$

where $h \equiv h_f + h_s$, take the constant values $a_n$, $d_n$, and $b_n$ if $a_n < z < a_{n+1}$. Let $\chi$ be the characteristic function of the liquid domain $\Omega_f$. Thus to define $\varphi$ on the whole interval $0 < z < L$, one should solve the equation

$$(\varepsilon\varphi_z)_z = -4\pi\chi(z)\sum_{\pm} c_i^- q_i \exp\left[\frac{q_i}{kT}(\varphi(H_d(z)) - \varphi(z))\right], \tag{2.13}$$

jointly with the conditions (2.7) and (2.8). Observe that the function $\xi_d = H_d(z) - z$ is periodic, and $\xi_d = h_f/2 - z$ on the interval of periodicity $0 < z < h$.

## 3. Scaling and identification of a small parameter

We look for an asymptotic solution of problem (2.13), (2.7),(2.8) assuming that the ratio

$$\frac{h}{L} = \frac{1}{N} = \delta$$

is a small parameter for some positive entire number $N$. We argue by the homogenization approach, so the interval $\Omega = \{0 < z < L\}$ is fixed and $\delta$ varies in $(0, 1)$. In that case

$$h(\delta) = \delta L, \quad h_f = \delta\bar{h}_f, \quad h_s = \delta\bar{h}_s, \quad \bar{h}_f + \bar{h}_s = L, \quad \Phi := \bar{h}_f/L.$$

Here, $\Phi$ is porosity.

We call $z \in \Omega$ a macro-variable and we introduce the micro-variable $y = z/(\delta L)$. With $\delta$ being small, the periodic functions $\varepsilon(z)$ and $\chi(z)$ oscillate strongly and they can be represented as functions of the micro-variable

$$\varepsilon(z) = \tilde{\varepsilon}\left(\frac{z}{\delta L}\right), \quad \chi(z) = \tilde{\chi}\left(\frac{z}{\delta L}\right),$$

where

$$\tilde{\varepsilon}(y) = \begin{cases} \varepsilon_f, & 0 < y < \Phi, \\ \varepsilon_s, & \Phi < y < 1, \end{cases} \quad \text{and} \quad \tilde{\chi}(y) = \begin{cases} 1, & 0 < y < \Phi, \\ 0, & \Phi < y < 1, \end{cases}$$

are periodic functions with period equal to 1. In what follows the functions

$$\tilde{\xi_a}(y) = -Ly, \quad \tilde{\xi_d}(y) = L(\Phi/2 - y), \quad \tilde{\xi_b}(y) = L(\Phi - y), \quad y \in Y \equiv (0, 1),$$

are extended periodically. One can verify easily that the functions $H_a(z)$, $H_d(z)$, and $H_b(z)$ defined in (2.12) can be represented as

$$H_a(z) = z + \delta\tilde{\xi_a}\left(\frac{z}{\delta L}\right), \quad H_d(z) = z + \delta\tilde{\xi_d}\left(\frac{z}{\delta L}\right), \quad H_b(z) = z + \delta\tilde{\xi_b}\left(\frac{z}{\delta L}\right).$$

With the above notations at hand, the function $\varphi(z)$ solves on the entire interval $0 < z < L$ the problem

$$(\tilde{\varepsilon}\left(\frac{z}{\delta L}\right)\varphi_z)_z = -f(\varphi), \tag{3.1}$$
$$f = 4\pi\tilde{\chi}\left(\frac{z}{\delta L}\right)\sum_{\pm} c_i^- q_i \exp\left(\frac{q_i}{kT}\left\{\varphi\left(z + \delta\tilde{\xi}_d\left(\frac{z}{\delta L}\right)\right) - \varphi(z)\right\}\right),$$

with the boundary conditions (2.8).

Let us perform scaling, using the bar-sign $\bar{f}$ for a reference value of the variable $f$ and the prime-sign $f'$ for a dimensionless value of $f$, i.e., $f = \bar{f}f'$. The special scaling notations are accepted for the following variables:

$$z = Lz', \quad c_i = \bar{c}c_i', \quad q_i = \bar{q}q_i', \quad \varphi = \bar{\varphi}\varphi', \quad H_d(z) = LH_d'(z').$$

The length

$$l_d = \left(\frac{\varepsilon_l kT}{2\bar{c}\bar{q}^2}\right)^{1/2} \tag{3.2}$$

is known as the Debye length. In terms of dimensionless variables, equation (3.1) in the fluid domain becomes

$$\left(\frac{l_d^2}{L^2}\right)_2\left(\frac{\bar{q}\bar{\varphi}}{kT}\right)_1\varphi'_{z'z'} \tag{3.3}$$
$$= -4\pi\sum_{\pm} c_i'q_i'\exp\left(q_i'\left(\frac{\bar{q}\bar{\varphi}}{kT}\right)_1\{\varphi'(H_d'(z')) - \varphi'(z')\}\right).$$

In the solid domain, equation (3.1) becomes $(\varepsilon_s)_3\varphi'_{z'z'} = 0$.

Assuming that the dimensionless quantities $(\cdot)_i$ satisfy the equalities

$$(\cdot)_i = \delta^{n_i}, \quad i = 1, 2, 3, \tag{3.4}$$

we obtain a hierarchy of problems to study. In this paper we restrict ourselves to the case when all the powers $n_i$ are equal to zero, i.e., $(\cdot)_i = O(1)$. The meaning of these hypotheses is the following. The relation $(\cdot)_1 = O(1)$ implies that electroosmotic force and thermal force are of the same order. Observe that the relation $(\cdot)_1 = O(1)$ holds, for example, for the symmetric electrolyte (where $z_+ = z_-$ and $c_+^- = c_-^-$) in water at $T = 298K$, $z = 1$, with the $\zeta$-potential equal to $25[mV]$ [5]. When $(\cdot)_1$ is not small, the Debye-Hückel linearization of the Poisson–Boltzmann equation does not work. Under the condition $(\cdot)_1 = O(1)$ the Debye length $l_d$ can no longer be compared to the electrical double layer, moreover double layer overlapping could occur. Indeed, it is a useful rule of thumb [5] that $l_d = 9.6/(z\sqrt{\bar{c}})$. For the above mentioned electrolyte with the counterion molar concentration $\bar{c} = 0,01[mM]$ we have $l_d = 100[nm]$, whereas the double electric layer is normally only a few nanometers thick [5] and the nanocapillary membrane may have a pore diameter of 15 [nm] [3]. For such cases the hypothesis $(\cdot)_2 = O(1)$ is natural. For water with low electrolyte concentration $\bar{c}$ the hypothesis $(\cdot)_3 = O(1)$, i.e., $\varepsilon_s/\varepsilon_f = O(1)$, is natural.

We close this section by recalling the Debye-Hückel approach to the Poisson–Boltzmann equation (2.11) in the single layer $z > 0$ with boundary conditions

$\varphi \to 0$ and $\varphi_z \to 0$ as $z \to \infty$ and $\varphi|_{z=0} = \zeta_0$. In the case of a symmetric electrolyte, the linearized equation (2.11), in the SI system of units where $4\pi$ is replaced by 1, becomes $l_d^2 \varphi_{zz} = -\varphi$, since the nonlocal term $\varphi(d)$ vanishes as $d \to \infty$. Clearly, $\varphi = \zeta_0 e^{-z/l_d}$ is a solution. This explains the notion (3.2).

## 4. A priori estimates

We proceed by returning to the dimensional variables. First, we study how solutions of problem (3.1),(2.8) depend on $\delta$. We call the electrolyte concave, linear or convex if

$$\sum_{\pm} c_i^- q_i > 0, \quad \sum_{\pm} c_i^- q_i = 0, \quad \sum_{\pm} c_i^- q_i < 0,$$

respectively. We recall that $q_1 = q_+ > 0$ and $q_2 = q_- < 0$. We write

$$p_1 = \frac{4\pi q_1 c_1^-}{\varepsilon_f}, \quad p_2 = \frac{4\pi |q_2| c_2^-}{\varepsilon_f}, \quad r_1 = \frac{q_1}{kT}, \quad r_2 = \frac{|q_2|}{kT}. \tag{4.1}$$

With these notations at hands, equation (2.11) in the fluid domain becomes

$$\varphi_{zz} = p_2 e^{r_2[\varphi(z) - \varphi(H_d(z))]} - p_1 e^{-r_1[\varphi(z) - \varphi(H_d(z))]}.$$

We prove that there is a positive constant $B_0$ such that

$$\int_0^L \varphi_z^2(z)dz \le B_0, \quad \max_{0 \le z \le L} |\varphi(z)| \le B_0. \tag{4.2}$$

Here and in what follows, constants $B_i$ do not depend on $\delta$.

**Linear electrolyte.** First, we consider the case when $p_1 = p_2 = p$. Given an interval $[a_n, b_n]$, we prove that $\varphi_z \neq 0$ for any $z \in [a_n, b_n]$ otherwise $\varphi = \text{const}$ on this interval. Assume that there is a point $z_0 \in [a_n, b_n]$, such that $\varphi_z(z_0) = 0$. When $z_0 < d_n$ the function $w^{(n)}(z) = \varphi(z) - \varphi(d_n)$ solves the problem

$$w_{zz}^{(n)} = g_1(w^{(n)}), \quad w_z^{(n)}(z_0) = 0, \quad w^{(n)}(d_n) = 0, \tag{4.3}$$

where

$$g_1(w) = \frac{dg(w)}{dw}, \quad g(w) = \frac{p}{r_1} e^{-r_1 w} + \frac{p}{r_2} e^{r_2 w} - \frac{p}{r_1} - \frac{p}{r_2}.$$

We multiply equation (4.3) by $w^{(n)}$ and integrate between the points $z_0$ and $d_n$ using the inequality $w g_1(w) \ge 0$. As a result we obtain

$$\int_{z_0}^{d_n} \left( |w_z^{(n)}|^2 + w^{(n)} g_1(w^{(n)}) \right) dz = 0 \quad \text{and} \quad \int_{z_0}^{d_n} |w_z^{(n)}|^2 dz \le 0.$$

Hence, $\varphi_z(d_n) = 0$. The case $z_0 > d_n$ can be considered similarly. Thus, $\varphi_z(d_n) = 0$ if $\varphi_z(z_0) = 0$ for some $z_0 \in [a_n, b_n]$.

It follows from $(4.3)_1$ that

$$\frac{d}{dz}\left( |w_z^{(n)}|^2 - 2g(w^{(n)}) \right) = 0. \tag{4.4}$$

Integrating between $d_n$ and $z$ we see that $w^{(n)}$ solves the Cauchy problem

$$w_z^2 = 2g(w), \quad w(d_n) = 0.$$

The function $\sqrt{g(w)}$ is Lipschitz continuous and $g(0) = 0$. By uniqueness, $w^{(n)} \equiv 0$ on the interval $[a_n, b_n]$.

Let us consider the next interval $[a_{n+1}, b_{n+1}]$. Due to the boundary condition

$$\varphi_z|_{b_n-} = \varphi_z|_{a_{n+1}+}, \tag{4.5}$$

the derivatives $w_z^{(n)}(b_n)$ and $w_z^{(n+1)}(a_{n+1})$ have the same sign.

By the above discussions, the solution $\varphi$ of the problem (2.7), (2.8), (2.13) is monotone on the whole interval $0 < z < L$. Hence,

$$\max_{0<z<L} |\varphi(z)| \le B_1, \quad B_1 = \max\{|\zeta_0|, |\zeta_L|\}. \tag{4.6}$$

We multiply the equation

$$(\varepsilon(z)\varphi_z)_z = -4\pi\chi(z) \sum_{\pm} c_i^- q_i \exp\left(\frac{q_i}{kT}\{\varphi(H_d(z)) - \varphi(z)\}\right) \equiv f(\varphi), \tag{4.7}$$

by the function $\varphi(z) - \varphi_0(z)$,

$$\varphi_0(z) = \frac{\zeta_L - \zeta_0}{L} z + \zeta_0,$$

and, because of the estimate (4.6), we obtain the inequality

$$\int_0^L \varepsilon(z)\varphi_z^2(z)dz \le B_2 + \int_0^L \varepsilon(z)\varphi_z\varphi_{0z}dz. \tag{4.8}$$

Now, to derive the first estimate in (4.2), it suffices to apply the Young inequality

$$|\varphi_z\varphi_{0z}| \le \frac{\mu}{2}\varphi_z^2 + \frac{1}{2\mu}\varphi_{0z}^2. \tag{4.9}$$

**Convex electrolyte.** We consider a "convex" electrolyte with the condition $p_2 > p_1$. Given a fluid interval $a_n < z < b_n$, the function $w = \varphi(z) - \varphi(d_n)$ is continuous and

$$w_{zz} = g_1(w), \quad w(d_n) = 0, \tag{4.10}$$

where

$$g_1(w) = \frac{dg(w)}{dw}, \quad g(w) = \frac{p_1}{r_1}e^{-r_1 w} + \frac{p_2}{r_2}e^{r_2 w} - \frac{p_1}{r_1} - \frac{p_2}{r_2}.$$

Observe, that the function $g$ is convex, $g'(0) = p_2 - p_1 > 0$, and

$$g(0) = g(w_*) = 0, \quad g|_{w>0, w<w_*} > 0, \quad \inf g = g(w_0) \equiv g_0 < 0, \quad g'(w_0) = 0,$$

for some $w_*$ and $w_0$, such that $w_* < w_0 < 0$.

It follows from (4.10) that

$$w_z = \pm\sqrt{\kappa^2 + 2g(w)}, \quad w_z(d_n) = \kappa. \tag{4.11}$$

*Step* 1. All the solutions of problem (4.10) can be arranged in seven types depending on the value of $\kappa$.

*Class* 1. When $\kappa = 0$ the solution is given by the representation formula

$$\int_0^{w(z)} \frac{d\omega}{\sqrt{2g(\omega)}} = |z - d_n|, \quad w_z = \sqrt{2g(w)}\,\mathrm{sign}(z - d_n). \tag{4.12}$$

Note that the above integral does converge. It is the crucial property of $w$ that $(z - d_n)w_z > 0$ and $w_z(d_n) = 0$.

*Class* 2. Assume that $\kappa > 0$ and $\kappa^2 + 2g(w_0) > 0$. Then

$$w_z = \sqrt{\kappa^2 + 2g(w)} > 0, \quad z \in (a_n, b_n). \tag{4.13}$$

Thus, $w$ is an increasing function.

*Class* 3. Assume that $\kappa > 0$ and $\kappa^2 + 2g(w_0) = 0$. Clearly, $w_z > 0$ for $z > d_n$ since $g(w) > 0$ for $w > 0$. If $w > w_0$ on the interval $[a_n, d_n]$, the derivative $w_z$ is positive and satisfies formula (4.13). In this case the solution $w$ belongs to Class 2. If there is a point $z_0 \in (a_n, d_n)$ such that $w(z_0) = w_0$, we have that $w_z > 0$ for $z > z_0$ and $w_z = 0$ for $z \le z_0$.

*Class* 4. Assume that $\kappa > 0$ and $\kappa^2 + 2g(w_0) < 0$. Again, $w_z > 0$ for $z > d_n$. If $\kappa^2 + 2g(w) > 0$ on the interval $[a_n, d_n]$, the solution $w$ belongs to Class 2. Let $z_c (< d_n)$ be a point closest to $d_n$ such that $w_z(z_c) = 0$ and $\kappa^2 + 2g(w_c) = 0$, where $w_c = w(z_c)$, $w_0 < w_c < 0$. Clearly, $(z - z_c)w_z > 0$. Moreover, the solution $w$ satisfies the formula

$$\int_{w_c}^{w(z)} \frac{d\omega}{\sqrt{2\left[g(\omega) - g(w_c)\right]}} = |z - z_c|, \quad z \in [a_n, b_n]. \tag{4.14}$$

Formula (4.14) coincides with (4.13) if $w_c = 0$. Hence, Class 2 can be treated as a subset of Class 4.

*Class* 5. Assume that $\kappa < 0$ and $\kappa^2 + 2g(w_0) > 0$. In that case $w_z < 0$ on the whole interval, i.e., $w$ is a decreasing function.

*Class* 6. Assume that $\kappa < 0$ and $\kappa^2 + 2g(w_0) = 0$. In that case the solution belongs to Class 5, otherwise there is a point $z_0 \in (d_n, b_n)$ such that $w(z_0) = w_0$, $w_z < 0$ for $z < z_0$, and $w_z = 0$ for $z \ge z_0$.

*Class* 7. Assume that $\kappa < 0$ and $\kappa^2 + 2g(w_0) < 0$. We have that $w_z < 0$ for $z < d_n$. If $\kappa^2 + 2g(w) > 0$ on the interval $[d_n, b_n]$, the solution $w$ belongs to Class 5. Let $z_c (> d_n)$ be a point closest to $d_n$ such that $w_z(z_c) = 0$ and $\kappa^2 + 2g(w_c) = 0$, where $w_c = w(z_c)$, $w_0 < w_c < 0$. Clearly, $(z - z_c)w_z > 0$. Moreover, the solution $w$ satisfies formula (4.14). One can treat Class 4 and Class 7 as identical.

To construct a global solution $\varphi(z)$ on the entire interval $[0, L]$, one should put together the above elements $\varphi^{(n)}(z)$, $z \in [a_n, b_n]$, taking into account the boundary conditions (4.5). Due to these conditions, there are two possibilities. The function $\varphi(z)$ is monotone (maybe not strictly) otherwise there is a unique interval $[a_n, b_n]$ and a point $z_c \in (a_n, b_n)$ such that

$$(z - z_c)\varphi_z(z) > 0, \quad \varphi_z(z_c) = 0. \tag{4.15}$$

Moreover, the corresponding function $w^{(n)}(z)$ is given by the representation formula (4.14).

*Step* 2. (Exclusion of solid domain.) Let us consider the alternative case (4.15). We exclude the solid domain by passing to a *rearrangement function* $\varphi'(z')$, $0 < z' < \Phi L = L'$ as follows. We divide the interval $0 < z' < L'$ into $N$ subintervals of the same length $\delta \bar{h}_l$ by the points $a'_i = i\delta\bar{h}_l$, $i = 0, \ldots, N$, $a'_0 = 0$, $a'_N = L'$. We define

$$\varphi'(z')|_{a'_0 < z' < a'_1} = \varphi(z'),$$

$$\varphi'(z')|_{a'_i < z' < a'_{i+1}} = \varphi(z' + a_i - a'_i) - \sum_{k=1}^{i} (\varphi(a_k) - \varphi(b_{k-1})), \quad \text{if} \quad i \geq 1.$$

Writing $d'_i = a'_i + \delta\bar{h}_l/2$ and taking into account the no-jump conditions (2.7), we see that the function $\varphi'(z')$ belongs to $C^1[0, L'] \cap W^{2,\infty}(0, L')$ and solves the problem

$$\begin{aligned} &\varphi'_{z'z'} = g_1\left(\varphi'(z') - \varphi'(H'_d(z'))\right), \\ &\varphi'(0) = \zeta_0, \quad \varphi'(L') = \zeta_L - \sum_{k=1}^{N} (\varphi(a_k) - \varphi(b_{k-1})), \end{aligned} \tag{4.16}$$

where $H'_d(z')$ is a step-wise function such that

$$H'_d(z') = d'_i \quad \text{if} \quad a'_i < z' < a'_{i+1}.$$

Because of the equalities

$$\varphi(a_k) - \varphi(b_{k-1}) = \int_{b_{k-1}}^{a_k} \varphi_z d\xi = \frac{\varepsilon_f \delta\bar{h}_s \varphi_z(b_{k-1} - 0)}{\varepsilon_s} = \frac{\varepsilon_f \delta\bar{h}_s \varphi'_{z'}(a'_k)}{\varepsilon_s},$$

the boundary condition at $z' = L'$ in (4.16) becomes

$$\varphi'(L') = \zeta_L - \frac{\varepsilon_f(1-\Phi)L}{\varepsilon_s} \sum_{k=1}^{N} \varphi'_{z'}(a'_k). \tag{4.17}$$

One can verify that correspondence between the functions $\varphi(z)$ and $\varphi'(z')$ is bijective.

The condition (4.15) means that

$$\varphi'_{z'}(z'_c) = 0, \quad (z' - z'_c)\varphi_{z'}(z') > 0 \quad \forall z'. \tag{4.18}$$

Clearly, the function $w'^{(n)}(z') = \varphi'(z') - \varphi'(d'_n)$ is given by the formula

$$\int_{w_c}^{w'^{(n)}(z')} \frac{d\omega}{\sqrt{2[g(\omega) - g(w_c)]}} = |z' - z'_c|, \quad w'^{(n)}_{z'} = \sqrt{2[g(w'^{(n)}) - g(w_c)]}\,\mathrm{sign}(z' - z'_c). \tag{4.19}$$

The above integral is well defined. The function $W'(z')$ defined by the formula

$$\int\limits_{w_c}^{W'(z')} \frac{d\omega}{\sqrt{2[g(\omega) - g(w_c)]}} = |z' - z'_c|, \quad 0 < z' < L', \tag{4.20}$$

is an extension of the function $w'^{(n)}(z')$ given by (4.19). Clearly, $W'(z')$ solves equation $w_{z'z'} = g_1(w)$ on the whole interval $0 < z < L'$.

*Step* 3. We claim that $W'_{z'}(z') \geq w'^{(m)}_{z'}(z') \geq 0$ for any interval $a'_m < z' < a'_{m+1}$, $m > n$, and $W'_{z'}(z') \leq w'^{(m)}_{z'}(z') \leq 0$ for any interval $a'_m < z' < a'_{m+1}$, $m < n$. To this end we first prove the following comparison inequalities.

**Lemma 4.1.** Let $w_1$ and $w_2$ be two solutions of the equation $w_{zz} = g_1(w)$ on the interval $a < z < b$ such that $w_1(a) > w_2(a)$ and $w_{1z}(a) \geq w_{2z}(a)$. Then $w_1(z) > w_2(z)$ and $w_{1z}(z) \geq w_{2z}(z)$ for any $a < z < b$.

*Proof.* Let us write $w = w_1 - w_2$. Given $z > a$, we integrate the equality $w_{zz} = \int_{w_2(z)}^{w_1(z)} g''(s)ds$ over the interval $(a, z)$ to obtain that

$$w_z(z) = w_z(a) + \int\limits_a^z ds \int\limits_{w_2(s)}^{w_1(s)} g''(\xi)d\xi, \quad w(z) = w(a) + \int\limits_a^z w_z(\xi)d\xi. \tag{4.21}$$

By continuity, the inequality $w_1(z) > w_2(z)$ is valid not only for $z = a$ but on some interval $a \leq z < z_0$. Due to convexity of the function $g(w)$, the inequality $w_z(z) \geq 0$ holds for $a \leq z \leq z_0$. Hence, $w(z_0) > 0$, $w_z(z_0) \geq 0$ and one can extend the interval $a \leq z < z_0$. Assume that the maximal interval $a \leq z < z_*$ does not coincide with the interval $a \leq z < b$. It means that $w(z_*) = 0$ and $w_z(z) \geq 0$ for $a \leq z \leq z_*$. But this claim contradicts the second equality in (4.21). The lemma is proved. $\square$

Let us compare $W'(z')$ and $w'^{(n+1)}(z')$ on the interval $a'_{n+1} < z' < a'_{n+2}$. Due to (4.18) and (4.19) we have $w'^{(n+1)}(a'_{n+1}) < 0$ and $W'(a'_{n+1}) > 0$. The continuity of $\varphi'_{z'}(z')$ implies that $W'_{z'}(a'_{n+1}) = w'^{(n)}_{z'}(a'_{n+1}) = w'^{(n+1)}_{z'}(a'_{n+1})$. By Lemma 5.1, $W'(z') > w'^{(n+1)}(z')$ and $W'_{z'}(z') \geq w'^{(n+1)}_{z'}(z')$ for $a'_{n+1} < z' < a'_{n+1}$. The same arguments are applied for the functions $W'_{z'}(z')$ and $w'^{(n+2)}(z')$ on the interval $a'_{n+2} < z' < a'_{n+3}$, and etc. Thus,

$$\int\limits_0^{L'} |\varphi'_{z'}|^2 dz' = \sum_{k=0}^{N-1} \int\limits_{a'_k}^{a'_{k+1}} |w'^{(k)}_{z'}|^2 dz' \leq \int\limits_0^{L'} |W'_{z'}|^2 dz' \leq \sup_{w_0 < w_c < 0} \int\limits_0^{L'} |W'_{z'}|^2 dz' \equiv B_1, \tag{4.22}$$

and

$$|\varphi'(z')| = |\zeta_0 + \int\limits_0^{z'} \varphi'_{z'}(s)ds| \leq B_2. \tag{4.23}$$

It follows from the identity

$$w^{(n)}(z) - w^{(n)}(d_n) = \int\limits_{d_n}^{z} w_z^{(n)}(s)ds$$

that $|w^{(n)}(z)| \le B_3$ for any interval $a_n < z < b_n$. Then one obtains estimates (4.2) as in the linear electrolyte case. The case of a concave electrolyte ($p_2 < p_1$) can be considered similarly.

One can write the Poisson–Boltzmann equation as

$$(\varepsilon(z)\varphi_z)_z = \varepsilon_f\chi(z)g_1(w(z)) = \varepsilon_f(p_2-p_1)\chi(z)+\varepsilon_f\chi(z)\Big(g_1(w(z))-g_1(0)\Big), \quad (4.24)$$

where $w(z) = \varphi(z) - \varphi(H_d(z))$ and $g_1(0) = p_2 - p_1$. It follows from (4.2) that

$$\sum_{n=0}^{N-1} \int\limits_{a_n}^{a_{n+1}} w_z^2(z)dz \le B_4, \quad \max_{0\le z\le L} |w(z)| \le B_5.$$

We prove that there is a constant $B_6$ independent of $\delta$ such that

$$|g_1(w(z)) - g_1(0)| \le B_6\delta. \quad (4.25)$$

To this end, we pass to the function $\varphi'(z')$ solving the problem (4.16). By definition,

$$g_1(w(z)) - g_1(0) = g_1(\varphi'(z') - \varphi'(H_d'(z'))) - g_1(0).$$

Let $z'$ be in the interval $a_n' < z' < a_{n+1}'$, then

$$g_1(w'(z')) - g_1(0) = \int\limits_{d_n'}^{z'} g_{ww}(w'(z'))\varphi'_{z'}(z')dz'. \quad (4.26)$$

We integrate the identity

$$\varphi'_{z'}(z') - \varphi'_{z'}(z_1') = \int\limits_{z_1'}^{z'} \varphi'_{z'z'}(x)dx$$

over the variable $z_1'$ to obtain that

$$L'\varphi'_{z'}(z') = \zeta_L - \zeta_0 - \sum_1^N \Big(\varphi(a_k) - \varphi(b_{k-1})\Big) + \int\limits_0^{L'} dz_1' \int\limits_{z_1'}^{z'} \varphi'_{z'z'}(x)dx.$$

Observe that

$$|\sum_1^N \Big(\varphi(a_k) - \varphi(b_{k-1})\Big)| \le \int\limits_0^L |\varphi_z|dz.$$

It follows from equation (4.16) that $|\varphi'_{z'z'}|$ is bounded uniformly in $\delta$, hence $|\varphi'_{z'}|$ verifies this property also. Now, estimate (4.25) results from (4.26) and from the inequality

$$\left|\int\limits_{d'_n}^{z'} g_{ww}(w'(z'))\varphi'_{z'}(z')dz'\right| \leq \delta\bar{h}_f \max_{|\omega|\leq 2B_1} |g_{ww}(\omega)| \max_{0<z'<L'} |\varphi'_{z'}(z')|/2.$$

## 5. Existence

Here we consider the question of solvability of problem (2.13), (2.7), (2.8) for any fixed value of $\delta$. We apply the Leray-Schauder fixed point theorem [6] and to this end we define operators $A_\lambda$, $0 \leq \lambda \leq 1$, as follows. Given a Hölder continuous function $v \in C^\alpha(\bar{\Omega})$, $0 < \alpha < 1/2$, we find a function $\varphi(z) \in H^1(\Omega)$, $\varphi = A_\lambda v$, as a unique solution of the linear boundary-value problem

$$\int\limits_{\Omega} \varepsilon(z)\varphi_z(z)\psi_z(z)dz = -\lambda\varepsilon_f \int\limits_{\Omega} \chi(z)g_1\Big(v(z) - v(H_d(z))\Big)\psi(z)dz, \tag{5.1}$$

for any test function $\psi \in H^1_0(\Omega)$ where $\varphi(0) = \zeta_0$ and $\varphi(L) = \zeta_L$. Clearly, $\varphi \in C^{2+\alpha}[a_n, b_n]$ and $\varphi \in C^{2+\alpha}[b_n, a_{n+1}]$ for any $n$. Thus, the operators $A_\lambda : C^\alpha(\bar{\Omega}) \to C^\alpha(\bar{\Omega})$ are well defined and a fixed point of $A_1$ solves the problem (2.13), (2.7), (2.8).

Because of the a priori estimates (4.2) and the continuous embedding of $H^1(\Omega)$ into $C^{1/2}(\bar{\Omega})$ there is a constant $M$ such that $\|\varphi_\lambda\|_{C^{1/2}} \leq M$ for any fixed point $\varphi_\lambda$ of $A_\lambda$. Given a constant $M' > M$, we introduce the ball

$$U = \{v \in C^\alpha(\bar{\Omega}) : \|v\| \leq M'\}$$

in the Banach space $C^\alpha(\bar{\Omega})$. The restrictions $A_\lambda : U \to C^\alpha(\bar{\Omega})$ enjoy the following properties. By construction, the boundary of $U$ does not contain fixed points of $A_\lambda$, $0 \leq \lambda \leq 1$. The set $\cup_{\lambda\in[0,1]} A_\lambda(U)$ is compact in $C^\alpha(\bar{\Omega})$ because of the compact imbedding of $C^{1/2}(\bar{\Omega})$ into $C^\alpha(\bar{\Omega})$ for $0 < \alpha < 1/2$. The family of maps $\{v \to A_\lambda v\}_{\lambda\in[0,1]}$ is equicontinuous on $U$. The mapping $(\lambda, v) \to A_\lambda v$ is continuous from $[0,1] \times U$ to $C^\alpha(\bar{\Omega})$. The operator $A_0$ has a unique fixed point in the interior of $U$, and the mapping $v \to v - A_0(v)$ has an inverse near this fixed point. This means that we have verified all the conditions of the Leray-Schauder theorem. Thus, problem (2.13), (2.7), (2.8) has a solution $\varphi \in H^1(\Omega)$ such that $\varphi \in C^{2+\alpha}[a_n, b_n]$ and $\varphi \in C^{2+\alpha}[b_n, a_{n+1}]$ for any $0 < n < N-1$. The derivation of a priori estimates (4.2) independent of $\delta$ is justified.

## 6. Two-scale compactness

We recall [7, 1] that a sequence $u^\delta \subset L^2(\Omega)$ is said to two-scale converge to a limit $u \in L^2(\Omega \times Y)$ if for any $\psi \in C^\infty(\Omega; C^\infty_{per}(Y))$ one has

$$\lim_{\delta \to 0} \int_\Omega u^\delta(z)\psi\left(z, \frac{z}{\delta}\right) dz \to \int_\Omega \int_Y u(z,y)\psi(z,y)dzdy. \tag{6.1}$$

The two-scale limit has the following property [7]. *From each bounded sequence in $L^2(\Omega)$ one can extract a subsequence which two-scale converges to a limit $u \in L^2(\Omega \times Y)$.*

As for derivatives, we will use the following assertion [7]. *Let $u^\delta(z)$ and $u^\delta_z(z)$ be bounded sequences in $L^2(\Omega)$. Then there exist functions $u \in L^2(\Omega)$, $w \in L^2(\Omega; H^1_{per}(Y))$ and a subsequence such that both $u^\delta(z)$ and $u^\delta_z(z)$ two-scale converge to $u(z)$ and $u_z(z) + w_y(z,y)$ respectively.*

Because of the estimates (4.2), there are a sequence $\varphi^\delta(z)$ and two functions $\varphi^0(z)$ and $\varphi^1(z,y)$ such that $\varphi^0 \in L^2(\Omega)$ and $\varphi^1 \in L^2(\Omega; H^1_{per}(Y))$ and

$$\int_\Omega \varphi^\delta(z)\psi\left(z, \frac{z}{\delta L}\right) dz \to \int_\Omega \int_Y \varphi^0(z)\psi(z,y)dzdy,$$

$$\int_\Omega \varphi^\delta_z(z)\psi\left(z, \frac{z}{\delta L}\right) dz \to \int_\Omega \int_Y \Big(\varphi^0_z(z) + \varphi^1_y(z,y)/L\Big)\psi(z,y)dzdy,$$

$\forall \psi \in C^\infty(\Omega; C^\infty_{\mathrm{per}}(Y))$ as $\delta \to 0$ [7]. Observe that [2], as $\delta \to 0$,

$$\int_\Omega f\left(\frac{z}{\delta L}\right)\psi\left(z, \frac{z}{\delta L}\right) dz \to \int_\Omega \int_Y f(y)\psi(z,y)dzdy, \quad \forall \psi \in C^\infty(\Omega; C^\infty_{\mathrm{per}}(Y)),$$

for any function $f(y)$, $f \in L^2(Y)$, extended periodically onto $\mathbb{R}$.

We have

$$\int_\Omega \tilde{\varepsilon}\left(\frac{z}{\delta L}\right)\varphi^\delta_z \Psi_z(z)dz = -\varepsilon_f \int_\Omega \tilde{\chi}\left(\frac{z}{\delta L}\right) g_1(w^\delta)\Psi(z)dz \quad \forall \Psi \in H^1_0(\Omega). \tag{6.2}$$

Taking $\Psi(z) = \delta\psi\left(\frac{z}{\delta L}\right)$, with $\psi \in H^1_{\mathrm{per}}(Y) \cap C^\infty_0(Y)$, and sending $\delta$ to 0, we obtain a micro-equation in the cell $Y$,

$$\int_\Omega \int_Y \tilde{\varepsilon}(y)\Big(\varphi^0_z(z) + \varphi^1_y(z,y)/L\Big)\psi_y(y)dzdy = 0. \tag{6.3}$$

With the function $\varphi^0(z)$ at hand, we find $\varphi^1(z,y)$ by the method of separation of variables in the form $\varphi^1(z,y) = \varphi^0_z(z)w_1(y)$, where $w_1(y)$ is a periodic solution of the problem

$$\frac{d}{dy}\left(\tilde{\varepsilon}(y)\left(1 + \frac{1}{L}\frac{dw_1}{dy}\right)\right) = 0, \quad \int_Y w_1(y)dy = 0. \tag{6.4}$$

Clearly, $w_1$ is defined uniquely and

$$\tilde{\varepsilon}(y)\left(1+\frac{1}{L}\frac{dw_1}{dy}\right)=\varepsilon_h(\Phi)=\text{const},\quad \varepsilon_h(\Phi)=\Big(\int_0^1 1/\tilde{\varepsilon}(y)dy\Big)^{-1}=\frac{1}{\frac{\Phi}{\varepsilon_f}+\frac{1-\Phi}{\varepsilon_s}}. \tag{6.5}$$

Taking in (6.2) $\Psi(z)=\psi(z)$, with $\psi\in H_0^1(\Omega)$, and sending $\delta$ to 0, we obtain a macroequation on the interval $\Omega$,

$$\int_\Omega\int_Y \tilde{\varepsilon}(y)\Big(\varphi_z^0(z)+\varphi_y^1(z,y)/L\Big)\psi_z(z)dzdy=-\varepsilon_f g_1(0)\int_\Omega\int_Y \tilde{\chi}(y)\psi(z)dzdy. \tag{6.6}$$

Hence, $\varphi^0(z)$ is a solution of the boundary value problem

$$\varepsilon_h(\Phi)\varphi_{zz}^0=-4\pi\Phi\sum_\pm c_i^- q_i,\quad \varphi^0(0)=\zeta_0,\quad \varphi^0(L)=\zeta_L. \tag{6.7}$$

## 7. A corrector and error estimates

The two-scale limit $\varphi^0(z)$ approximates the function $\varphi^\delta(z)$ for small values of $\delta$. Here, we improve the approximation by finding a corrector to the function $\varphi^0(z)$. We argue by the formal expansion series [10] approach. To this end we assume that there is a function $\varphi^2(z,y)$ defined for $0<z<L$ and 1-periodic in $y$ such that

$$\varphi^\delta(z)=\varphi^0(z)+\delta\varphi_z^0(z)w_1(y)+\delta^2\varphi^2(z,y)+o(\delta^2), \tag{7.1}$$

where $\varphi^0(z)$ and $w_1(y)$ are defined by (6.7) and (6.4) respectively.

We introduce the flux $F^\delta(z)$ as follows:

$$F^\delta(z)=\tilde{\varepsilon}\left(\frac{z}{\delta L}\right)\frac{d}{dz}\varphi^\delta(z),\quad \frac{d}{dz}F^\delta=\varepsilon_f\tilde{\chi}\left(\frac{z}{\delta L}\right)g_1(w^\delta(z)), \tag{7.2}$$

where $w^\delta(z)\equiv\varphi^\delta(z)-\varphi^\delta(H_d(z))$, and we represent it also by the expansion series

$$F^\delta(z)=F^0(z,y)+\delta F^1(z,y)+o(\delta),\quad y=z/(\delta L). \tag{7.3}$$

Applying the derivative formula

$$\frac{d}{dz}\varphi^2\left(z,\frac{z}{\delta L}\right)=\varphi_z^2\left(z,\frac{z}{\delta L}\right)+\frac{1}{\delta L}\varphi_y^2\left(z,\frac{z}{\delta L}\right),$$

we insert the expansions (7.1) and (7.3) into the first equality in (7.2) arriving at an equality

$$\sum_{-1}^{1}\delta^k(\cdots)_k=o(\delta). \tag{7.4}$$

We conclude that $(\cdots)_k=0$ for each $k=-1,0,1,\ldots$. These equalities imply that

$$\varphi_y^0(z,y)=0,\quad F^0(z,y)=\varphi_z^0(z)\tilde{\varepsilon}(y)(1+w_{1y}(y)/L), \tag{7.5}$$

$$F^1(z,y)=\tilde{\varepsilon}(y)(\varphi_{zz}^0(z)w_1(y)+\varphi_y^2(z,y)/L). \tag{7.6}$$

Then we insert the expansions (7.1) and (7.3) into the second equality in (7.2). Similarly, we obtain (paying attention to the powers $\delta^{-1}$ and $\delta^0$) the equalities

$$\varphi^0_z(z)\frac{\partial}{\partial y}\left\{\tilde{\varepsilon}(y)\Big(1+w_{1y}(y)/L\Big)\right\}=0, \tag{7.7}$$

$$\varphi^0_{zz}\tilde{\varepsilon}(y)(1+w_{1y}(y)/L)+\frac{1}{L}\frac{\partial}{\partial y}\left\{\tilde{\varepsilon}(y)\Big(\varphi^0_{zz}w_1(y)+\frac{1}{L}\varphi^2_y(z,y)\Big)\right\}=\varepsilon_f\tilde{\chi}(y)g_1(0). \tag{7.8}$$

We find $\varphi^2(z,y)$ by the method of separation of variables assuming that there is a function $w_2(y)$ such that $\varphi^2(z,y)=\varphi^0_{zz}(z)w_2(y)$. Inserting this representation formula into (7.8), we obtain that $w_2(y)$ should be a periodic solution of the equation

$$\varepsilon_h\varphi^0_{zz}+\frac{1}{L}\varphi^0_{zz}\frac{d}{dy}\left\{\tilde{\varepsilon}(y)\Big(w_1(y)+\frac{1}{L}\frac{d}{dy}w_2(y)\Big)\right\}=-4\pi\chi(y)\sum_{\pm}c_i^-q_i. \tag{7.9}$$

Clearly, this equation has a unique solution satisfying the equality $\int_0^1 w_2dy=0$.

Let us introduce a two-scale corrector

$$\varphi^c(z,y)=\varphi^0(z)+\delta\varphi^0_z(z)w_1(y)+\delta^2\varphi^0_{zz}(z)w_2(y).$$

Its crucial property is that

$$\left(\tilde{\varepsilon}\left(\frac{z}{\delta L}\right)\varphi^{c,\delta}_z\right)_z=\varepsilon_f\tilde{\chi}\left(\frac{z}{\delta L}\right)g_1(0),\quad \varphi^{c,\delta}\equiv\varphi^c\left(z,\frac{z}{\delta L}\right). \tag{7.10}$$

This equality can be verified in a straightforward manner.

The difference $h=\varphi-\varphi^{c,\delta}$ solves the problem

$$\int_\Omega\tilde{\varepsilon}\left(\frac{z}{\delta L}\right)h_z(z)\psi_z(z)dz=\varepsilon_f\int_\Omega\tilde{\chi}\left(\frac{z}{\delta L}\right)\psi(z)\Big(g_1(w^\delta)-g_1(0)\Big)dz, \tag{7.11}$$

$$h(0)=-\delta\varphi^0_z(0)w_1(0)-\delta^2\varphi^2_{zz}(0)w_2(0)\equiv h_0,$$

$$h(L)=-\delta\varphi^0_z(L)w_1(1)-\delta^2\varphi^2_{zz}(L)w_2(1)\equiv h_L,$$

for any $\psi\in H^1_0(\Omega)$. Writing

$$h_0(z)=z(h_L-h_0)/L+h_0,$$

we insert the function $\psi_0=h-h_0\in H^1_0(\Omega)$ into (7.11) to obtain that

$$\int_\Omega\tilde{\varepsilon}\psi^2_{0z}dz=-\int_\Omega\tilde{\varepsilon}\psi_{0z}dz-\varepsilon_f\int_\Omega\tilde{\chi}\psi_0\Big(g_1(w^\delta)-g_1(0)\Big)dz.$$

Applying the Young inequality and inequality (4.25), we conclude that

$$\|\psi_0\|_{H^1_0(\Omega)}\le B\delta\quad\text{and}\quad\|\varphi^\delta-\varphi^{c,\delta}\|_{H^1_0(\Omega)}\le B\delta. \tag{7.12}$$

## 8. A two-scale model

We can summarize the results as follows. There is a function $\varphi^c(z,y)$ of the macro- and micro-variables such that the expansion

$$\varphi^\delta(z) = \varphi^c\left(z, \frac{z}{\delta L}\right) + O(\delta), \tag{8.1}$$

holds in $H^1(\Omega)$. The equations (6.4), (6.7), (7.9) constitute a two-scale model.

Being more simple, the two-scale model allows us to distinguish between micro- and macro-variables. The macro-variable is the mean value of $\varphi^c(z,y)$ over the micro-variable $y$:

$$\varphi^0(z) = \int_Y \varphi^c(z,y)dy. \tag{8.2}$$

These definitions are natural since, as it follows from the definition of the two-scale convergence, the above mean value is the weak limit in $L^2$ of $\varphi^\delta$ as $\delta \to 0$.

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# Superconducting Vortices: Chapman Full Model

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*In Memory of A.V. Kazhikhov*

**Abstract.** In the article the II-type superconducting mean-field model is investigated. We consider the physical boundary conditions for this model. Namely the magnetic field is given on the entire boundary of the domain and on the inflow part of the boundary an extra condition is required for the vorticity. This part of the boundary is unknown before resolving the problem. In fact we investigate the "free boundary problem".

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**Keywords.** Mean-field model, superconducting vortices, flux, solvability.

## 1. Introduction. General model

We study one and two-dimensional reductions of the three-dimensional mean field model for the motion of superconducting vortices (see, for instance the formulaes (72), (73), (76), (78) of [9] or the articles [10], [11]). Considering the case in which all the vortices are rectilinear, aligned and oriented with the $x_3$-direction along with the magnetic field $\mathbf{H}$, we have that $\mathbf{H} = (0, 0, h(\mathbf{x}, t))$ with $\mathbf{w} = (0, 0, \omega(\mathbf{x}, t))$ being the three-dimensional vortex density. The evolution of non-zero components $h = h(\mathbf{x}, t)$ and $\omega = \omega(\mathbf{x}, t)$ for $\mathbf{x} := (x_1, x_2)$ is governed in a bounded domain of $\Omega \subset \mathbb{R}^2$ by a system of differential equations (see, the formulaes (93)-(95) of [9])

$$\begin{cases} \omega_t + \operatorname{div}(\omega \mathbf{v}) = 0 \quad \text{with} \quad \mathbf{v} = -\operatorname{sign}(\omega)\,\nabla h, \\ -\Delta h + h = \omega \quad \text{for} \quad (\mathbf{x}, t) \in \Omega_T := \Omega \times (0, T), \end{cases} \tag{1.1}$$

which has to be closed by the natural condition for the magnetic field on the boundary $\Gamma$ of the domain $\Omega$,

$$h = a(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \Gamma_T := \Gamma \times (0, T) \tag{1.2}$$

and by the initial condition on the vorticity

$$\omega(\mathbf{x},0) = \omega_0(\mathbf{x}), \quad \mathbf{x} \in \Omega. \tag{1.3}$$

Since the first equation in the system is a first-order hyperbolic equation, the necessity for an additional boundary condition on $\omega$ depends on whether the characteristics are directed into or out of $\Omega$ on the boundary $\Gamma$. If $(\mathbf{v}\cdot\mathbf{n}) > 0$, vortices leave the sample, and on this outflow section of boundary no extra boundary conditions are required. Here $\mathbf{n}$ denotes the outward normal to $\Gamma$.

However if $(\mathbf{v}\cdot\mathbf{n}) < 0$, the vortices move into the sample and in [6], [8], the following boundary condition for the flux of vorticity has been suggested:

$$-(\mathbf{v}\cdot\mathbf{n})\ \omega = k\,\frac{F_n(|\nabla h|)}{|\nabla h|}\,\frac{\partial h}{\partial \mathbf{n}}$$

where $k$ depends on the material, a physical positive constant, and

$$F_n := \max(|\nabla h| - J_n, 0) \ \text{ with } \ J_n = J_n(\mathbf{x},t) > 0$$

being a so-called function of the nucleation of the vortices on the boundary. Analyzing this condition we can generalize it to the following one: if $(\mathbf{v}\cdot\mathbf{n}) < 0$ on $\Gamma_T$, then

$$\omega = \operatorname{sign}\left(\frac{\partial h}{\partial \mathbf{n}}\right)\ b\left(\mathbf{x},t,\frac{\partial h}{\partial \mathbf{n}}\right) \tag{1.4}$$

for a given positive function $b = b(\mathbf{x},t,\tau)$, which is limited by some function $f = f(\mathbf{x},t)$ independent of the parameter $\tau$, i.e.,

$$0 \leqslant b(\mathbf{x},t,\tau) \leqslant f(\mathbf{x},t) \qquad \text{for } \forall\tau\in\mathbb{R} \ \text{ and } \ (\mathbf{x},t)\in\Gamma_T.$$

Let us note that we do not know beforehand, where the vortices go into $\Omega$, since the restriction $(\mathbf{v}\cdot\mathbf{n}) < 0$ depends on $\mathbf{v}$. In fact we have a "free moving" boundary condition on $\Gamma_T$ for $\omega$, that is one of the main difficulties in this considered problem.

For the interested reader we note that hyperbolic-elliptic type systems, such as (1.1), arise in many mathematical models of continuum mechanics and have been considered in [2, 3, 4, 5, 17].

### 1.1. Model for positive vorticity. The existence result

In the following sections we study the superconducting vortex model just for positive values of vorticity. Therefore our model is rewritten as the system

$$\begin{cases} \omega_t + \operatorname{div}(\omega\mathbf{v}) = 0 \quad \text{with} \quad \mathbf{v} = -\nabla h, \\ \quad -\Delta h + h = \omega \quad \text{for} \quad (\mathbf{x},t)\in\Omega_T \end{cases} \tag{1.5}$$

with the boundary conditions

$$h = a(\mathbf{x},t), \quad (\mathbf{x},t)\in\Gamma_T, \tag{1.6}$$

$$\omega = b\left(\mathbf{x},t,\frac{\partial h}{\partial \mathbf{n}}\right), \quad (\mathbf{x},t)\in\Gamma_T^-(\mathbf{v}) \tag{1.7}$$

and the initial condition

$$\omega(\mathbf{x},0) = \omega_0(\mathbf{x}), \quad \mathbf{x}\in\Omega, \tag{1.8}$$

where we write

$$\Gamma_T^-(\mathbf{v}) := \{(\mathbf{x},t) \in \Gamma_T : \quad (\mathbf{v}\cdot\mathbf{n})\,(\mathbf{x},t) < 0\}\,.$$

Let us assume the following regularity on the data for our problem (1.5)–(1.8). *The boundary* $\Gamma \in C^2$ *and the function* $b = b(\mathbf{x},t,\tau)$ *is measurable for a.e.* $(\mathbf{x},t) \in \Gamma_T$ *and continuous on* $\tau \in \mathbb{R}$. *Moreover there exists a function* $f = f(\mathbf{x},t)$, *such that for some* $p \in (3,\infty]$,

$$\begin{cases} f(\mathbf{x},t) \in L_\infty(0,T;L_p(\Omega)), \\ 0 \leqslant b(\mathbf{x},t,\tau) \leqslant f(\mathbf{x},t) \quad \text{ on } \ \Gamma_T \quad \text{ and } \quad \forall \tau \in \mathbb{R}\,; \\ a \in L_\infty(0,T;W_p^2(\Gamma)) \cap W_\infty^1(0,T;L_p(\Gamma)); \\ \omega_0 \geqslant 0 \quad \text{a.e. in } \ \Omega \quad \text{and} \quad \omega_0 \in L_p(\Omega). \end{cases} \tag{1.9}$$

In this article we show the following existence result.

**Theorem 1.** *If the data* $a$, $b$, $\omega_0$ *satisfy* (1.9), *then there exists at least one weak solution* $\{\omega, h\}$ *of the problem* (1.5)–(1.8), *such that* $\omega \in L_\infty(0,T;L_p(\Omega))$, $h \in L_\infty(0,T;\ W_q^2(\Omega))$ (*for* $q := p$ *if* $p < \infty$ *and for arbitrary* $q < \infty$ *if* $p = \infty$) *and the equalities*

$$-\Delta h + h = \omega \quad \text{a.e. in } \Omega_T, \tag{1.10}$$

$$\mathbf{v} = -\nabla h \quad \text{a.e. in } \Omega_T, \tag{1.11}$$

$$h = a \quad \text{a.e. on } \Gamma_T, \tag{1.12}$$

$$\int_{\Omega_T} \{\omega\psi_t + \omega\,(\mathbf{v}\cdot\nabla)\psi\}\, d\mathbf{x}dt + \int_\Omega \omega_0\ \psi(\mathbf{x},0)\ d\mathbf{x} = \int_{\Gamma_T} (\mathbf{v}\cdot\mathbf{n})\ b(\mathbf{x},t,\frac{\partial h}{\partial \mathbf{n}})\ \psi\ d\mathbf{x}dt, \tag{1.13}$$

*hold for any function* $\psi \in TF(\mathbf{v})$. *Here we define the set of* <u>*test*</u> *functions* $TF(\mathbf{v})$, *related with the function* $\mathbf{v} = \mathbf{v}(\mathbf{x},t) \in C(\overline{\Omega}\times[0,T])$ *as*

$$TF(\mathbf{v}) := \{\varphi \in C^{1,1}(\overline{\Omega}\times[0,T]) : \varphi(\cdot,T) = 0 \ \text{ on } \Omega \ \text{ and } \varphi(\mathbf{x},t) = 0 \ \text{ on } \Gamma_T^*(\mathbf{v})\},$$

*where*

$$\Gamma_T^*(\mathbf{v}) := \{(\mathbf{x},t) \in \Gamma_T : (\mathbf{v}\cdot\mathbf{n})(\mathbf{x},t) \geqslant 0\}.$$

**Remark 1.** *To satisfy the boundary condition* (1.7), *because we do not know beforehand* $\Gamma_T^-(\mathbf{v})$, *the entrance of the flux of vortices into* $\Omega$, *we have been compelled to introduce a set of test functions* $TF(\mathbf{v})$ <u>*depending on the*</u> <u>*solution*</u> $\mathbf{v}$.

Let us remark that problems similar to (1.5)–(1.8) were considered in [1] and [12], but for other boundary conditions.

## 2. Approximation problem. Leray-Schauder fixed point argument

Let us formulate a well-known result from approximation theory [7].

**Lemma 1.** *Let $Q \subseteq \mathbb{R}^n$, $\quad n \geqslant 1$, be an open set and $s \in [1, +\infty]$. Then for any $g \in L_s(Q)$ there exist functions $g^\theta \in C^\infty(Q)$, satisfying the following properties:*

$$||g^\theta||_{L_s(Q)} \leqslant C||g||_{L_s(Q)}\,,$$
$$||g^\theta - g||_{L_r(Q)} \longrightarrow 0 \ \ \textit{when} \ \ \theta \to 0, \tag{2.1}$$

*for $r := s$, if $s < \infty$ and for any $r < \infty$, if $s = \infty$. The constant $C$ is independent of the parameter $\theta$.*

Using Lemma 1, we approximate the data $a, \omega_0, b, f$ by $C^\infty$-functions $a^\theta, \omega_0^\theta$, $b^\theta$, $f^\theta$, satisfying the relations (2.1) in the corresponding space $L_q(Q)$, defined by the regularity conditions (1.9). We can assume that any derivatives of these smooth approximations are bounded by constants, depending only on a positive parameter $\theta$. Moreover the functions $b^\theta$, $\omega_0^\theta$ satisfy a so-called *compatibility* condition

$$b^\theta(\cdot, t, \cdot) = 0 \quad \text{for } t \in [0, \theta] \quad \text{and} \quad \omega_0^\theta(\mathbf{x}) = 0 \quad \text{for } \mathbf{x} \in U_\theta(\Gamma).$$

Here we define the neighborhood of $\Gamma$ as $U_\theta(\Gamma) := \{\mathbf{x} \in \Omega : \ d(\mathbf{x}) < \theta\}$ and the distance function $d = d(\mathbf{x})$ on $\Gamma$ for any $\mathbf{x} \in \overline{\Omega}$ as $d(\mathbf{x}) := \inf_{\mathbf{y} \in \mathbb{R}^2 \setminus \Omega} |\mathbf{x} - \mathbf{y}|$.

In Sections 2–4 we shall work with these approximations $a^\theta, \omega_0^\theta, b^\theta, f^\theta$, instead of our data $a$, $\omega_0$, $b$, $f$, but for simplicity of notation, we suppress the index $\theta$ and continue to write $a$, $\omega_0$, $b$, $f$, respectively.

Now we fix a parameter $\varepsilon \in (0, 1)$ and study the solvability of the following problem.

**Problem $\mathbf{P}_\varepsilon$.** *Find $\omega \in W_2^{2,1}(\Omega_T)$ and $h \in L_\infty(0, T; W_2^2(\Omega))$ satisfying the coupled two systems*

$$\begin{cases} \omega_t + \operatorname{div}(\omega \mathbf{v}) = \ \varepsilon \Delta \omega \ \text{ for } \ (\mathbf{x}, t) \in \Omega_T \ \text{ with } \ \mathbf{v} := -\nabla h; \\ \omega(\mathbf{x}, t) = b(\mathbf{x}, t, \frac{\partial h}{\partial \mathbf{n}}), \ (\mathbf{x}, t) \in \Gamma_T; \quad \omega(\mathbf{x}, 0) = \omega_0(\mathbf{x}), \ \mathbf{x} \in \Omega \end{cases} \tag{2.2}$$

*and*

$$\begin{cases} -\Delta h + h = \omega, & (\mathbf{x}, t) \in \Omega_T; \\ h = a(\mathbf{x}, t), & (\mathbf{x}, t) \in \Gamma_T. \end{cases} \tag{2.3}$$

The solution $\{\omega, h\}$ of **Problem $\mathbf{P}_\varepsilon$** depends on the parameters $\varepsilon$, $\theta$, but for the simplicity of presentation in the sequel we indicate the dependence of functions and constants on the parameters $\varepsilon$, $\theta$, if it will be necessary. We show the solvability of **Problem $\mathbf{P}_\varepsilon$**, using the Leray-Schauder fixed point argument (see [15], p. 286, Theorem 11.6). To do so, we consider the following problem, depending also on an auxiliary parameter $\lambda \in [0, 1]$.

**Problem $\mathbf{P}_{\varepsilon,\lambda}$.** *Find* $\omega \in W_2^{2,1}(\Omega_T)$ *and* $h \in L_\infty(0,T;W_2^2(\Omega))$ *satisfying the coupled two systems*

$$\begin{cases} \omega_t + \lambda \operatorname{div}(\omega \mathbf{v}) = \varepsilon \Delta \omega \ \text{ for } \ (\mathbf{x},t) \in \Omega_T \ \text{ with } \ \mathbf{v} := -\nabla h; \\ \omega = \lambda b(\mathbf{x}, t, \frac{\partial h}{\partial \mathbf{n}}), \ (\mathbf{x},t) \in \Gamma_T; \quad \omega(\mathbf{x},0) = \lambda \omega_0(\mathbf{x}), \ \mathbf{x} \in \Omega \end{cases} \tag{2.4}$$

*and*

$$\begin{cases} -\Delta h = \lambda\left(\omega - h\right), & (\mathbf{x},t) \in \Omega_T; \\ h = \lambda a(\mathbf{x},t), & (\mathbf{x},t) \in \Gamma_T \end{cases} \tag{2.5}$$

*for any fixed* $\lambda \in [0,1]$.

To apply the Leray-Schauder fixed point argument, first we assume the existence of a solution of (2.4)–(2.5) and deduce a priori estimates which do not depend on $\varepsilon$ and $\lambda$. Below until the end of Section 3, all constants $C$ will be independent of $\varepsilon$ and $\lambda$, but *may depend on the parameter* $\theta$.

**Lemma 2.** *Let us write*

$$m := \max\left(||a||_{L_\infty(\Gamma_T)}, ||f||_{L_\infty(\Gamma_T)}, ||\omega_0||_{L_\infty(\Omega)}\right) < \infty.$$

*For any* $\lambda \in [0,1]$ *the solution* $\{\omega, h\}$ *of* Problem $\mathbf{P}_{\varepsilon,\lambda}$ *fulfills the estimates*

$$0 \leqslant \omega(\mathbf{x},t) \leqslant 2m \qquad \text{for} \quad (\mathbf{x},t) \in \Omega_T; \tag{2.6}$$

$$\|h\|_{L_\infty(0,T;W_q^2(\Omega))}, \quad \|h\|_{L_\infty(0,T;C^{1+\alpha}(\overline{\Omega}))} \leqslant C\left(m + \|a\|_{L_\infty(0,T;W_q^2(\Gamma))}\right) \tag{2.7}$$

*for any* $q \in (2,\infty)$ *and any* $\alpha \in [0,1)$, *where* $C$ *depends only on* $q$ *and* $\Omega$.

*Proof.* The positivity of $\omega$ follows from a maximum principle and the positivity of $f$, $\omega_0$, in view of (1.9). The details of the proof can be found in Lemma 3 of [1]. Now let us introduce the functions $h_m := \max(h - m, 0)$, $\omega_m := \max(\omega - m, 0)$ and rewrite the equation (2.5) in the form

$$-\Delta h + \lambda(h - m) = \lambda(\omega - m).$$

Multiplying the last one by $h_m^{k-1}$ for any integer $k \geqslant 2$, we obtain

$$(k-1)\int_\Omega h_m^{k-2}\left|\nabla h_m\right|^2 d\mathbf{x} + \lambda\left\|h_m\right\|_{L_k(\Omega)}^k \leqslant \lambda\left\|h_m\right\|_{L_k(\Omega)}^{k-1}\left\|\omega - m\right\|_{L_k(\Omega)},$$

which implies for any $t \in [0,T]$,

$$\|h_m(\cdot,t)\|_{k,\Omega} \leqslant \|\omega - m\|_{k,\Omega} \leqslant \left(\|\omega_m(\cdot,t)\|_{L_k(\Omega)}^k + m^k\left|\Omega\right|\right)^{\frac{1}{k}}. \tag{2.8}$$

Next, multiplying the equation of (2.4) by $\omega_m^{k-1}$, we have that

$$\frac{1}{k}\frac{d}{dt}\int_\Omega \omega_m^k d\mathbf{x} + \varepsilon(k-1)\int_\Omega \omega_m^{k-2}\left|\nabla\omega_m\right|^2 d\mathbf{x}$$
$$+\lambda\int_\Omega(\omega - m)z_m d\mathbf{x} = \lambda\int_\Omega(h - m)z_m d\mathbf{x}, \tag{2.9}$$

where $z_m := (1-\frac{1}{k})\omega_m^k + m\omega_m^{k-1}$. By the Hölder inequality and the positivity of $h_m$, we derive that

$$\int_\Omega (h-m) z_m d\mathbf{x} \leqslant \left(1-\frac{1}{k}\right) \|h_m\|_{L_{k+1}(\Omega)} \|\omega_m\|^k_{L_{k+1}(\Omega)} + m\,\|h_m\|_{L_k(\Omega)} \|\omega_m\|^{k-1}_{L_k(\Omega)}\,.$$

Substituting this result in (2.9) and taking into account (2.8), we deduce

$$\frac{1}{k}\frac{d}{dt}\int_\Omega \omega_m^k d\mathbf{x} \leqslant 2m^{k+1}\,|\Omega|\,.$$

Integrating over the time variable $t$ and passing to the limit as $k\to\infty$, we obtain (2.6).

The estimates (2.7) follow the classical estimates for elliptic equations (see the book [19])

$$||h||_{W^2_q(\Omega)} \leqslant C(||\omega||_{L_q(\Omega)} + \|a\|_{W^2_q(\Gamma)}) \qquad \text{for any} \quad q\in(2,\infty) \tag{2.10}$$

applied to the problem (2.5) with a given $\omega$, and from the embedding theorem $W^1_q(\Omega) \hookrightarrow C^\alpha(\bar{\Omega})$ for $\alpha := 1-\frac{2}{q}$. The constant $C$ depends only on $q$ and $\Omega$. Above we have studied the case when $\lambda>0$. The case $\lambda=0$ is trivial. □

**Lemma 3.** *For any fixed $\varepsilon>0$ there exists at least one solution $\{\omega,h\}$ of* Problem $\mathbf{P}_\varepsilon$, *such that*

$$\omega\in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T]), \qquad h\in C^{2+\alpha,\alpha/2}(\overline{\Omega}\times[0,T]) \tag{2.11}$$

*for some $\alpha\in(0,1)$.*

*Proof.* Now we construct a compact operator, which fulfills the Leray-Schauder fixed point theorem. First we choose some function $\widetilde{\omega}\in C(\overline{\Omega}\times[0,T])$, satisfying (2.6). For any fixed $\lambda\in[0,1]$, the elliptic problem

$$\begin{cases} -\Delta\widetilde{h} = \lambda\left(\widetilde{\omega}-\widetilde{h}\right), & (\mathbf{x},t)\in\Omega_T; \\ \widetilde{h} = \lambda a(\mathbf{x},t), & (\mathbf{x},t)\in\Gamma_T \end{cases} \tag{2.12}$$

has a unique solution $\widetilde{h}\in L_\infty(0,T;C^{1+\alpha}(\overline{\Omega}))$, satisfying (2.7) by (2.10). We consider the linear parabolic problem

$$\begin{cases} \omega_t = \varepsilon\triangle\omega - \lambda\,\mathrm{div}\,(\mathbf{g})\,, \quad (\mathbf{x},t)\in\Omega_T \ \text{ with } \mathbf{g}:=\widetilde{\omega}\widetilde{\mathbf{v}},\ \widetilde{\mathbf{v}}:=-\nabla\widetilde{h}; \\ \omega = \lambda b(\mathbf{x},t,\frac{\partial\widetilde{h}}{\partial\mathbf{n}}),\ (\mathbf{x},t)\in\Gamma_T; \quad \omega(\mathbf{x},0)=\lambda\omega_0(\mathbf{x}),\ \mathbf{x}\in\Omega. \end{cases} \tag{2.13}$$

According to (2.6), (2.7)

$$\|\mathbf{g}\|_{L^\infty(\Omega_T)} \leqslant C \tag{2.14}$$

and Theorem 10.1 of [18], the system (2.13) has a unique solution $\omega\in C^{\alpha,\alpha/2}(\overline{\Omega}\times[0,T])$ for *some* $\alpha\in(0,1)$, such that

$$\|\omega\|_{C^{\alpha,\alpha/2}(\overline{\Omega}\times[0,T])} \leqslant \lambda\,\mathbf{C}(\varepsilon)\,(\|b\|_{C^{1,1}(\overline{\Omega}\times[0,T])} + \|\omega_0\|_{C^1(\overline{\Omega})} + \|\mathbf{g}\|_{L^\infty(\Omega_T)}) \tag{2.15}$$

with the constant $\mathbf{C}(\varepsilon)$ being dependent on $\varepsilon$. Really the smoothness of solution for (2.13) is defined only by the smoothness of the boundary and initial data. Hence we have constructed the operator

$$\omega := B\left[\widetilde{\omega}, \lambda\right], \tag{2.16}$$

having the fixed point $\omega := B\left[\omega, \lambda\right]$, which is a weak solution of **Problem** $\mathbf{P}_{\varepsilon,\lambda}$. Let us note that the estimate (2.6) is valid for all solutions of **Problem** $\mathbf{P}_{\varepsilon,\lambda}$ and any $\lambda \in [0,1]$. By (2.15) $B$ is a *compact continuous* operator on the Banach space $C(\overline{\Omega} \times [0,T]) \times [0,1]$ into $C(\overline{\Omega} \times [0,T])$. If $\lambda = 0$ the system (2.13) has only the zero solution. Therefore $B$ fulfills all conditions of the Leray-Schauder fixed point theorem and **Problem** $\mathbf{P}_{\varepsilon,1} \equiv \mathbf{P}_{\varepsilon}$ is solvable in the Banach space $C(\overline{\Omega} \times [0,T])$. Moreover, taking into account (2.15) and applying classical results of [19] for elliptic problem (2.3), we obtain that $h \in C^{2+\alpha,\alpha/2}(\overline{\Omega} \times [0,T])$. As a consequence of it and Theorem 12.1 of [18], we derive that $\omega \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0,T])$. □

**Lemma 4.** *The solution* $\{\omega, h\}$ *of* Problem $\mathbf{P}_{\varepsilon}$ *fulfills the estimates*

$$\sqrt{\varepsilon}||\nabla\omega||_{L_2(\Omega_T)} \leqslant C, \tag{2.17}$$

$$||\partial_t\left(\nabla h\right)||_{L_2(\Omega_T)} \leqslant C \tag{2.18}$$

*with the constants* $C$ *being independent of* $\varepsilon$.

*Proof.* By *extension* results [21], there exists an extension $\widehat{\omega} = \widehat{\omega}(\mathbf{x},t) \in C^{\infty}\left(\Omega_T\right)$ of the boundary condition $b(\mathbf{x},t,\frac{\partial h}{\partial \mathbf{n}})$ and the initial condition $\omega_0(\mathbf{x})$ into the domain $\Omega_T$, such that

$$\begin{cases} \widehat{\omega}\big|_{t=0} = \omega_0 \ \text{ and } \ \widehat{\omega} = b(\mathbf{x},t,\frac{\partial h}{\partial \mathbf{n}}(\mathbf{x},t)) \ \text{ for } \ (\mathbf{x},t) \in \Gamma_T, \\ ||\widehat{\omega}||_{C^{2,1}\left(\overline{\Omega}\times[0,T]\right)} \leqslant C. \end{cases} \tag{2.19}$$

The function $z := \omega - \widehat{\omega}$ satisfies the system

$$\begin{cases} \partial_t z + \operatorname{div}(z\,\mathbf{v}) = \varepsilon\Delta z + F \qquad \text{in} \quad \Omega_T, \\ z\big|_{\Gamma_T} = 0 \quad \text{and} \quad z\big|_{t=0} = 0, \end{cases} \tag{2.20}$$

where $F := \varepsilon\Delta\widehat{\omega} - \partial_t\widehat{\omega} - \operatorname{div}(\widehat{\omega}\mathbf{v})$, such that

$$||F||_{L_\infty(\Omega_T)} \leqslant C. \tag{2.21}$$

Multiplying the equation of (2.20) by $z$ and integrating over $\Omega_T$ with the help of (2.6), (2.7), we derive (2.17).

In view of (2.3) the function $h_t$ satisfies for $t \in (0,T)$ the elliptic problem

$$\begin{cases} -\Delta h_t + h_t = \operatorname{div}\mathbf{G} \ \text{ on } \ \Omega, \\ h_t\big|_{\Gamma} = a_t, \end{cases}$$

with $\mathbf{G} := \varepsilon\nabla\omega - \mathbf{v}\omega$, such that $||\mathbf{G}||_{L_2(\Omega_T)} \leqslant C$. Then applying the classical results of [19], we have

$$\|\partial_t\left(\nabla h\right)\|_{L_2(\Omega)} \leqslant C\left(\|\mathbf{G}\|_{L_2(\Omega)} + \|a_t\|_{L_2(\Gamma)}\right) \ \text{ for } \ t \in (0,T),$$

which gives the estimate (2.18). □

By virtue of (2.7) and (2.18), we have that

$$\nabla h \in \Re\,[\Omega] := L_2\big(0,T; W_2^1(\Omega)\big) \cap W_2^1\big(0,T; L_2(\Omega)\big),$$

such that

$$||\nabla h||_{\Re[\Omega]} \leqslant C.$$

**Lemma 5.** *For the trace value of $\frac{\partial h}{\partial \mathbf{n}}$ on $\Gamma$, we have that*

$$\frac{\partial h}{\partial \mathbf{n}} \in \mathcal{P}\,[\Gamma] := L_2\big(0,T; W_2^{\frac{1}{2}}(\Gamma)\big) \cap W_2^{\frac{1}{2}}\big(0,T; L_2(\Gamma)\big)$$

*and*

$$||\frac{\partial h}{\partial \mathbf{n}}||_{\mathcal{P}[\Gamma]} \leqslant C \tag{2.22}$$

*with the constant $C$ being independent of $\varepsilon$.*

*Proof.* Let us choose a neighborhood $G$ of $\Gamma$ and an orthogonal coordinate system $(y_1, y_2)$, such that $G = \{\mathbf{y} = (y_1, y_2) : (y_1, 0) \in \Gamma,\ y_2 \in [0, \eta]\}$ for some $\eta > 0$ and this new coordinate system coincides with the tangential-normal coordinate system along $\Gamma$. The space $\Re\,[G]$ can be rewritten as

$$\Re\,[G] = \Big\{ \phi(y_2, t, y_1) \in L_2(0,\eta;\ L_2(0,T; W_2^1(\Gamma))) \cap W_2^1(0,T; L_2(\Gamma)) : \\ \partial_{y_2} \phi \in L_2\big(0,\eta; L_2(0,T; L_2(\Gamma))\big) \Big\}.$$

By virtue Theorem 3.2 of [20], the mapping $A : \phi \to \phi|_{y_2=0}$ is well-defined on $\Re\,[G]$. Furthermore, the operator $\phi \to A(\phi)$ from $\Re\,[G]$ to

$$\Big[L_2\big(0,T; W_2^1(\Gamma)\big) \cap W_2^1\big(0,T; L_2(\Gamma)\big), L_2\big(0,T; L_2(\Gamma)\big)\Big]_{\frac{1}{2}} \equiv \mathcal{P}\,[\Gamma]$$

is continuous and surjective. Here $[X, Y]_\delta$, $\delta \in [0,1]$, are the intermediate spaces of Banach spaces $X$ and $Y$ as defined in [20]. Therefore the value of $\frac{\partial h}{\partial \mathbf{n}}|_\Gamma$ belongs to the space $\mathcal{P}\,[\Gamma]$ and satisfies the estimate (2.22). □

## 3. Limit transition on $\varepsilon$

In this section the parameter $\theta$ continues to be fixed and the limit on $\varepsilon \to 0$ of the solutions $\{\omega_\varepsilon, h_\varepsilon\}$ for **Problem** $\mathbf{P}_\varepsilon$ will be considered. In view of the estimates (2.6)–(2.7), (2.17)–(2.18) and (2.22), there exists a subsequence of $\{\omega_\varepsilon, h_\varepsilon\}$, such that

$$\begin{aligned} h_\varepsilon &\rightharpoonup h \quad \text{weakly} - * \text{ in } L_\infty(0,T; W_q^2(\Omega)) \quad \text{for } \forall q < \infty, \\ \varepsilon \nabla \omega_\varepsilon &\to 0 \quad \text{strongly in } L_2(\Omega_T), \\ \omega_\varepsilon &\rightharpoonup \omega \quad \text{weakly} - * \text{ in } L_\infty(\Omega_T), \\ \mathbf{v}_\varepsilon &\to \mathbf{v} \qquad \text{strongly in } L_\infty(0,T; L_2(\Omega)), \end{aligned} \tag{3.1}$$

and

$$\frac{\partial h_\varepsilon}{\partial \mathbf{n}} \longrightarrow \frac{\partial h}{\partial \mathbf{n}} \quad \text{strongly in } L_2(\Gamma_T). \tag{3.2}$$

Obviously the triple $\{\omega, h, \mathbf{v}\}$ satisfies the estimates (2.6)–(2.7) and the relations (1.10)–(1.12).

Now we prove that the pair $\{\omega, \mathbf{v}\}$ also satisfies (1.13). First we choose an arbitrary test function $0 \leqslant \psi \in TF(\mathbf{v})$, related with <u>the found $\mathbf{v}$</u> and introduce the approximation of the unit function on $\Omega$ as

$$\mathbf{1}_\sigma(\mathbf{x}) := \begin{cases} 0, & \text{if } 0 \leqslant d(\mathbf{x}) < \sigma, \\ \frac{d-\sigma}{\sigma}, & \text{if } \sigma \leqslant d(\mathbf{x}) < 2\sigma, \\ 1, & \text{if } \mathbf{x} \in \Omega \text{ and } d(\mathbf{x}) \geqslant 2\sigma. \end{cases} \tag{3.3}$$

Multiplying the equation of (2.2) by $\eta_\sigma := \mathbf{1}_\sigma\, \psi$ and integrating it over the domain $\Omega_T$, we obtain

$$\begin{aligned} 0 = \left\{ \int_{\Omega_T} [\omega_\varepsilon(\psi_t + (\mathbf{v}_\varepsilon \cdot \nabla)\psi)]\mathbf{1}_\sigma - \varepsilon\, (\nabla\omega_\varepsilon \cdot \nabla\eta_\sigma)\, d\mathbf{x}dt + \int_\Omega \omega_0(\mathbf{x})\, \eta_\sigma(\mathbf{x}, 0)\, d\mathbf{x} \right\} \\ + \frac{1}{\sigma} \int_0^T \int_{[\sigma<d<2\sigma]} \omega_\varepsilon\, (\mathbf{v}_\varepsilon \cdot \nabla)d\, \psi\, d\mathbf{x}dt = K^{\varepsilon,\sigma} + L^{\varepsilon,\sigma}. \end{aligned}$$

Using (3.1), we have

$$\lim_{\sigma\to 0} \left[ \lim_{\varepsilon\to 0} K^{\varepsilon,\sigma} \right] = \int_{\Omega_T} \omega(\psi_t + (\mathbf{v} \cdot \nabla)\psi)d\mathbf{x}dt + \int_\Omega \omega_0\psi(\mathbf{x}, 0)\, d\mathbf{x}.$$

To take the limit transition $\varepsilon \to 0$ in the term $L^{\varepsilon,\sigma}$ we need an additional result. We have

$$\begin{aligned} L^{\varepsilon,\sigma} &= \frac{1}{\sigma} \int_0^T \int_{[\sigma<d<2\sigma]} (\mathbf{v}_\varepsilon \cdot \nabla)d\, z_\varepsilon\, \psi\, d\mathbf{x}dt \\ &+ \frac{1}{\sigma} \int_0^T \int_{[\sigma<d<2\sigma]} (\mathbf{v}_\varepsilon \cdot \nabla)d\, \widehat{\omega}_\varepsilon\, \psi\, d\mathbf{x}dt = L_1^{\varepsilon,\sigma} + L_2^{\varepsilon,\sigma}. \end{aligned}$$

The functions $\widehat{\omega}_\varepsilon$ are defined by (2.19). Multiplying equation (2.20) by $\operatorname{sign}_\delta(z_\varepsilon) := \frac{z_\varepsilon}{|z_\varepsilon|_\delta}$ with $|z_\varepsilon|_\delta := \sqrt{z_\varepsilon^2 + \delta^2}$, we obtain

$$\partial_t(\, |z_\varepsilon|_\delta) + \operatorname{div}(|z_\varepsilon|_\delta\, \mathbf{v}_\varepsilon) = \left[\varepsilon\Delta z_\varepsilon + F_\varepsilon\right]\ \operatorname{sign}_\delta(z).$$

Hence multiplying this equality by an arbitrary test function $0 \leqslant \varphi \in TF(\mathbf{v})$, integrating over $\Omega_T$ and taking the limit on $\delta \to 0$, we derive the inequality

$$-\int_{\Omega_T} (\mathbf{v}_\varepsilon \cdot \nabla\varphi)\, |z_\varepsilon|\, d\mathbf{x}dt \leqslant C \int_{\Omega_T} (|F_\varepsilon|\varphi + |\varphi_t| + \varepsilon|\Delta\varphi|)\, d\mathbf{x}dt. \tag{3.4}$$

For $\varphi := (1 - \mathbf{1}_\sigma)\, \psi$ with the above chosen $\psi$ this inequality implies

$$\lim_{\sigma\to 0} \left[ \overline{\lim_{\varepsilon\to 0}}\, \frac{1}{\sigma} \int_0^T \int_{[\sigma<d<2\sigma]} (\mathbf{v}_\varepsilon \cdot \nabla)\, d\, |z_\varepsilon|\ \psi\, d\mathbf{x}dt \right] = 0, \tag{3.5}$$

that is $\lim_{\sigma\to 0}\ \left[ \overline{\lim}_{\varepsilon\to 0} L_1^{\varepsilon,\sigma} \right] = 0$.

By (2.7) the set of functions $(\mathbf{v}_\varepsilon \cdot \nabla)\, d$ is uniformly continuous on $\overline{\Omega} \times [0,T]$, independently of $\varepsilon$ and $t \in [0,T]$, such that $(\mathbf{v}_\varepsilon \cdot \nabla)\, d = -(\mathbf{v}_\varepsilon \cdot \mathbf{n})$ on $\Gamma_T$ and the smooth functions $\widehat{\omega}_\varepsilon$ are equal to $b(\cdot,\cdot,\frac{\partial h_\varepsilon}{\partial \mathbf{n}})$ on $\Gamma_T$, hence from (3.2) we have

$$\lim_{\sigma\to 0}\left[\lim_{\varepsilon\to 0}\; L_2^{\varepsilon,\sigma}\right] = -\int_0^T \int_{\Gamma_T} (\mathbf{v}\cdot\mathbf{n})\, b\left(\mathbf{x}, t, \frac{\partial h}{\partial \mathbf{n}}\right)\; \psi\, d\mathbf{x}dt.$$

Therefore $\{\omega, \mathbf{v}\}$ satisfies the equality (1.13).

## 4. Limit transition on $\theta \to 0$

In this section, we shall complete the proof of our main result through the passage to the limit, when $\theta \to 0$ of the triple $\{\omega_\theta, h_\theta, \mathbf{v}_\theta\}$ constructed in Section 3. First in Subsection 4.1 we see the case when our data satisfy (1.9) with $p = \infty$ and later on in Subsection 4.2 we investigate the case $p \in (3, \infty)$.

### 4.1. The case $p = \infty$. The limit transition on $\theta \to 0$

Let us consider that $f$, $b$, $a$, $\omega_0$ satisfy (1.9) with $p = \infty$. Hence the approximated data $f^\theta$, $b^\theta$, $a^\theta$, ${\omega_0}^\theta$ fulfill the conditions (2.1) for $s = \infty$, i.e., the pair $\{\omega_\theta, h_\theta\}$, constructed in Section 3, satisfies the estimates (2.6)–(2.7) with the constants $m, C$ being independent of $\theta$. The function $\partial_t h_\theta$ satisfies the elliptic problem for a.e. $t \in (0,T)$,

$$-\triangle\, \partial_t h_\theta + \partial_t h_\theta = \operatorname{div} \mathbf{G}_\theta \quad \text{on} \quad \Omega,$$
$$\partial_t h_\theta\big|_\Gamma = \partial_t a_\theta$$

with $\mathbf{G}_\theta := -\mathbf{v}_\theta \omega_\theta$, such that $||\mathbf{G}_\theta||_{L_\infty(0,T;L_q(\Omega))} \leqslant C$ and $C$ is independent of $\theta$. Therefore applying the classical results of [19] we have

$$\|\partial_t\, (\nabla h_\theta)\|_{L_q(\Omega)} \leqslant C\left(\|\mathbf{G}_\theta\|_{L_q(\Omega)} + \|\partial_t a_\theta\|_{L_q(\Gamma)}\right)$$

for a.e. $t \in (0,T)$ and $\forall q < \infty$, which implies

$$\|\partial_t(\nabla h_\theta)\|_{L_\infty(0,T;L_q(\Omega))} \leqslant C \tag{4.1}$$

with $C = C(q)$, independent of $\theta$. As a consequence of this estimate (4.1) and (2.6)–(2.7), there exists a subsequence of $\{\omega_\theta, h_\theta, \mathbf{v}_\theta\}$, such that

$$\begin{aligned} &h_\theta \rightharpoonup h \text{ weakly} - * \text{ in } L_\infty(0,T; W_q^2(\Omega)),\\ &\omega_\theta \rightharpoonup \omega \text{ weakly} - * \text{ in } L_\infty(\Omega_T),\\ &\mathbf{v}_\theta \to \mathbf{v} \text{ strongly in } L_\infty(0,T; L_q(\Omega)) \end{aligned} \tag{4.2}$$

for $\forall q < \infty$. Using the same argument of Lemma 5, we have also

$$(\mathbf{v}_\theta \cdot \mathbf{n}) \equiv \frac{\partial h_\theta}{\partial \mathbf{n}} \to (\mathbf{v}\cdot\mathbf{n}) \equiv \frac{\partial h}{\partial \mathbf{n}} \text{ strongly in } L^2(\Gamma_T).$$

Obviously the found triple $\{\omega, h, \mathbf{v}\}$ fulfills the relations (1.10)–(1.12).

Let us show that $\{\omega, \mathbf{v}\}$ satisfies (1.13) too. To do it we have to prove $\{\omega, \mathbf{v}\}$ satisfies the equality (1.13) for any test function $\psi \in TF(\mathbf{v})$, related with this found $\mathbf{v}$. Let us fix a small number $\delta > 0$ and introduce a subset of $TF(\mathbf{v})$ as

$$TF(\mathbf{v}, \delta) := \{\varphi \in C^{1,1}(\overline{\Omega} \times [0,T]) : \varphi(\cdot, T) = 0 \text{ on } \Omega \\ \text{and } \ \varphi(x,t) = 0 \text{ on } \Gamma_T(\mathbf{v}, \delta)\},$$

where

$$\Gamma_T(\mathbf{v}, \delta) := \{(x,t) \in \Gamma_T : (\mathbf{v} \cdot n)(x,t) \geqslant -\delta\}.$$

In view of (2.7), (4.1) and the embedding theorem $W^1_q(\Omega_T) \hookrightarrow C^{\alpha,\alpha}(\overline{\Omega} \times [0,T])$ with $\alpha := 1 - \frac{3}{q}$ for any $q > 3$, there exists $0 < \theta_0 = \theta_0(\delta)$, such that $\forall \theta : 0 < \theta < \theta_0$, and we have

$$TF\left(\mathbf{v}_\theta, \frac{\delta}{2}\right) \supset TF(\mathbf{v}, \delta). \tag{4.3}$$

Let us choose an arbitrary $\psi \in TF(\mathbf{v}, \delta)$. By (4.3) the pair $\{\omega_\theta, \mathbf{v}_\theta\}$ fulfills (1.13) with the selected $\psi$ for any $\theta < \theta_0$. Therefore with the help of (4.2), taking the limit transition on $\theta \to 0$ in (1.13) written for $\{\omega_\theta, \mathbf{v}_\theta\}$ and this chosen $\psi$, we derive that the pair $\{\omega, \mathbf{v}\}$ fulfills (1.13) for $\forall\ \psi \in TF(\mathbf{v}, \delta)$. Since any function $\psi \in TF(\mathbf{v})$ can be approximated by a sequence $\psi_\delta \in TF(\mathbf{v}, \delta)$, such that

$$||\psi_\delta - \psi||_{C^{1,1}(\overline{\Omega} \times [0,T])} \to 0, \qquad \text{when } \delta \to 0,$$

we deduce that the pair $\{\omega, \mathbf{v}\}$ fulfills (1.13) for any $\psi \in TF(\mathbf{v})$ too. This concludes the proof that $\{\omega, h\}$ is the solution of our problem (1.5)–(1.8) in the case $p = \infty$.

### 4.2. The case $p \in (3, \infty)$. Estimates independent of $\theta$ and the limit transition on $\theta \to 0$

In this subsection we study the case $p \in (3, \infty)$. Hence the data $f$, $b$, $a$, $\omega_0$ satisfy (1.9) with $p \in (3, \infty)$ and the approximated data $f^\theta$, $b^\theta$, $a^\theta$, ${\omega_0}^\theta$ fulfill the conditions (2.1) for $s := p$, i.e. the pair $\{\omega_\theta, h_\theta\}$, constructed in Section 3, satisfies the estimates (2.6)–(2.7) with the constants $m, C$ being dependent on $\theta$. By the last reason we will derive estimates in $L_p$-spaces which are independent of the parameter $\theta$. First we prove the following lemma.

**Lemma 6.** *For any fixed $\theta > 0$ the vorticity $\omega_\theta$ has traces on $\Gamma_T(\mathbf{v}_\theta)$ and for $t = 0$, which are equal to $b^\theta(\mathbf{x}, t, \frac{\partial h_\theta}{\partial \mathbf{n}})$ and $\omega_0^\theta$, respectively, in the following sense:*

$$\frac{1}{\sigma} \int_0^t \int_{[\sigma < d < 2\sigma]} (\mathbf{v}_\theta \cdot \nabla)\, d\, |\omega_\theta|^p\, \psi\, d\mathbf{x} d\tau \underset{\sigma \to 0}{\longrightarrow} \int_0^t \int_\Gamma (\mathbf{v}_\theta \cdot \mathbf{n}) \left| b^\theta\left(\mathbf{x}, t, \frac{\partial h_\theta}{\partial \mathbf{n}}\right)\right|^p \psi\, d\mathbf{x} d\tau, \tag{4.4}$$

$$\frac{1}{\sigma} \int_0^\sigma \int_\Omega |\omega_\theta|^p\ \psi\, d\mathbf{x} d\tau \underset{\sigma \to 0}{\longrightarrow} \int_\Omega |\omega_0^\theta|^p\ \psi\, d\mathbf{x} \tag{4.5}$$

*for any test function $0 \leqslant \psi \in TF(\mathbf{v}_\theta)$, $t \in [0,T]$ and $p \in (3, \infty)$.*

*Proof.* By the extension results [21], there exists an extension $\widehat{\omega} = \widehat{\omega}(\mathbf{x},t) \in C^\infty(\Omega_T)$ of the boundary condition $b^\theta(\mathbf{x},t,\frac{\partial h_\theta}{\partial \mathbf{n}})$ and the initial condition $\omega_0^\theta(\mathbf{x})$ into the domain $\Omega_T$, such that

$$\begin{cases} \widehat{\omega}\big|_{t=0} = \omega_0^\theta \text{ and } \widehat{\omega} = b^\theta(\mathbf{x},t,\frac{\partial h_\theta}{\partial \mathbf{n}}(\mathbf{x},t)) \text{ for } (\mathbf{x},t) \in \Gamma_T, \\ ||\widehat{\omega}||_{C^{1,1}(\overline{\Omega}\times[0,T])} \leqslant C = C(\theta). \end{cases}$$

We see that $z := \omega_\theta - \widehat{\omega}$ satisfies the equality

$$\int_{\Omega_T} z\,(\varphi_t + \mathbf{v}_\theta\cdot\nabla\varphi)\,d\mathbf{x}dt = \int_{\Omega_T} F_\theta\,\varphi\,d\mathbf{x}dt,$$

for $\forall\varphi \in TF(\mathbf{v}_\theta)$ and $F := \partial_t\widehat{\omega}_\theta + \operatorname{div}(\widehat{\omega}_\theta\mathbf{v}_\theta) \in L_\infty(0,T;L_p(\Omega))$. Applying the methods of [14] on transport equations, we deduce that the function $z$ satisfies the equality

$$\int_{\Omega_T} |z|^p\,(\varphi_t+\mathbf{v}_\theta\cdot\nabla\varphi)\,d\mathbf{x}dt = \int_{\Omega_T}\big(p\,F_\theta\,|z|^{p-1}\operatorname{sign}(z) + (p-1)|z|^p\operatorname{div}(\mathbf{v}_\theta)\big)\,\varphi\,d\mathbf{x}dt.$$

Now, considering the suitable test function $\varphi = (1-\mathbf{1}_\sigma(\mathbf{x}))\psi(\mathbf{x},t)$ with $\mathbf{1}_\sigma$ defined by (3.3) and $\psi \in TF(\mathbf{v}_\theta)$, we deduce that

$$\lim_{\sigma\to 0}\frac{1}{\sigma}\left[\int_0^T\int_{[\sigma<d<2\sigma]}(\mathbf{v}_\theta\cdot\nabla)\,d\,|z|^p\,\psi\,d\mathbf{x}d\tau\right] = 0,$$

which implies the assertion (4.4). Analogously if we choose the test function $\varphi = (1-1_\sigma(t))\psi(\mathbf{x},t)$, with $1_\sigma(t)$ defined by

$$1_\sigma(t) := \begin{cases} 0, & \text{if } -\infty < t < \sigma, \\ \frac{t-\sigma}{\sigma}, & \text{if } \sigma \leqslant t < 2\sigma, \\ 1, & \text{if } 2\sigma \leqslant t < +\infty, \end{cases}$$

we obtain

$$\frac{1}{\sigma}\int_0^\sigma\int_\Omega |z|^p\;\psi\,d\mathbf{x}d\tau \underset{\sigma\to 0}{\longrightarrow} 0,$$

which implies (4.5). □

As a consequence of the previous lemma we can establish a Gronwall type inequality for $\omega_\theta$.

**Lemma 7.** *For any fixed $\theta > 0$ and all $t \in [0,T]$, we have*

$$\int_\Omega |\omega_\theta(\mathbf{x},t)|^p\,d\mathbf{x} + (p-1)\int_0^t\int_\Omega [\omega_\theta - h_\theta]\;|\omega_\theta|^p\;d\mathbf{x}d\tau$$
$$\leqslant \int_\Omega |\omega_0^\theta|^p\,d\mathbf{x} + \int_0^t\int_{\Gamma(\mathbf{v}_\theta)}|(\mathbf{v}_\theta\cdot\mathbf{n})|\left|b^\theta\left(\mathbf{x},t,\frac{\partial h_\theta}{\partial\mathbf{n}}\right)\right|^p\,d\mathbf{x}d\tau. \qquad (4.6)$$

*Here $\Gamma(\mathbf{v}_\theta)$ is the part of $\Gamma$ where $(\mathbf{v}_\theta\cdot\mathbf{n}) < 0$.*

*Proof.* Using again the methods of [14], we can verify that the function $\omega_\theta$ satisfies the equality

$$\int_{\Omega_T} |\omega_\theta|^p \left(\varphi_t + \mathbf{v}_\theta \cdot \nabla\varphi\right) d\mathbf{x}dt = (p-1)\int_{\Omega_T} |\omega_\theta|^p \operatorname{div}(\mathbf{v}_\theta)\varphi \;\; d\mathbf{x}dt,$$

for any function $\varphi \in C^{1,1}(\Omega_T)$ with compact support in $\Omega_T$.

The inequality (4.6) follows if we consider the appropriate test function $\varphi(\mathbf{x},\tau) = (1_\delta(\tau) - 1_\delta(\tau+\delta-t))\mathbf{1}_\sigma(\mathbf{x})$ and pass to the limit when $\delta \to 0$ and $\sigma \to 0$ according to Lemma 6. □

Now we show the following result.

**Lemma 8.** *The triple $\{\omega_\theta, h_\theta, \mathbf{v}_\theta\}$ satisfies the estimates*

$$||\omega_\theta||_{L_\infty(0,T;\, L_p(\Omega))} \leqslant C, \tag{4.7}$$

$$\|h_\theta\|_{L_\infty(0,T;W_p^2(\Omega))} \leqslant C, \tag{4.8}$$

$$||\partial_t\left(\nabla h_\theta\right)||_{L_\infty(0,T;\, L_p(\Omega))} \leqslant C. \tag{4.9}$$

*Here and below constants $C$ do not depend on $\theta$.*

*Proof.* Let $\overline{h}$ be the solution of the elliptic problem

$$\begin{cases} -\triangle\, \overline{h}_\theta + \overline{h}_\theta = \omega_\theta \qquad \text{on} \quad \Omega, \\ \qquad\qquad \overline{h}_\theta\big|_\Gamma = 0 \end{cases} \tag{4.10}$$

for a.e. $t \in (0,T)$. Multiplying the equation of (4.10) by $\overline{h}^p$ and integrating over $\Omega$, we obtain

$$||\overline{h}_\theta||_{L_{p+1}(\Omega)} \leqslant ||\omega_\theta||_{L_{p+1}(\Omega)},$$

which with the help of (1.9), (2.1) and (4.6) implies (4.7). The estimates (4.8) follow the classical results for elliptic equations (see the book [19])

$$||h_\theta||_{W_p^2(\Omega)} \leqslant C(||\omega_\theta||_{L_p(\Omega)} + \|a_\theta\|_{W_p^2(\Gamma)})$$

for any given $p \in (2,\infty)$, applied to the problem (2.3) with a given $\omega$. The estimate (4.9) is a direct consequence of (4.1). □

From (1.9), (2.1), (4.7)–(4.9) and Corollary 9 of [22], we conclude that there exists a subsequence of $\{\omega_\theta, h_\theta, \mathbf{v}_\theta\}$, such that

$$\begin{aligned} h_\theta &\rightharpoonup h && \text{weakly} - * \text{ in } L_\infty(0,T;\, W_p^2(\Omega)), \\ \omega_\theta &\rightharpoonup \omega && \text{weakly} - * \text{ in } L_\infty(0,T; L_p(\Omega)), \\ \mathbf{v}_\theta &\rightharpoonup \mathbf{v} && \text{strongly in } L_\infty(0,T; W_p^1(\Omega)). \end{aligned}$$

Using the same argument of Lemma 5, we derive

$$(\mathbf{v}_\theta \cdot \mathbf{n}) \equiv \frac{\partial h_\theta}{\partial \mathbf{n}} \to (\mathbf{v}\cdot\mathbf{n}) \equiv \frac{\partial h}{\partial \mathbf{n}} \text{ strongly in } L^2(\Gamma_T).$$

Obviously the limit triple $\{\omega,\, h,\, \mathbf{v}\}$ fulfills the relations (1.10)–(1.12). By the same method as in subsection 4.1, we show that $\{\omega, \mathbf{v}\}$ satisfies (1.13) too.

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# Augmented Lagrangian Method and Compressible Visco-plastic Flows: Applications to Shallow Dense Avalanches

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*Dedicated to the memory of Professor Alexandre V. Kazhikov*

**Abstract.** In this paper we propose a well-balanced finite volume/augmented Lagrangian method for compressible visco-plastic models focusing on a compressible Bingham type system with applications to dense avalanches. For the sake of completeness we also present a method showing that such a system may be derived for a shallow flow of a rigid-viscoplastic incompressible fluid, namely for incompressible Bingham type fluid with free surface. When the fluid is relatively shallow and spreads slowly, lubrication-style asymptotic approximations can be used to build reduced models for the spreading dynamics, see for instance [N.J. Balmforth *et al.*, J. Fluid Mech (2002)] . When the motion is a little bit quicker, shallow water theory for non-Newtonian flows may be applied, for instance assuming a Navier type boundary condition at the bottom. We start from the variational inequality for an incompressible Bingham fluid and derive a shallow water type system. In the case where Bingham number and viscosity are set to zero we obtain the classical Shallow Water or Saint-Venant equations obtained for instance in [J.F. Gerbeau, B. Perthame, DCDS (2001)]. For numerical purposes, we focus on the one-dimensional in space model: We study associated static solutions with sufficient conditions that relate the slope of the bottom with the Bingham number and domain dimensions. We also propose a well-balanced finite volume/augmented Lagrangian method. It combines well-balanced finite volume schemes for spatial discretization with the augmented Lagrangian method to treat the associated optimization problem. Finally, we present various numerical tests.

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**Keywords.** Compressible flows, shallow-water systems, Bingham flows, avalanches, mixed finite volume/augmented Lagrangian, well-balanced scheme, visco-plastic flows.

## 1. Introduction

Avalanches are natural phenomena that occur in mountainous regions such as the Alps in France. During these last few years, we have joined with others in real efforts devoted to the physical understanding of avalanche formation and motion in complex topography, see [2], [40]. This paper is an attempt to derive a compressible visco-plastic system from depth-averaged processes for dense avalanches and to provide an accurate numerical scheme for such a model. Our results consist of two parts: A generalization to compressible flows of the Augmented Lagrangian method for incompressible Bingham visco-plastic flow initiated by R. Glowinski, see [22]; A simple method to derive a shallow-water type model for an incompressible Bingham flow with free surface. Note that our numerical scheme may be used in other applications such as numerical modeling of projectile penetration into compressible rigid visco-plastic media, see for instance models in [15].

Note that it is very difficult to postulate a constitutive relation for the stress tensor in terms of a deformation measure that correctly describes avalanche behaviour, see for instance [1]. This explains, for instance, why instead of prescribing a detailed constitutive relation, a Coulomb dry friction law for the basal friction and a Mohr-Coulomb yield criterion for the interior behavior have been used by several authors, see [40]. The information obtained in this way is sufficient to derive, more easily, dynamic equations that describe the spatio-temporal evolution of the height and the depth-averaged horizontal velocity component of a moving avalanche pile. In our paper, we propose to consider a shallow flow of a rigid viscoplastic incompressible fluid, namely a Bingham fluid. More general constitutive relations may be studied such as those included in [15]. Note also that depending on the basal boundary condition (slip boundary condition or non-slip boundary condition), various shallow-water type equations may be obtained, see for instance [28] and [10] for models coming from incompressible Navier-Stokes equations and [26] for models coming from Bingham type equations with Dirichlet boundary condition at the bottom. Here, we consider boundary conditions in the spirit of [28] namely Navier boundary conditions at the bottom. This is dedicated to quicker flows replacing Dirichlet boundary conditions by a wall law boundary condition taking into account the boundary layer. Assuming a Navier boundary condition, we start with the variational inequality for incompressible Bingham fluid and prove that a shallow-water type system may be obtained using adequate test functions. Several numerical simulations of avalanching flows in simple configuration are then proposed to compare our proposed scheme to previous ones. More precisely in our study, we propose a well-balanced finite volume/augmented Lagrangian method. It combines well-balanced finite volume schemes for spatial discretization with the augmented Lagrangian method to treat the optimization problem. The key point in our result is that there exists a real interaction between the finite volume scheme and the augmented Lagrangian procedure. This gives a real well-balanced scheme that allows us to simulate initiation and run-out problems capturing interesting stationary solutions. Let us say that our numerical scheme will be soon tested in a two-dimensional space interacting with C. Ancey's group for experimental data. Readers interested by a theoretical studies linked to compressible Bingham type models are referred to [7], [6], [5], [43] and more recently [36].

This paper is organized as follows: in Section 2 we present the equations that define the 3D free-surface problem. In Section 3 we deduce the depth-averaged model from the variational inequality for incompressible Bingham fluid. In Section 4 we present the associated 1D system and sufficient conditions to identify stationary solutions of the model. The numerical scheme based on the combination of finite volumes methods and the use of the augmented Lagrangian is shown in Section 5. Finally, in Section 6 we present three numerical tests. In the first one we study the convergence to a stationary solution when the initial profile of the free surface is a rectangular pulse. In this test we compare the results that we obtain with the proposed numerical scheme with non-well-balanced numerical schemes. In the second numerical test we study the transition between two different stationary solutions corresponding to two different Bingham numbers. And in the third test we present the case of an avalanche over all the domain.

Let us finish the introduction by mentioning that this paper is dedicated to the Memory of Professor Alexandre V. Kazhikov, one of the most inventive applied mathematicians in compressible fluid mechanics.

## 2. Statement of the 3D-problem

We consider here the evolution equations in the time interval $(0, T)$, $T > 0$ describing the flow of an inhomogeneous Bingham fluid in a domain $\mathcal{D}(t) \subset \mathbb{R}^3$ with a smooth boundary $\partial\mathcal{D}(t)$. In the following, the space and time coordinates as well as all mechanical fields are non-dimensional. The notation $\boldsymbol{u}$ stands for the velocity field, $\boldsymbol{\sigma}$ denotes the Cauchy stress tensor field, $p = -\operatorname{trace}(\boldsymbol{\sigma})/3$ represents the pressure and $\boldsymbol{\sigma}' = \boldsymbol{\sigma} + pI$ is the deviatoric part of the stress tensor. The momentum balance law in Eulerian coordinates reads

$$\rho\Big(\mathrm{St}\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\Big) - \mathbf{div}\,\boldsymbol{\sigma}' + \mathrm{B}\nabla p = \frac{1}{\mathrm{Fr}^2}\rho\boldsymbol{f} \quad \text{in } \mathcal{D}(t), \tag{1}$$

where $\rho = \rho(t, x) \geq \underline{\rho} > 0$ is the mass density distribution and $\boldsymbol{f}$ denotes the body forces. We have denoted by $\mathrm{St} = L/(V_c T_c), \mathrm{Fr}^2 = V^2/(L f_c)$ the Strouhal and Froude numbers and we introduce $\mathrm{B} = \kappa_c/(\rho_c V_c^2)$, where $\rho_c, V_c, L, \kappa_c, T_c$ are the characteristic density, velocity, length, stress and time respectively. Since we deal with an incompressible fluid, we get

$$\operatorname{div}\,\boldsymbol{u} = 0 \quad \text{in } \mathcal{D}(t). \tag{2}$$

The conservation of mass becomes

$$St\frac{\partial \rho}{\partial t} + \boldsymbol{u}\cdot\nabla\rho = 0 \quad \text{in } \mathcal{D}(t). \tag{3}$$

We notice from the above equation that, excepting some special cases, the flow of an incompressible fluid with inhomogeneous mass density is not stationary.

If we denote by $\boldsymbol{D}(\boldsymbol{u}) = (\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}^T\boldsymbol{u})/2$ the rate of deformation tensor, the constitutive equation of the Bingham fluid can be written as follows:

$$\boldsymbol{\sigma}' = \frac{2}{\mathrm{Re}}\eta_1 \boldsymbol{D}(\boldsymbol{u}) + \eta_2 \mathrm{B}\frac{\boldsymbol{D}(\boldsymbol{u})}{|\boldsymbol{D}(\boldsymbol{u})|} \qquad \text{if } |\boldsymbol{D}(\boldsymbol{u})| \neq 0, \tag{4}$$

$$|\boldsymbol{\sigma}'| \leq \eta_2 \mathrm{B} \qquad \text{if } |\boldsymbol{D}(\boldsymbol{u})| = 0, \tag{5}$$

where $\eta_1 \geq \eta_0 > 0$ is the non-dimensional viscosity distribution depending on $\rho$ and $\eta_2 \geq 0$ is a non-negative continuous function which stands for the non-dimensional yield limit distribution in $\mathcal{D}(t)$. Here $\mathrm{Re} = \rho_c V_c L/\eta_c$ is the Reynolds number and $\eta_c$ is a characteristic viscosity. Note that if $\kappa_c$ is the characteristic yield stress, then $\mathrm{B} = \mathrm{Bi}/\mathrm{Re}$, where $\mathrm{Bi} = \kappa_c L/(\eta_c V_c)$ is the Bingham number. The type of behavior described by equations (4–5) can be observed in the case of some oils or sediments used in the process of oil drilling. The Bingham model, also denominated "Bingham solid" (see for instance [38]) was considered in order to describe the deformation of many solid bodies. Recently, the inhomogeneous (or density-dependent) Bingham fluid was chosen in landslides modeling [20, 14].

When considering a density-dependent model, the viscosity coefficient $\eta_1$ and the yield limit $\eta_2$ depend on the density $\rho$ through two constitutive functions, *i.e.*,

$$\eta_1 = \eta_1(\rho), \quad \eta_2 = \eta_2(\rho). \tag{6}$$

In order to complete equations (1–6) with the boundary conditions we assume that $\partial\mathcal{D}(t)$ is divided into two disjoint parts so that $\partial\mathcal{D}(t) = \Gamma_b(t) \cup \Gamma_s(t)$. On the boundary $\Gamma_b(t)$, which corresponds to the bottom part of the fluid, we consider a Navier condition with a friction coefficient $\alpha$ and a no-penetration condition

$$\boldsymbol{\sigma}_t = -\alpha \boldsymbol{u}_t, \qquad \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_b. \tag{7}$$

Here $\boldsymbol{n}$ stands for the outward unit normal on $\partial\mathcal{D}(t)$ and we have adopted the following notation for the tangential and normal decomposition of any velocity field $\boldsymbol{u}$ and any density of surface forces $\boldsymbol{\sigma n}$:

$$\boldsymbol{u} = u_n\boldsymbol{n} + \boldsymbol{u}_t, \;\text{ with }\; u_n = \boldsymbol{u}\cdot\boldsymbol{n}, \qquad \boldsymbol{\sigma n} = \sigma_n\boldsymbol{n} + \boldsymbol{\sigma}_t \;\text{ with }\; \sigma_n = \boldsymbol{\sigma n}\cdot\boldsymbol{n}.$$

The (unknown) boundary $\Gamma_s(t)$ is a free surface, i.e., we assume a no-stress condition

$$\boldsymbol{\sigma n} = 0 \qquad \text{on} \quad \Gamma_s(t), \tag{8}$$

and the fact that the fluid region is advected by the flow, which can be expressed by

$$St\frac{\partial 1_{\mathcal{D}(t)}}{\partial t} + \boldsymbol{u}\cdot\nabla 1_{\mathcal{D}(t)} = 0, \tag{9}$$

where $1_{\mathcal{D}(t)}$ is the characteristic function of the domain $\mathcal{D}(t)$.

Finally the initial conditions are given by

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \quad \rho|_{t=0} = \rho_0. \tag{10}$$

Setting

$$\mathcal{V}(t) = \left\{\boldsymbol{\Phi} \in H^1(\mathcal{D}(t))^3,\ \mathrm{div}\ \boldsymbol{\Phi} = 0 \ \text{ in } \mathcal{D}(t), \quad \boldsymbol{\Phi}\cdot\boldsymbol{n} = 0 \ \text{ on } \Gamma_b(t)\right\},$$

we give the variational formulation of (1), (2), (4), (5) and (7–8) for the velocity field (see [24]), namely

$$
\begin{aligned}
&\forall t \in (0,T), \qquad \boldsymbol{u}(t,\cdot) \in \mathcal{V}(t), \qquad \forall \boldsymbol{\Phi} \in \mathcal{V}(t),\\
&\int_{\mathcal{D}(t)} \rho\Big(St\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\Big)\cdot(\boldsymbol{\Phi}-\boldsymbol{u})\\
&+\frac{1}{Re}\int_{\mathcal{D}(t)} 2\eta_1(\rho)\boldsymbol{D}(\boldsymbol{u}) : (\boldsymbol{D}(\boldsymbol{\Phi})-\boldsymbol{D}(\boldsymbol{u}))\\
&+B\int_{\mathcal{D}(t)} \eta_2(\rho)\left(|\boldsymbol{D}(\boldsymbol{\Phi})|-|\boldsymbol{D}(\boldsymbol{u})|\right)\\
&+\int_{\Gamma_b} \alpha \boldsymbol{u}_t\cdot(\boldsymbol{\Phi}_t-\boldsymbol{u}_t) \geq \frac{1}{Fr^2}\int_{\mathcal{D}(t)} \rho \boldsymbol{f}\cdot(\boldsymbol{\Phi}-\boldsymbol{u}).
\end{aligned}
\tag{11}
$$

We can formulate the same problem in terms of velocity and pressure by using the space

$$
\mathcal{W}(t) = \big\{\boldsymbol{\Phi} \in H^1(\mathcal{D}(t))^3, \quad \boldsymbol{\Phi}=0, \quad \boldsymbol{\Phi}\cdot\boldsymbol{n}=0 \ \text{ on } \ \Gamma_b(t)\big\},
$$

to deduce

$$
\left\{
\begin{aligned}
&\forall t \in (0,T), \quad \boldsymbol{u}(t,\cdot) \in \mathcal{W}(t),\ p(t,\cdot) \in L^2(\mathcal{D}(t)), \quad \forall \boldsymbol{\Phi} \in \mathcal{W}(t),\ \forall q \in L^2(\mathcal{D}(t)),\\
&\int_{\mathcal{D}(t)} \rho\Big(St\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\Big)\cdot(\boldsymbol{\Phi}-\boldsymbol{u}) - \int_{\mathcal{D}(t)} p(\operatorname{div}\boldsymbol{\Phi}-\operatorname{div}\boldsymbol{u})\\
&+\frac{1}{Re}\int_{\mathcal{D}(t)} 2\eta_1(\rho)\boldsymbol{D}(\boldsymbol{u}) : (\boldsymbol{D}(\boldsymbol{\Phi})-\boldsymbol{D}(\boldsymbol{u})) + B\int_{\mathcal{D}(t)} \eta_2(\rho)\left(|\boldsymbol{D}(\boldsymbol{\Phi})|-|\boldsymbol{D}(\boldsymbol{u})|\right)\\
&+\int_{\Gamma_b(t)} \alpha \boldsymbol{u}_t\cdot(\boldsymbol{\Phi}_t-\boldsymbol{u}_t) \geq \frac{1}{Fr^2}\int_{\mathcal{D}(t)} \rho \boldsymbol{f}\cdot(\boldsymbol{\Phi}-\boldsymbol{u}),\\
&\int_{\mathcal{D}(t)} q\operatorname{div}\boldsymbol{u} = 0.
\end{aligned}
\right.
\tag{12}
$$

Finally the problem of the flow of an inhomogeneous Bingham fluid becomes:

*Find the velocity field* $\boldsymbol{u}$ *and the mass density field* $\rho$ *such that conditions* (3), (6), (10) *and* (11) *hold.*

or in an equivalent form

*Find the velocity field* $\boldsymbol{u}$*, the pressure* $p$ *and the mass density field* $\rho$ *such that conditions* (3), (6), (10) *and* (12) *hold.*

As far as we know there does not exist any uniqueness result for this problem. Note recent mathematical studies in [7], [6], [5] and [43] dedicated to non-homogeneous incompressible Bingham flows and compressible Bingham flows in 1-D space.

## 3. The plane slope case

We consider here the case of a plane slope. For this let $\Omega \subset \mathbb{R}^2$ be a fixed bounded domain and

$$\mathcal{D}(t) = \{(x,z)\ ;\ x \in \Omega,\ 0 < z < h(t,x)\},$$

where $h(t,x)$ is the thickness of the fluid and $x = (x_1, x_2)$. We define by

$$\Gamma_s(t) = \{(x,z)\ ;\ x \in \Omega,\ z = h(t,x)\}, \quad \Gamma_b(t) = \partial\mathcal{D}(t) \setminus \Gamma_s(t)$$

the free and bottom surfaces. We denote by $\boldsymbol{v} = (v_1, v_2)$ the horizontal components of the velocity field and by $w$ the vertical one, i.e., $\boldsymbol{u} = (\boldsymbol{v}, w)$.

*Penalization condition averaging.* Let us remark that equation (9), for this choice of the flow geometry, reads

$$St\frac{\partial h}{\partial t} + \boldsymbol{v}\cdot\nabla h - w = 0, \quad \text{for } z = h(t,x). \tag{13}$$

If we choose $q = q(x)$ dependent only on $x$ in (12) we get

$$0 = \int_{\mathcal{D}(t)} q \operatorname{div} \boldsymbol{u} = \int_\Omega q(x)\left(\int_0^{h(t,x)} \operatorname{div} \boldsymbol{v}(t,x,z)\, dz + w(t,x,h(t,x))\right) dx$$

$$= \int_\Omega q(x)\left(\operatorname{div}\left(\int_0^{h(t,x)} \boldsymbol{v}(t,x,z)\, dz\right) - \boldsymbol{v}(t,x,h(t,x))\cdot\nabla h(t,x) + w(t,x,h(t,x))\right) dx,$$

and using the kinematic conditions (13) we get

$$\int_\Omega q\left(St\frac{\partial h}{\partial t} + \operatorname{div}(h\overline{\boldsymbol{v}})\right) dx = 0, \qquad \text{for all } \ q \in L^2(\Omega), \tag{14}$$

where $\overline{\boldsymbol{v}}(t,x) := \dfrac{1}{h(t,x)}\displaystyle\int_0^{h(t,x)} \boldsymbol{v}(t,x,z)\, dz$ is the vertical mean value of the horizontal velocity.

*Mass equation averaging.* Using the same technique as before one can deduce from the mass equation (3) that

$$\int_\Omega q\left(St\frac{\partial \overline{\rho} h}{\partial t} + \operatorname{div}(h\overline{\rho\boldsymbol{v}})\right) dx = 0, \qquad \text{for all } \ q \in L^2(\Omega), \tag{15}$$

where $\overline{\rho}(t,x) := \dfrac{1}{h(t,x)}\displaystyle\int_0^{h(t,x)} \rho(t,x,z)\, dz$ is the vertical mean value of the mass density and $\overline{\rho\boldsymbol{v}}(t,x) := \dfrac{1}{h(t,x)}\displaystyle\int_0^{h(t,x)} \rho(t,x,z)\boldsymbol{v}(t,x,z)\, dz$ is the vertical mean value of the mass flux.

On the other hand the divergence free condition (2) reads

$$w(t,x,z) = -\int_0^z \operatorname{div} \boldsymbol{v}(t,x,s)\, ds, \qquad \text{for all } \ (x,z) \in \mathcal{D}(t). \tag{16}$$

*Momentum equation rescaling.* In order to deduce an asymptotic model in the shallow flow approximation, we consider $\varepsilon \ll 1$ a small parameter representing the aspect ratio of the thickness. Following a standard scaling technique, we write

$$\begin{aligned}
&X := x, \quad Z := \frac{z}{H(t,x)\varepsilon},\\
&H := \frac{h}{\varepsilon}, \quad P = \mathrm{Fr}^2 p, \quad \beta(t,x,Z) := \frac{\alpha(t,x,z)}{\varepsilon},\\
&\boldsymbol{V}(t,X,Z) := \boldsymbol{v}(t,x,z), \quad W(t,X,Z) := \frac{w(t,x,z)}{\varepsilon}.
\end{aligned} \tag{17}$$

We denote by $\Omega^0$ the domain defined by $\mathcal{D}_f \times (0,1)$. In these new variables the equations (14-15) read

$$St\frac{\partial H}{\partial t} + \mathrm{div}(H\overline{\boldsymbol{V}}) = 0, \tag{18}$$

$$St\frac{\partial(\overline{\rho}H)}{\partial t} + \mathrm{div}(H\overline{\rho\boldsymbol{V}}) = 0, \tag{19}$$

where

$$\overline{\boldsymbol{V}}(t,x) := \frac{1}{H(t,x)}\int_0^{H(t,x)} \boldsymbol{V}(t,x,z)\,dz$$

and

$$\overline{\rho\boldsymbol{V}}(t,x) := \frac{1}{H(t,x)}\int_0^{H(t,x)} \rho(t,x,z)\boldsymbol{V}(t,x,z)\,dz$$

are mean values on the thickness.

We write now each term of the variational inequality (12) in the scaled variables. For this we choose the same scaling for the test functions $\boldsymbol{\Phi} = (\boldsymbol{\Psi}, \varepsilon\theta)$. We decompose the left-hand side of (12) in five terms $I_i$ for $i = 1, \ldots, 5$. They read

$$\begin{aligned}
I_1 = \varepsilon \int_{\Omega^0} H\rho\Big(\mathrm{St}\frac{\partial \boldsymbol{V}}{\partial t}\cdot(\boldsymbol{\Psi}-\boldsymbol{V}) + \varepsilon^2\mathrm{St}\frac{\partial W}{\partial t}(W-\theta)\Big)dXdZ&\\
+\varepsilon\int_{\Omega^0} H\rho(\boldsymbol{V}\cdot\nabla_x\boldsymbol{V} + \frac{1}{H}W\partial_Z\boldsymbol{V})\cdot(\boldsymbol{\Psi}-\boldsymbol{V})dXdZ&\\
+\varepsilon^3\int_{\Omega^0} H\rho(\boldsymbol{V}\cdot\nabla_x W + \frac{1}{H}W\partial_Z W)(W-\theta)dXdZ&,
\end{aligned} \tag{20}$$

$$I_2 = \frac{\varepsilon}{\mathrm{Fr}^2}\int_{\Omega^0} HP\Big(\mathrm{div}_x\boldsymbol{\Psi} + \frac{1}{H}\partial_Z\theta - \mathrm{div}V - \frac{1}{H}\partial_Z W\Big)dXdZ, \tag{21}$$

$$\begin{aligned}
I_3 = \frac{\varepsilon}{\mathrm{Re}}\int_{\Omega^0} 2\eta_1(\rho)\Big(HD(\boldsymbol{V}):(D(\boldsymbol{\Psi})-D(\boldsymbol{V})) + \frac{1}{H}\partial_Z W(\partial_Z\theta - \partial_Z W)\Big)dXdZ&\\
+\frac{\varepsilon}{\mathrm{Re}}\int_{\Omega^0}\left(\sum_{i=1}^{2}\eta_1(\rho)\left(\varepsilon\partial_{x_i}W + \frac{1}{\varepsilon H}\partial_Z V_i\right)\right.&\\
\left.\times\left(\frac{1}{\varepsilon}\partial_Z(\boldsymbol{\Psi}_i - V_i) + \varepsilon H\partial_{x_i}(\theta - W_i)\right)\right)dXdZ&,
\end{aligned} \tag{22}$$

$$I_4 = -\varepsilon B \int_{\Omega^0} H\eta_2(\rho)\Big(\sqrt{|D(\boldsymbol{V})|^2 + (\frac{1}{H}\partial_Z W)^2 + \frac{1}{2}\sum_{i=1}^{2}(\varepsilon\partial_{x_i}W + \frac{1}{\varepsilon H}\partial_Z V_i)^2}$$
$$- \sqrt{|D(\boldsymbol{\Psi})|^2 + (\frac{1}{H}\partial_Z\theta)^2 + \frac{1}{2}\sum_{i=1}^{2}(\varepsilon\partial_{x_i}\theta + \frac{1}{\varepsilon H}\partial_Z\boldsymbol{\Psi}_i)^2}\Big)dX\,dZ, \tag{23}$$

$$I_5 = \int_{\mathcal{D}_f} \varepsilon\beta\boldsymbol{V}\cdot(\boldsymbol{\Psi}-\boldsymbol{V})dX.$$

Concerning the right-hand side named $I_6$, we get

$$I_6 = \varepsilon\int_{\Omega^0}\Big(\rho H f_H\cdot(\boldsymbol{V}-\boldsymbol{\Psi}) + \varepsilon H\rho f_v(W-\theta)\Big)dX\,dZ.$$

### 3.1. Momentum equation asymptotic

Let us assume that

$$\mathrm{St} = \mathrm{Re} = \mathrm{B} = \mathrm{Fr} = 0(1), \qquad \varepsilon \ll 1.$$

We also assume that the external forces $f_H$ and $f_v$ verifies

$$f_H = \mathcal{O}(\varepsilon) \quad \text{and} \quad f_v = \mathcal{O}(1).$$

Dividing the variational equation by $\varepsilon$, let us search for solutions, in the rescaled formulation, under the form

$$\begin{aligned} \boldsymbol{V} &= \boldsymbol{V}_0 + \varepsilon^2\boldsymbol{V}_1 + \cdots, & W &= W_0 + \varepsilon^2 W_1 + \cdots, \\ P &= P_0 + \varepsilon^2 P_1 + \cdots, & \rho &= \rho_0 + \varepsilon^2\rho_1 + \cdots. \end{aligned} \tag{24}$$

We write $\boldsymbol{V}_0 = (\boldsymbol{V}_{0,1}, \boldsymbol{V}_{0,2})$. In what follows we first focus on the terms of order $1/\varepsilon^2$, after we write the terms of order 1 plus the terms of order $\varepsilon$.

☐ **Terms of order $1/\varepsilon^2$.**

We get

$$\int_{\Omega^0} \frac{\eta_1(\rho_0)}{H}\partial_Z\boldsymbol{V}_0\cdot\partial_Z(\boldsymbol{\Psi}-\boldsymbol{V}_0) = 0.$$

Assuming $\eta_1 > c > 0$ in $\Omega^0$, this gives, using the boundary conditions,

$$\partial_Z V_{0,1} = \partial_Z V_{0,2} = 0. \tag{25}$$

☐ **Terms of order 1 plus terms of order $\varepsilon$.**

Coming from (18)–(19), we get

$$St\frac{\partial H}{\partial t} + \mathrm{div}(H\boldsymbol{V}_0) = 0, \tag{26}$$

$$St\frac{\partial(\overline{\rho_0}H)}{\partial t} + \mathrm{div}(H\overline{\rho_0}\boldsymbol{V}_0) = 0. \tag{27}$$

Moreover we get

$$\int_{\Omega^0} H\rho_0\Big(\mathrm{St}\partial_t \boldsymbol{V}_0\cdot(\boldsymbol{\Psi}-\boldsymbol{V}_0)+\boldsymbol{V}_0\cdot\nabla_x\boldsymbol{V}_0(\boldsymbol{\Psi}-\boldsymbol{V}_0)\Big)dX\,dZ$$
$$+\int_{\mathcal{D}f}\beta\boldsymbol{V}_0\cdot(\boldsymbol{\Psi}-\boldsymbol{V}_0)dX$$
$$+\int_{\Omega^0}\Big(\frac{2}{\mathrm{Re}}H\eta_1(\rho_0)D(\boldsymbol{V}_0):D(\boldsymbol{\Psi}-\boldsymbol{V}_0)+\frac{2}{\mathrm{Re}}\frac{1}{H}\eta_1(\rho_0)\partial_Z W_0(\partial_Z\theta-\partial_Z W_0)\Big)dX\,dZ$$
$$+\int_{\Omega^0} H\mathrm{B}\eta_2(\rho_0)\Big(\sqrt{|D(\boldsymbol{\Psi})|^2+(\frac{1}{H}\partial_Z\theta)^2}-\sqrt{|D(\boldsymbol{V}_0)|^2+(\frac{1}{H}\partial_Z W_0)^2}\Big)dX\,dZ$$
$$+\frac{1}{\mathrm{Fr}^2}\int_{\Omega^0}HP_0(\mathrm{div}\boldsymbol{\Psi}+\frac{1}{H}\partial_Z\theta)dX\,dZ-\frac{1}{\mathrm{Fr}^2}\int_{\Omega^0}HP_0(\mathrm{div}V_0+\frac{1}{H}\partial_Z W_0)dX\,dZ$$
$$+\int_{\Omega_0}\sum_{i=1}^{2}\Big(\eta_1(\rho_0)\partial_{X_i}W\partial_Z(\boldsymbol{\Psi}_i-V_{0,i})+\partial_Z V_{0,i}\partial_{X_i}(\theta-W)\Big)dX\,dZ$$
$$\geq\frac{1}{\mathrm{Fr}^2}\int_{\Omega^0}H\rho_0 f_H(\boldsymbol{\Psi}-\boldsymbol{V}_0)dX\,dZ+\frac{1}{\mathrm{Fr}^2}\int_{\Omega^0}\varepsilon H\rho_0 f_v(\theta-W_0)dX\,dZ. \qquad (28)$$

Using (25), $\mathrm{div}(\boldsymbol{V}_0,W_0)=0$ and the boundary conditions, we have

$$W_0=-ZH\mathrm{div}_x V_0.$$

Moreover, in what follows we choose $\boldsymbol{\Psi}$ independent of $Z$. Finally, we choose the same relation for the test functions, that is $\theta=-ZH\mathrm{div}\boldsymbol{\Psi}$. As $\Omega^0=\mathcal{D}_f\times(0,1)$ we can also integrate in $Z\in(0,1)$. We write

$$\overline{\eta_i(\rho_0)}=\int_0^1\eta_i(\rho_0(Z))dZ,\quad i=1,2,$$
$$\overline{\rho_0 f_H}=\int_0^1\rho_0(Z)f_H(Z)dZ,\quad \overline{Z\rho_0 f_v}=\int_0^1 Z\rho_0(Z)f_v(Z)dZ.$$

And we obtain

$$\begin{aligned}
&\int_{\mathcal{D}_f}H\overline{\rho_0}\Big(\mathrm{St}\partial_t\boldsymbol{V}_0\cdot(\boldsymbol{\Psi}-\boldsymbol{V}_0)+\boldsymbol{V}_0\cdot\nabla_x\boldsymbol{V}_0(\boldsymbol{\Psi}-\boldsymbol{V}_0)\Big)dX\\
&+\int_{\mathcal{D}^f}\beta\boldsymbol{V}_0\cdot(\boldsymbol{\Psi}-\boldsymbol{V}_0)dX\\
&+\int_{\mathcal{D}^f}\frac{2}{\mathrm{Re}}H\overline{\eta_1(\rho_0)}D(\boldsymbol{V}_0):D(\boldsymbol{\Psi}-\boldsymbol{V}_0)dX\\
&+\int_{\mathcal{D}^f}\frac{2}{\mathrm{Re}}H\overline{\eta_1(\rho_0)}\mathrm{div}_x\boldsymbol{V}_0(\mathrm{div}_x\boldsymbol{\Psi}-\mathrm{div}_x\boldsymbol{V}_0)dX\\
&+\int_{\mathcal{D}^f}\mathrm{B}H\overline{\eta_2(\rho_0)}\Big(\sqrt{|D(\boldsymbol{\Psi})|^2+(\mathrm{div}\boldsymbol{\Psi})^2}-\sqrt{|D(\boldsymbol{V}_0)|^2+(\mathrm{div}_x\boldsymbol{V}_0)^2}\Big)dX\\
&\geq\frac{1}{\mathrm{Fr}^2}\int_{\mathcal{D}^f}H\overline{\rho_0 f_H}\cdot(\boldsymbol{\Psi}-\boldsymbol{V}_0)-\frac{\varepsilon}{\mathrm{Fr}^2}\int_{\mathcal{D}^f}(H)^2\overline{Z\rho_0 f_v}(\mathrm{div}\boldsymbol{\Psi}-\mathrm{div}\boldsymbol{V}_0)dX.
\end{aligned}\qquad (29)$$

Equation (29) with (26) and (27) gives a viscous shallow water formulation of Bingham type.

**Remark 1.** From (29), when $\eta_2 = 0$ we obtain an equality. In this case we obtain the 2D viscous Shallow Water equations.

## 4. One-dimensional system and stationary solutions

In this section we present the one-dimensional in space system and its stationary solutions. In the next section, we present a numerical scheme to discretize this system and we study its well-balanced properties, *i.e.*, conditions allowing us to preserve the stationary solution of the associated system.

From (26), (27) and (29) we obtain the one-dimensional in space model, if $H = H(x,t)$, $\overline{\rho_0} = \overline{\rho_0}(x,t)$ and $\boldsymbol{V}_0 = \boldsymbol{V}_0(x,t)$, with $(x,t) \in [0,L] \times [0,T]$.

We consider the external forces

$$f_H = -\sin\theta, \quad f_v = -\cos\theta.$$

Then the one-dimensional in space model is defined by:

$$St\frac{\partial H}{\partial t} + \frac{\partial (H\boldsymbol{V}_0)}{\partial x} = 0, \tag{30}$$

$$St\frac{\partial(\overline{\rho_0}H)}{\partial t} + \frac{\partial(H\overline{\rho_0}\boldsymbol{V}_0)}{\partial_x} = 0, \tag{31}$$

$$\begin{aligned}
&\int_0^L H\overline{\rho_0}\Big(\mathrm{St}\partial_t\boldsymbol{V}_0(\boldsymbol{\Psi}-\boldsymbol{V}_0) + \frac{1}{2}\partial_x(\boldsymbol{V}_0^2)(\boldsymbol{\Psi}-\boldsymbol{V}_0)\Big)dx \\
&+ \int_0^L \beta\boldsymbol{V}_0(\boldsymbol{\Psi}-\boldsymbol{V}_0)dx + \int_0^L \frac{4}{\mathrm{Re}}H\overline{\eta_1(\rho_0)}\partial_x(\boldsymbol{V}_0)\partial_x(\boldsymbol{\Psi}-\boldsymbol{V}_0)dx \\
&+ \int_0^L \mathrm{B}H\overline{\eta_2(\rho_0)}\sqrt{2}\Big(|\partial_x(\boldsymbol{\Psi})| - |\partial_x(\boldsymbol{V}_0)|\Big)dx \\
&\geq \frac{-1}{\mathrm{Fr}^2}\int_0^L H\overline{\rho_0}\sin\theta(\boldsymbol{\Psi}-\boldsymbol{V}_0) + \frac{\varepsilon}{\mathrm{Fr}^2}\int_0^L \frac{\cos\theta}{2}(H)^2\overline{\rho_0}(\partial_x\boldsymbol{\Psi}-\partial_x\boldsymbol{V}_0)dx.
\end{aligned} \tag{32}$$

In what follows we study sufficient conditions to ensure that a solution over an inclined slope is a stationary solution, with velocity being equal to zero.

By (32) we obtain that such a stationary solution must verify

$$\int_0^L \mathrm{B}H\overline{\eta_2(\rho_0)}\sqrt{2}|\partial_x(\boldsymbol{\Psi})| \geq \frac{-1}{\mathrm{Fr}^2}\int_0^L H\overline{\rho_0}\sin\theta\boldsymbol{\Psi} + \frac{\varepsilon}{\mathrm{Fr}^2}\int_0^L \frac{\cos\theta}{2}(H)^2\overline{\rho_0}\partial_x\boldsymbol{\Psi}. \tag{33}$$

Furthermore, we will focus on two types of solutions. The first one corresponds to material at rest, *i.e.* a stationary solution with velocity being equal zero and a horizontal free surface (See Figure 1). Let us recall that we did the change of variable $x = \varepsilon\widetilde{x}$. Consequently, the property of horizontal free surface is defined by

$$\sin\theta\, x + \varepsilon H\cos\theta = cst,$$

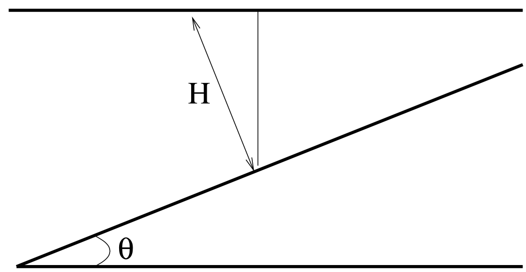



FIGURE 1. Stationary solution with horizontal free surface

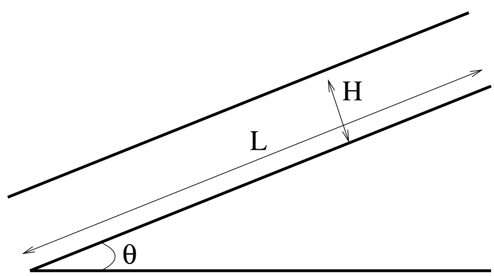



FIGURE 2. Stationary solution with constant height

where *cst* is the height of the free surface. Then, for this stationary solution we obtain that the right-hand side of (33) is equal to zero for all $\Psi$.

By the way, we deduce that the stationary solution corresponding to material at rest is a stationary solution of the model for all values of $\eta_2$ and $\theta$.

Secondly, we want to study sufficient conditions which ensure that a solution with a constant height over an inclined slope is stationary (See Figure 2).

We define

$$\mathcal{F}(x) = \frac{1}{\mathrm{Fr}^2}\int_0^L H\overline{\rho_0}\sin\theta dx + \frac{\varepsilon}{\mathrm{Fr}^2}\frac{\overline{\rho_0}}{2}H^2\cos\theta.$$

Then (33) can be rewritten as

$$\int_0^L H\overline{\eta_2(\rho_0)}|\partial_x\mathbf{\Psi}|dx \geq \int_0^L \mathcal{F}(x)\partial_x\mathbf{\Psi}dx.$$

And this inequality is satisfied if

$$|\mathcal{F}(x)| \leq H\overline{\eta_2(\rho_0)}\sqrt{2}. \tag{34}$$

If $H$ is constant, then

$$\mathcal{F}(x) = \frac{1}{\mathrm{Fr}^2}H\overline{\rho_0}\sin\theta\, x + \frac{\varepsilon}{\mathrm{Fr}^2}\frac{\overline{\rho_0}}{2}H^2\cos\theta + cst,$$

where $cst$ is a constant. Since we assume $H$ to be constant, we can then set

$$cst = -\frac{1}{\mathrm{Fr}^2} H\overline{\rho_0} \sin\theta \, \frac{L}{2} - \frac{\varepsilon}{\mathrm{Fr}^2} \frac{\overline{\rho_0}}{2} H^2 \cos\theta.$$

Finally, we deduce that (34) and then (33) hold if

$$\frac{1}{\mathrm{Fr}^2} \overline{\rho_0} \sin\theta \frac{L}{2} \le \overline{\eta_2(\rho_0)} \sqrt{2}. \tag{35}$$

Observe that if $\varepsilon = H/L$, the previous condition can then be written as

$$\frac{\overline{\rho_0}}{\mathrm{Fr}^2} H \sin\theta \le 2\sqrt{2}\,\varepsilon\, \overline{\eta_2(\rho_0)}.$$

## 5. A well-balanced finite volume/augmented Lagrangian algorithm

In this section we propose a numerical scheme to discretize the 1D model presented in the previous section. We consider the case with constant density, in such a way that the model reduces to the equations (30) and (32) which contain the main difficulties.

First, we write the semi-discretization in time. Then, following [27], we observe that the problem (32) can be seen as an optimization problem and we use an augmented Lagrangian formulation to rewrite (32). This classically leads to the resolution of a saddle-point problem which involves an iterative method where are successively solved:

- a linear system associated to a problem on the speed (let us denote it as SU),
- and a minimization problem associated to the Lagrange multiplier (the solution of this problem is known explicitly).

The key-point of the present paper consists in realizing that the global problem, which couples "indirectly" (30) and SU, implies that the numerical algorithms solving these two problems must also be coupled. Namely, their spatial discretizations have terms in common. This is not obvious a priori, but if we look at the global problem, (30) and SU lead to an underlying Shallow-Water system with source terms (linked to topography and Lagrangian terms). Consequently, if one wants the global scheme to preserve stationary solutions, philosophy inspired by so-called "well-balanced" methods for Shallow-Water system with source terms, must be used. Adopting these methods, SU is complemented with terms linked to the augmented Lagrangian, inducing a coupling which is – to our knowledge – not mentioned in previous works. In the following, we show that this method allows us to perform simulations which preserve various stationary solutions (contrary to previous approaches for which we show that they are not well-balanced). Let us note that, actually, this method can be seen as a generalization to compressible flows of the augmented Lagrangian method for incompressible Bingham visco-plastic flow (applied, in the present paper, to Shallow-Water type equations).

Let us now make more precise the underlying Shallow-Water ideas which inspired our approach.

The accurate solution of hyperbolic systems with source terms requires numerical solvers with specific properties. Indeed, an upwind discretization of the source term, compatible with the one of the flow term, must be performed. Otherwise, a first-order error in space, stemming from the numerical diffusion terms, takes place. This error, after time iteration, may yield large errors in wave amplitude and speed. Roe in [41] studies the relation between the choice of quadrature formulae to approximate the average of the source term and the property of preserving the stationary solutions.

Bermúdez and Vázquez-Céndón introduce in [8] some numerical solvers – with an upwind treatment of the source term for 1D Shallow-Water equations (1D SWE) – which preserve water at rest. This work originated the so-called "well-balanced" solvers, in the sense that the discrete source terms balance the discrete flux terms when computed on some (or all) of the steady solutions of the continuous systems. Several sequels of this work for 1D SWE followed. See, *e.g.*, Greenberg-Leroux [29], Le Veque [35], Castro et al. [13].

This section is organized as follows. In Subsection 5.1 we present the semi-discretization in time of the model. In Subsection 5.2, we present the associated reformulation using an augmented Lagrangian method. We also discuss the link between the system obtained and the classical Shallow Water equations. In Subsection 5.3 we detail the iterative algorithm of the augmented Lagrangian. In Subsection 5.4 we introduce the spatial discretization. And finally, in Subsection 5.4.3 we study the well-balanced properties of the proposed numerical scheme.

### 5.1. Semi-discretization in time

We denote the variables with superscript $n$ to denote the approximation in time $t = t^n$ and with superscript $n+1$ for the time $t = t^n + \Delta t$.

Then, we consider the following semi-discretization in time of (30)–(32) :

$$St\frac{H^{n+1} - H^n}{\Delta t} + \frac{\partial(H^n \boldsymbol{V}_0^n)}{\partial x} = 0, \tag{36}$$

$$\begin{aligned}
&\int_0^L H^n\overline{\rho_0}\Big(\mathrm{St}\frac{\boldsymbol{V}_0^{n+1} - \boldsymbol{V}_0^n}{\Delta t}(\boldsymbol{\Psi} - \boldsymbol{V}_0^{n+1}) + \frac{1}{2}\partial_x((\boldsymbol{V}_0^n)^2)(\boldsymbol{\Psi} - \boldsymbol{V}_0^{n+1})\Big)dx \qquad (37)\\
&\quad + \int_0^L \beta\boldsymbol{V}_0^{n+1}(\boldsymbol{\Psi} - \boldsymbol{V}_0^{n+1})dx + \int_0^L \frac{4}{\mathrm{Re}}H^n\overline{\eta_1(\rho_0)}\partial_x(\boldsymbol{V}_0^{n+1})\partial_x(\boldsymbol{\Psi} - \boldsymbol{V}_0^{n+1})dx\\
&\quad + \int_0^L \mathrm{B}H^n\overline{\eta_2(\rho_0)}\sqrt{2}\Big(|\partial_x(\boldsymbol{\Psi})| - |\partial_x(\boldsymbol{V}_0^{n+1})|\Big)dx\\
&\geq \frac{-1}{\mathrm{Fr}^2}\int_0^L H^n\overline{\rho_0}\sin\theta(\boldsymbol{\Psi} - \boldsymbol{V}_0^{n+1}) + \frac{\varepsilon}{\mathrm{Fr}^2}\int_0^L \frac{\cos\theta}{2}(H^n)^2\overline{\rho_0}(\partial_x\boldsymbol{\Psi} - \partial_x\boldsymbol{V}_0^{n+1})dx.
\end{aligned}$$

**5.2. Rewriting the system: augmented Lagrangian**

We now follow Fortin and Glowinski [27] and rewrite equation (37) as an optimization problem: $\boldsymbol{V}_0^{n+1}$ is the solution of the minimization problem

$$\mathcal{J}(\boldsymbol{V}_0^{n+1}) = \min_{\boldsymbol{V}} \mathcal{J}(\boldsymbol{V}),$$

where

$$\mathcal{J}(\boldsymbol{V}) = F(\mathcal{B}(\boldsymbol{V})) + G(\boldsymbol{V}),$$

with

$$\begin{array}{lll} \mathcal{B} : \mathcal{V} \to \mathcal{H}, & F : \mathcal{H} \to R, & \mathcal{V} = H_0^1([0, L]), \\ \mathcal{B}(\boldsymbol{V}) = \partial_x \boldsymbol{V}, & F(\lambda) = \int_0^L \mathrm{B} H \overline{\eta_2(\rho_0)} |\lambda| dx, & \mathcal{H} = L^2([0, L]), \end{array}$$

and

$$\begin{aligned} G(\boldsymbol{V}) = & \int_0^L H^n \overline{\rho_0} \left( \mathrm{St} \frac{\boldsymbol{V}^2/2 - \boldsymbol{V}_0^n \boldsymbol{V}}{\Delta t} + \frac{1}{2} \partial_x((\boldsymbol{V}_0^n)^2) \boldsymbol{V} \right) dx \\ & + \int_0^L \beta \frac{\boldsymbol{V}^2}{2} dx + \int_0^L \frac{4}{\mathrm{Re}} H^n \overline{\eta_1(\rho_0)} \frac{1}{2} (\partial_x \boldsymbol{V})^2 dx \\ & + \frac{1}{\mathrm{Fr}^2} \int_0^L H^n \overline{\rho_0} \sin\theta \boldsymbol{V} + \frac{\varepsilon}{\mathrm{Fr}^2} \int_0^L \frac{\cos\theta}{2} (H^n)^2 \overline{\rho_0} \partial_x \boldsymbol{V} dx. \end{aligned}$$

Then, we define the Lagrangian by

$$\begin{aligned} & \mathcal{L} : \mathcal{V} \times \mathcal{H} \times \mathcal{H} \to R, \\ & \mathcal{L}(\boldsymbol{V}, q, \mu) = F(q) + G(\boldsymbol{V}) + \int_0^L H^n \mu (\mathcal{B}(\boldsymbol{V}) - q) dx, \end{aligned}$$

and the augmented Lagrangian, for a given value $r \in \mathbb{R}$ $(r > 0)$, is defined by

$$\mathcal{L}_r(\boldsymbol{V}, q, \mu) = \mathcal{L}(\boldsymbol{V}, q, \mu) + \frac{r}{2} \int_0^L H^n (\mathcal{B}(\boldsymbol{V}) - q)^2 dx. \tag{38}$$

Consequently the initial optimization problem consists now in characterizing the saddle point of $\mathcal{L}_r(\boldsymbol{V}, q, \mu)$. On the one hand, let us now begin by deriving with respect to $\boldsymbol{V}$ in (38). It reads

$$\mathcal{M}(\boldsymbol{V}, q, \mu, \boldsymbol{\Psi}) = 0, \quad \forall \boldsymbol{\Psi},$$

where

$$
\begin{aligned}
\mathcal{M}(\boldsymbol{V},q,\mu,\boldsymbol{\Psi}) = &\int_0^L H^n\overline{\rho_0}\Big(\mathrm{St}\frac{\boldsymbol{V}^{n+1}-\boldsymbol{V}_0^n}{\Delta t}\boldsymbol{\Psi}+\frac{1}{2}\partial_x((\boldsymbol{V}_0^n)^2)\boldsymbol{\Psi}\Big)dx\\
&+\int_0^L \beta\boldsymbol{V}\boldsymbol{\Psi}dx+\int_0^L\frac{4}{\mathrm{Re}}H^n\overline{\eta_1(\rho_0)}\partial_x(\boldsymbol{V})\partial_x(\boldsymbol{\Psi})dx\\
&+\frac{1}{\mathrm{Fr}^2}\int_0^L H^n\overline{\rho_0}\sin\theta\,\boldsymbol{\Psi}dx\\
&-\frac{\varepsilon}{\mathrm{Fr}^2}\int_0^L\frac{\cos\theta}{2}(H^n)^2\overline{\rho_0}\partial_x\boldsymbol{\Psi}dx+\int_0^L\mu H^n\mathcal{B}(\boldsymbol{\Psi})dx\\
&+r\int_0^L H^n(\mathcal{B}(\boldsymbol{V})-q)\,\mathcal{B}(\boldsymbol{\Psi})dx.
\end{aligned}
\tag{39}
$$

On the other hand, as the problem is non-differentiable with respect to $q$, we obtain the following variational inequality:

$$
\int_0^L H^n rq(p-q)+H^n\mathrm{B}\eta_2\sqrt{2}(|p|-|q|)-H^n(\mu+rB(\boldsymbol{V}))(p-q)\geq 0,\quad \forall p\in\mathcal{H}
$$

which can be rewritten as the following minimization problem: find $q\in\mathcal{H}$, a solution of

$$
\min_{p\in\mathcal{H}}\left(\frac{H^n r}{2}p^2+H^n\mathrm{B}\overline{\eta_2(\rho_0)}|p|-H^n(\mu+r\mathcal{B}(\boldsymbol{V}))p\right). \tag{40}
$$

But this problem can be directly solved and the solution is

$$
q=\begin{cases} 0 & \text{if } |\mu+r\mathcal{B}(\boldsymbol{V})|<\mathrm{B}\overline{\eta_2(\rho_0)},\\ \frac{1}{r}\Big((\mu+r\mathcal{B}(\boldsymbol{V}))-\mathrm{B}\eta_2\sqrt{2}\mathrm{SGN}(\mu+r\mathcal{B}(\boldsymbol{V}))\Big) & \text{otherwise.}\end{cases} \tag{41}
$$

Moreover, from (39) we deduce that $\boldsymbol{V}$ verifies

$$
\begin{aligned}
&H^n\overline{\rho_0}\Big(\mathrm{St}\frac{\boldsymbol{V}-\boldsymbol{V}_0^n}{\Delta t}+\frac{1}{2}\partial_x((\boldsymbol{V}_0^n)^2)\Big)+\partial_x(\frac{\varepsilon}{2\mathrm{Fr}^2}(H^n)^2\overline{\rho_0}\cos\theta)=-\beta\boldsymbol{V}\\
&-\frac{1}{\mathrm{Fr}^2}H^n\overline{\rho_0}\sin\theta+\partial_x(H^n(\mu-rq))+4\partial_x\left(\frac{H^n\overline{\eta_1(\rho_0)}}{\mathrm{Re}}\partial_x\boldsymbol{V}\right)+r\partial_x(H^n\partial_x\boldsymbol{V}).
\end{aligned}
\tag{42}
$$

Finally, we observe that the new form of system (36)-(37) exhibits a coupling (through an iterative process described in the next subsection) of the following equations:

$$
\begin{cases}
\partial_t H+\partial_x(H\boldsymbol{V})=0,\\
H\overline{\rho_0}\mathrm{St}\partial_t\boldsymbol{V}+\dfrac{H\overline{\rho_0}}{2}\partial_x(\boldsymbol{V}^2)+\partial_x(\dfrac{\varepsilon}{2\mathrm{Fr}^2}H^2\overline{\rho_0}\cos\theta)=-\beta\boldsymbol{V},\\
-\dfrac{1}{\mathrm{Fr}^2}H\overline{\rho_0}\sin\theta+\partial_x(H(\mu-rq))+4\partial_x\left(\dfrac{H\overline{\eta_1(\rho_0)}}{\mathrm{Re}}\partial_x\boldsymbol{V}\right)+r\partial_x(H\partial_x\boldsymbol{V}).
\end{cases}
\tag{43}
$$

These are precisely the Shallow Water equations in formulation $(H, \boldsymbol{V})$, with viscosity, the source term defined by the topography and an extra source term linked to the augmented Lagrangian, namely :

$$\partial_x(H(\mu - rq)), \tag{44}$$

where $\mu$ is the Lagrange multiplier and $q$ is the solution of the optimization problem (40), defined by (41).

### 5.3. Iterative algorithm for the saddle-point

We now present (still following [27]) the iterative algorithm used to compute the saddle-point mentioned in the previous section.

We denote with superscripts $k$ and $k+1$ the variables involved in the iterative algorithm. Let us recall that, with superscript $n$ and $n+1$, we denote the approximations of the variables at time $t = t^n$ and $t = t^n + \Delta t$, respectively. The iterative algorithm is defined through the following steps:

**Step I.0**: We consider that we know $\boldsymbol{V}_0^n$, $H^n$, $\mu^n$ and $q^n$. Then, we impose for $k = 0$, $\boldsymbol{V}^k = \boldsymbol{V}_0^n$, $\mu^k = \mu^n$, $q^k = q^n$.

**Step I.1**: Compute $q^{k+1}$ via:

$$d^{k+1} = \mu^k + rB(\boldsymbol{V}^k),$$

$$q^{k+1} = \begin{cases} 0 & \text{if } \|d^{k+1}\| \le B\overline{\eta_2}, \\ \frac{1}{r}\left(d^{k+1} - B\overline{\eta_2}\frac{d^{k+1}}{\|d^{k+1}\|}\right) & \text{if } \|d^{k+1}\| > B\overline{\eta_2}. \end{cases} \tag{45}$$

**Step I.2**: Compute $\boldsymbol{V}^{k+1}$ via :

$$\begin{aligned} H^n\left[\overline{\rho_0}\mathrm{St}\frac{\boldsymbol{V}^{k+1} - \boldsymbol{V}_0^n}{\Delta t} + \partial_x\left(\frac{\overline{\rho_0}(\boldsymbol{V}_0^n)^2}{2} + \frac{\varepsilon}{\mathrm{Fr}^2}H^n\overline{\rho_0}\cos\theta\right)\right] &= -\beta\boldsymbol{V}^{k+1} \\ - \frac{1}{\mathrm{Fr}^2}H^n\overline{\rho_0}\sin\theta + \partial_x(H^n(\mu^k - rq^{k+1})) + \partial_x&\left(H^n(\frac{\overline{4\eta_1(\rho_0)}}{\mathrm{Re}} + r)\partial_x\boldsymbol{V}^{k+1}\right). \end{aligned} \tag{46}$$

**Step I.3**: Update $\mu^{k+1}$ via:

$$\mu^{k+1} = \mu^k + r(B(\boldsymbol{V}^{k+1}) - q^{k+1}). \tag{47}$$

**Loop Steps I.1–I.3**: $k \to k+1$, until (*e.g.*, with $tol = 10^{-2}$)

$$\frac{\|\mu^{k+1} - \mu^k\|}{\|\mu^k\|} \le tol. \tag{48}$$

**At convergence**: We now have determined the value of $\boldsymbol{V}$ at time $t^{n+1}$, we just have to set

$$\boldsymbol{V}_0^{n+1} = \boldsymbol{V}^{k+1}, \tag{49}$$

and we also set $\mu^{n+1} = \mu^k$, $q^{n+1} = q^{k+1}$.

### 5.4. Spatial discretization

In this subsection we describe the discretization in space of equations (36) and (46). That is, the spatial discretization of the following two equations:

$$\begin{cases} St\dfrac{H^{n+1}-H^n}{\Delta t}+\dfrac{\partial(H^n\boldsymbol{V}_0^n)}{\partial x}=0, \\ H^n\left[\overline{\rho_0}\mathrm{St}\dfrac{\boldsymbol{V}^{k+1}-\boldsymbol{V}_0^n}{\Delta t}+\partial_x\left(\dfrac{\overline{\rho_0}(\boldsymbol{V}_0^n)^2}{2}+\dfrac{\varepsilon}{\mathrm{Fr}^2}H^n\overline{\rho_0}\cos\theta\right)\right] \\ \qquad =-\beta\boldsymbol{V}^{k+1}-\underline{\dfrac{1}{\mathrm{Fr}^2}}H^n\overline{\rho_0}\sin\theta+\partial_x(H^n(\mu^k-rq^{k+1})) \\ \qquad +\partial_x\left(H^n(\dfrac{4\eta_1(\rho_0)}{\mathrm{Re}}+r)\partial_x\boldsymbol{V}^{k+1}\right). \end{cases} \tag{50}$$

We observe that usually, to discretize a Bingham system where two equations are involved, the discretization in space of both equations is uncoupled. See, *e.g.*, [16] for a compressible Bingham system with variable density. Nevertheless, we propose a spatial discretization that contains a coupling between previous equations. As we mentioned previously, we want to obtain well-balanced numerical schemes. Basically, the difficulty in treating the spatial discretization in the present model comes from the extra source terms, which depends on $\mu$ and $q$.

In this section, we want to design a numerical scheme that preserves the following stationary solutions which can be encountered when using the present model, namely:

- Case $\eta_2 = 0$: the model degenerates to Shallow Water equations (SWE) and the so called "water at rest" test case – where the velocity is equal to zero and the free surface is horizontal – is a classical solution. We thus want our scheme to degenerate to one of the well-balanced schemes for SWE;
- Case $\eta_2 \neq 0$:
  - Case "material at rest": the solution to be captured for all $\eta_2 > 0$ is such that the velocity is equal to zero and the free surface is horizontal (*cf.* Figure 1);
  - Case where the free surface has a constant height on an inclined slope (*cf.* Figure 2) and the velocity is equal to zero, which is also a stationary solution under condition (35).

We note that the solution of the "water at rest" case and the one of the "material at rest" case are actually the same. Since our global model degenerates to the SWE when $\eta_2 = 0$, we want that, not only the solution in the case of a free surface with a constant height on an inclined slope is rigorously captured by our scheme, but also that the solution of the "material at rest" case (which seems to be rarely studied in the context of a Bingham model). And it is worth noting that, on the one hand, designing a scheme that captures one or the other of these two solutions is quite easily achievable. On the other hand, a consistent scheme for the present model must preserve *both* solutions and the difficulty of its design lies behind this feature.

Let us go back for a while to the case $\eta_2 = 0$ – when the model reduces to SWE – and follow the paper [13]. Namely, we remark that for the Shallow Water model the source term linked to the topography is only present in the momentum equations and does not appear in the mass conservation equation. Nevertheless, in well-balanced schemes, the topography term induces a contribution in the discretization of the mass conservation equation.

Taking into account the "Shallow Water structure" (mentioned in Subsection 5.2) of the present model, we borrow the aforementioned idea by taking into account all the source terms of our momentum equation and plugging their contribution in the discretization of the mass conservation equation. In particular, and this is the key point of present approach, Lagrangian terms $\mu$ and $q$ will be in the discretization of the first equation of (50).

First, we rewrite the second equation of (50) as

$$H^n\overline{\rho_0}\mathrm{St}\frac{\boldsymbol{V}^{k+1}}{\Delta t} + \beta\boldsymbol{V}^{k+1} - \partial_x\left(H^n(\frac{4\overline{\eta_1(\rho_0)}}{\mathrm{Re}} + r)\partial_x\boldsymbol{V}^{k+1}\right)$$

$$= -H^n\left[\partial_x\left(\frac{\overline{\rho_0}(\boldsymbol{V}_0^n)^2}{2} + \frac{\varepsilon}{\mathrm{Fr}^2}H^n\overline{\rho_0}\cos\theta\right) - \overline{\rho_0}\mathrm{St}\frac{\boldsymbol{V}_0^n}{\Delta t} - \frac{1}{\mathrm{Fr}^2}\overline{\rho_0}\sin\theta\right]$$

$$+\partial_x(H^n(\mu^k - rq^{k+1})). \tag{51}$$

For the right-hand side of the previous equation, we denote by $b_i$ $\forall i$, a given approximation at the point $x_i$, *i.e.*,

$$b_i \approx \left\{ -H^n\left[\partial_x\left(\frac{\overline{\rho_0}(\boldsymbol{V}_0^n)^2}{2} + \frac{\varepsilon}{\mathrm{Fr}^2}H^n\overline{\rho_0}\cos\theta\right)\right.\right.$$

$$\left.\left.-\overline{\rho_0}\mathrm{St}\frac{\boldsymbol{V}_0^n}{\Delta t} - \frac{1}{\mathrm{Fr}^2}\overline{\rho_0}\sin\theta\right] + \partial_x(H^n(\mu^k - rq^{k+1}))\right\}\Bigg|_{x=x_i}. \tag{52}$$

In Subsection 5.4.1 we will introduce the approximation that we consider to define $b_i$.

Then, we define the vector $\boldsymbol{b}$ with all components, $\boldsymbol{b} := (b_i)_i$. And we solve the linear system $A\boldsymbol{V} = \boldsymbol{b}$, where $A$ is the matrix of the system induced by (51). To define $A$ we consider a second-order finite difference to approximate the left-hand side of (51) and it reads

$$\left(\frac{H_i^n\overline{\rho_0}St}{\Delta t} + \beta + \left(\frac{4\overline{\eta_1(\rho_0)}}{Re} + r\right)\frac{H_{i+1/2}^n + H_{i-1/2}^n}{\Delta x^2}\right)\boldsymbol{V}_i^{k+1}$$

$$-\left(\frac{4\overline{\eta_1(\rho_0)}}{Re} + r\right)\frac{H_{i+1/2}^n}{\Delta x^2}\boldsymbol{V}_{i+1}^{k+1} - \left(\frac{4\overline{\eta_1(\rho_0)}}{Re} + r\right)\frac{H_{i-1/2}^n}{\Delta x^2}\boldsymbol{V}_{i-1}^{k+1}$$

where $H_{i+1/2}^n = (H_i^n + H_{i+1}^n)/2$.

**5.4.1. Definition of *b*.** The definition of the right-hand side in the linear system is fundamental in the design of the numerical scheme. For example, in relation with the stationary solutions of the system with velocity zero, $\boldsymbol{b}$ must be zero for all components.

We use a finite volume method to define $\boldsymbol{b}$. In (52) we distinguish a component that is constant for the iterative algorithm in $k$:

$$-H^n\left[\partial_x\left(\frac{\overline{\rho_0}(\boldsymbol{V}_0^n)^2}{2}+\frac{\varepsilon}{\mathrm{Fr}^2}H^n\overline{\rho_0}\cos\theta\right)-\overline{\rho_0}\mathrm{St}\frac{\boldsymbol{V}_0^n}{\Delta t}-\frac{1}{\mathrm{Fr}^2}\overline{\rho_0}\sin\theta\right]. \tag{53}$$

And the first equation of (50) also contains the terms evaluated in $t=t^n$:

$$-St\frac{H^n}{\Delta t}+\frac{\partial(H^n\boldsymbol{V}_0^n)}{\partial x}. \tag{54}$$

In the following we denote by $F(H^n,\boldsymbol{V}^n)$ the flux function contained in equations (53)-(54):

$$F(W)=\begin{pmatrix} H\boldsymbol{V} \\ \overline{\rho}\boldsymbol{V}^2/2+\varepsilon H\overline{\rho_0}\cos\theta/\mathrm{Fr}^2\end{pmatrix},\quad \text{with}\quad W=\begin{pmatrix} H \\ \boldsymbol{V}\end{pmatrix},$$

and by $\phi$ a numerical flux function that approximates $F$. We begin by considering a family of numerical flux functions defined by (see [13]):

$$\phi_{i+1/2}=\phi(W_i,W_{i+1})=\frac{F(W_i)+F(W_{i+1})}{2}-\frac{1}{2}D_{i+1/2}(W_{i+1}-W_i), \tag{55}$$

where $D_{i+1/2}$ is a defined or semi-defined positive matrix. For example, the Lax-Friedrichs scheme corresponds to the definition $D_{i+1/2}:=\frac{\Delta x}{\Delta t}I$, where $I$ is the identity matrix. If we denote the Roe matrix by $\mathcal{J}_{i+1/2}$, then Roe method is obtained for $D_{i+1/2}=|\mathcal{J}_{i+1/2}|$, the absolute value of the Roe matrix associated to $F$. For the numerical results that we present in Section 6 we consider this scheme.

In the sequel, we denote by $\phi^H$ the first component of the numerical flux function and by $\phi^V$ the second one. The numerical flux function that we use is introduced in the following subsection and is defined in equation (59).

By using the notation introduced previously we propose the following approximation of (52) to define $b_i$:

$$b_i=-H_i^n\left[\frac{\phi_{i+1/2}^{V^n}-\phi_{i-1/2}^{V^n}}{\Delta x}-\frac{1}{\mathrm{Fr}^2}\overline{\rho_0}\sin\theta-St\frac{\overline{\rho_0}\boldsymbol{V}_0^n}{\Delta t}\right]+\frac{G_{i-1/2}^V+G_{i+1/2}^V}{2}$$

with

$$G_{i+1/2}^V=H_{i+1/2}^n\frac{\mu_{i+1}^k-rq_{i+1}^{k+1}-(\mu_i^k-rq_i^{k+1})}{\Delta x}. \tag{56}$$

**5.4.2. Approximation of $H^{n+1}$.** The first possible choice to define an approximation of $H^{n+1}$ is to use directly the first component of the flux function defined by (55), In this case we obtain

$$StH_i^{n+1}=StH_i^n+\frac{\Delta t}{\Delta x}(\phi_{i+1/2}^H-\phi_{i-1/2}^H). \tag{57}$$

In the following, we denote the numerical schemes obtained in this case by **(Non-WB 1)**, *i.e.*, the first non well-balanced scheme. Since, actually, this scheme does not preserve the two types of stationary solutions. Namely, it is easy to prove that the scheme preserves the stationary solutions with constant height over an inclined plane. But, it does not preserve the stationary solutions with horizontal free surface.

Following [13] we can conclude that the source term which introduces the topography must be taken into account in the definition of $\phi^H$. If we write

$$G_{\text{topo}} = \begin{pmatrix} 0 \\ -\frac{1}{\text{Fr}^2}\overline{\rho_0} \sin\theta \end{pmatrix},$$

then $\phi^H$ is defined as the first component of

$$\phi_{\text{topo},i+1/2} = \frac{F(W_i) + F(W_{i+1})}{2} - \frac{1}{2} D_{i+1/2}(W_{i+1} - W_i - \mathcal{J}_{i+1/2}^{-1} G_{\text{topo}}). \tag{58}$$

There are several techniques which are applied in the case where $\mathcal{J}_{i+1/2}$ is not invertible. For instance, one can define the eigenvalues of the generalized inverse matrix by zero if the corresponding eigenvalue to be inverted is null or smaller than a certain value of tolerance.

We can now introduce a second choice by defining $\phi^H$ as the first component of (58). This leads to another numerical scheme, which is denoted as **(Non-WB 2)**, since it is not a well-balanced scheme. As a matter of fact, by introducing the technique proposed in [13] for SWE, we obtain that the scheme preserves the stationary solutions with horizontal free surface. But a constant height over an inclined plane is not a stationary solution of SWE, and we can prove that the obtained numerical scheme is not able to preserve these solutions.

Finally, we propose another discretization to define $\phi^H$ which leads to a scheme denoted as **(WB-B)**. As we mentioned at the beginning of this section, the main difference between the scheme that we propose and previous ones is to treat Lagrangian variables $\mu$ and $q$ in the same manner as in the well-balanced schemes for SWE. Consequently, we propose to define $\phi_H$ by taking into account the term defined in a function of $\mu + r\,q$ as a source term. Namely, if the iterative algorithm ends for index $k_e$, we approximate previous terms by $\mu^{ke+1} + r\,q^{ke+1}$. And we thus define $\phi^H$ as the first component of

$$\phi_{\mu,q,i+1/2} = \frac{F(W_i) + F(W_{i+1})}{2} - \frac{1}{2} D_{i+1/2}(W_{i+1} - W_i - \mathcal{J}_{i+1/2}^{-1}(G_{topo,i+1/2} + G_{\mu,q,i+1/2})) \tag{59}$$

where

$$G_{\mu,q,i+1/2} = \begin{pmatrix} 0 \\ H_{i+1/2}^n(\mu_{i+1}^{ke+1} - rq_{i+1}^{ke+1}) - H_{i-1/2}^n(\mu_i^{ke+1} - rq_i^{ke+1}) \end{pmatrix}.$$

**5.4.3. Well-balanced properties of the proposed scheme.** If we consider a domain of length $L$, we obtain the following result:

**Theorem 5.1.** *If we consider the following initialization for $\mu$ and $q$:*

$$\mu(x) = \frac{-1}{\mathrm{Fr}^2}\overline{\rho_0}\sin\theta\,(x - L/2) - \frac{\varepsilon}{\mathrm{Fr}^2}\overline{\rho_0}\cos\theta(H(x) - H(L/2)), \quad q(x) = 0 \quad x \in [0, L],$$

*then, the numerical scheme* **(WB-B)** *exactly preserves the stationary solution of material at rest, and also exactly preserves the stationary solution of constant height over an inclined plane verifying* (35)*.*

*Proof.* To prove this result, it is enough to prove the following two items:

i) $\boldsymbol{b} = 0$. In this case the solution of the linear system is $\boldsymbol{V}^{k+1} = 0\ \forall k$. So, if $\boldsymbol{V}^n = 0$ we obtain that $\boldsymbol{V}^{n+1} = 0$.

ii) $H^n_{i+1} - H^n_i - [\mathcal{J}^{-1}_{i+1/2}(G_{topo,i+1/2} + G_{\mu,q,i+1/2})]_1 = 0\ \forall i$. In this case, since $\boldsymbol{V}^{n+1} = 0$, we obtain that $\phi^H_{i+1/2}$ defined by (59) is zero for all $i$, then by (57) we have that $H^{n+1} = H^n$.

Consequently, if i) and ii) are verified, the given stationary solution is exactly preserved. The verification of i) and ii) is an easy computation, so that, for sake of brevity, we omit it. □

Although we refer to two types of stationary solutions, in Section 6 we observe that the proposed scheme preserves other types of stationary solutions, even when there is a bump in the free surface and the Bingham number is relatively huge. In the Test 1 presented in Section 6, we compare the results obtained with **(WB-B)** and the non-well-balanced schemes **(Non-WB 1)** and **(Non-WB 2)**.

## 6. Numerical tests

In this section we present three numerical tests. In the first one we study the convergence of a rectangular pulse towards the stationary solution, and the dependence of the stationary state on various Bingham numbers is explored. In the second numerical test we present the transition between two stationary solutions when the rigid properties of the material change. In the third test we present the case of an avalanche over the considered domain.

For the tests we set the parameters $St = 1$, $B = 1$, $\varepsilon = 1$, $\eta_1 = 1$ and $Fr = 0.3193$. Moreover, CFL condition is equal to 0.8, $r = \Delta x\,\eta_2/\eta_1$ and $\Delta x = 0.01$. We consider different values of $\eta_2$ in the following numerical tests. For the boundary conditions we impose the velocity to be zero.

### 6.1. Test 1: convergence to a stationary solution

In this subsection we present a test where the free surface of the initial condition is a rectangular pulse and the initial velocity is equal to zero in the entire domain. Furthermore, the bottom is supposed to be an inclined plane with an angle of 5

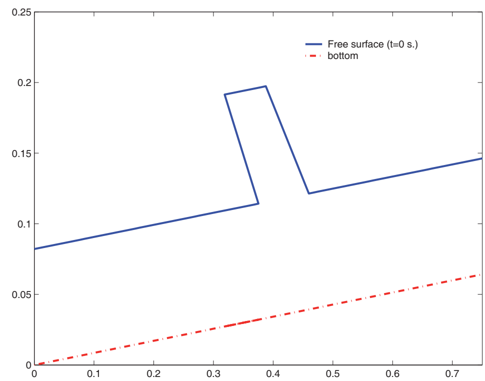



FIGURE 3. Test 1: Initial condition

degrees (*cf.* Figure 3). We study the final stationary profile of the material surface and the dependence of the shape with respect to different Bingham numbers.

The domain considered is $[0,1]$ and $\Delta x = 0.01$. And the height of the material is defined by:

$$H(x,0) = \begin{cases} 0.2 & \text{if } x \in [0.5, 0.6], \\ 0.1 & \text{otherwise.} \end{cases}$$

We study the evolution towards a stationary solution for different values of the Bingham number. Namely, we consider $\eta_2 = 10,\ 2,\ 0.1$ and $0.01$. The goal is to study the rigid properties of the material with respect to the Bingham number. When the Bingham number is nearly zero, the material is more similar to a fluid like the water. In fact, it can be remembered that when $\eta_2 = 0$ the proposed model reduces to Shallow Water equations with viscosity terms.

First, in Figure 4, we present the evolution at four different times of the material surface for $\eta_2 = 10$. Figure 4(d) corresponds to $t = 5$ s., where the solution is stationary. We observe that the stationary solution presents a bump, *i.e.* the material is sufficiently rigid to support the gradient of pressure produced by the gradient of the surface.

In Figure 5 we compare the numerical result obtained for $\eta_2 = 10$ with the proposed well-balanced scheme (continuous black line) and the results obtained without a well-balanced treatment for the discretization of $H$. Namely, we compare with the schemes **(Non-WB 1)** (red doted lines of Figure 5) and **(Non-WB 2)** (blue dashed lines of Figure 5) presented in Section 5. Any of both schemes are not able to preserve the stationary solution with a bump in the free surface.

In Figure 6 we present the evolution of the surface for the four different values of $\eta_2$ ($\eta_2 = 10,\ 1,\ 0.1$ and $0.01$). In Figure 7 we present the stationary surfaces obtained with these values of $\eta_2$.

If we compare Figures 7(a) with 7(b), corresponding to $\eta_2 = 10$ and $\eta_2 = 2$, respectively, we observe that in both cases the stationary solution presents a bump in the surface. Nevertheless the bump is smaller when $\eta_2$ decreases.

Figures 7(c) and 7(d) correspond to $\eta_2 = 0.1$ and $\eta_2 = 0.01$, respectively. We observe that when $\eta_2$ converges to zero, the stationary solution converges to the stationary solution of water at rest over an inclined plane. Namely, they converges to the stationary solution that we obtain for this test with Shallow Water equations, *i.e.*, the model obtained for $\eta_2 = 0$.

In Figure 8 we present the velocity at different times obtained with the four different values of $\eta_2$. We observe that the smaller values of velocities are obtained when the Bingham number is greater. Namely, for $t = 0.5$ s. (Figure 8(a)) we observe that the velocity is nearly zero for $\eta_2 = 10$. And for $t = 1$ s. (Figure 8(a)) the solution corresponding to $\eta_2 = 10$ is vanishing, while for the other values of $\eta_2$ the velocity is, by comparison, far from zero. For $t = 5$ s. (Figure 8(c)) the solution for $\eta_2 = 2$ exhibits a vanishing motion. We observe that for $t = 20$ s. (Figure 8(d)) the velocities corresponding to $\eta_2 = 0.1$ and $\eta_2 = 0.01$ tend much more slowly to zero than the ones for $\eta_2 = 2$ and $\eta_2 = 10$.

Finally, in Figure 9 we present two comparisons. In Figure 9(a), we compare the four stationary solutions corresponding to the fours considered values of $\eta_2$ . And, in Figure 9(b), we present the values of $\mu$ obtained at the final time, when the solution is stationary, for these values of $\eta_2$. We can easily identify that $\mu$ follows the same type of profile as the surface. Actually, as we mentioned previously, $\mu$ is a term whose role is to introduce an equilibrium in the pressure. If we observe for instance the case $\eta_2 = 10$, the material surface exhibits a gradient, which in the model of Shallow Water corresponds to a pressure gradient. This pressure gradient for stationary solutions is compensated, in the proposed model, by $\mu$. That is why, in the case of a stationary solution, the profile of the surface is of the same type as the one of $\mu$. Equivalently we observe that for $\eta_2 = 0.01$ where the material surface is nearly flat, the gradient of pressure is nearly zero, then the pressure gradient which has to be compensated is small, hence the profile of $\mu$ is close to zero.

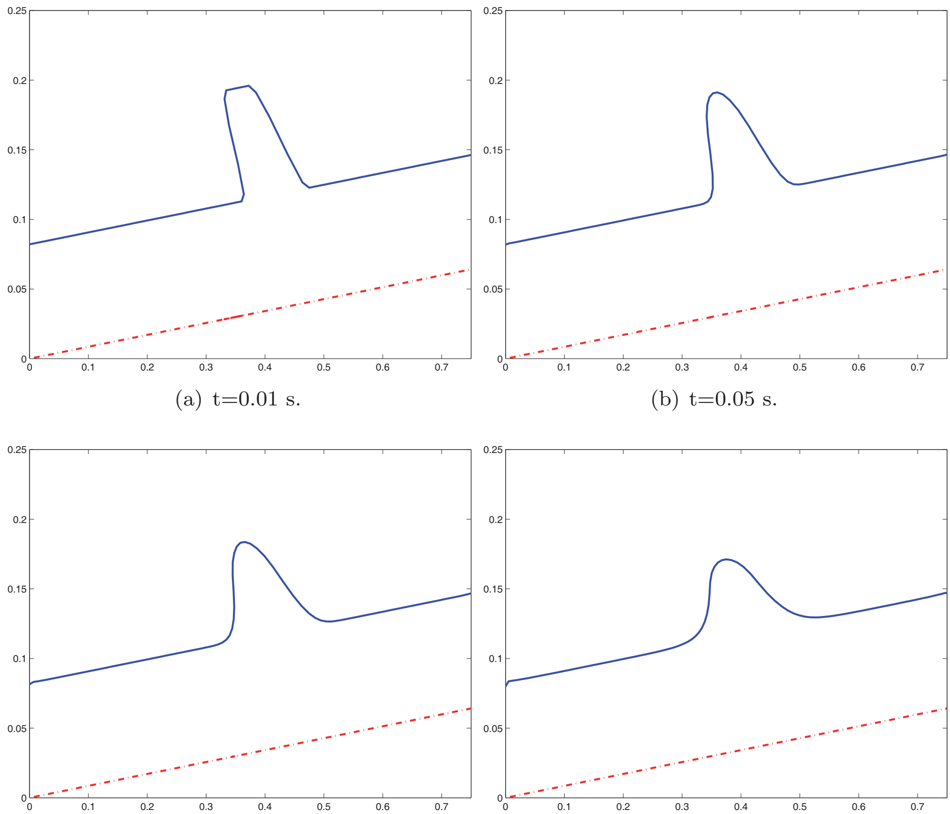
### 6.2. Test 2: Transition between two stationary solutions

The objective of this test is to observe the behaviour of the material when its rigid properties change. For instance, if we think about the snow, many phenomena can modify this material and introduce this type of change in its properties.

We consider a domain of 2 meters over an inclined slope of 20 degrees. First we compute numerically the stationary solution for $\eta_2 = 10$ when the initial condition is

$$H(x,0) = \begin{cases} 2.1 & \text{if } x \in [1.5, 1.6], \\ 0.1 & \text{otherwise.} \end{cases}$$

Then, we consider this stationary solution as initial condition for the same problem, except we change the value of $\eta_2$. Namely, we consider $\eta_2 = 5$ and $\eta_2 = 2$. In Figure 10, we see the transitions between the stationary solution (continuous black line, labeled “initial condition”) when we change the rigid properties of the material.



(a) t=0.01 s. (b) t=0.05 s.

(c) t=0.1 s. (d) t=5 s.

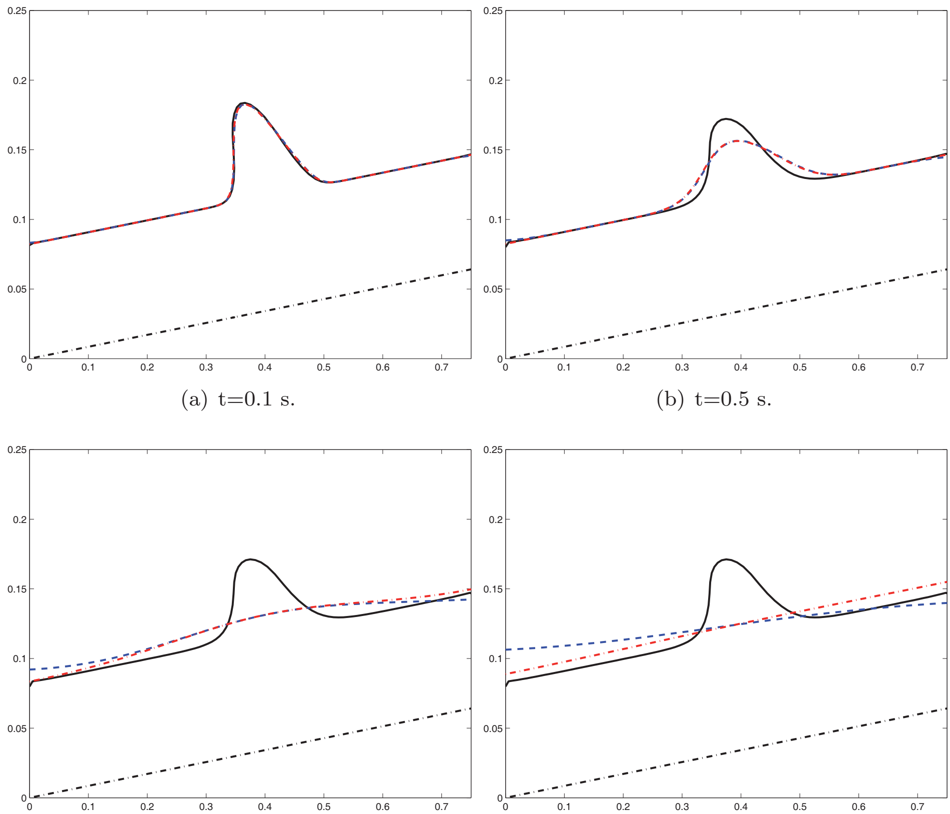
FIGURE 4. Test 1: Evolution of the material surface $\eta_2 = 10$

For $\eta_2 = 2$ (Figure 10(a)) we observe that the transition is from a bump to a stationary solution defined by two areas of horizontal free surface and an inclined surface connecting these areas. For $\eta_2 = 5$ (Figure 10(b)) the transition leads to a similar bump shape with only a change of the height and the width of the bump.

### 6.3. Test 3: avalanche

In this test we consider a domain of 10 meters. The final time is $T = 2$ s. The angle of the inclined plane that defines the bottom is 30 degrees, and $\eta_2 = 10$. As initial conditions we consider a velocity which is equal to zero in all the domain and $H$ defined by

$$H(x,0) = \begin{cases} 10.1 & \text{if } x \geq 9.5, \\ 0.1 & \text{otherwise.} \end{cases}$$



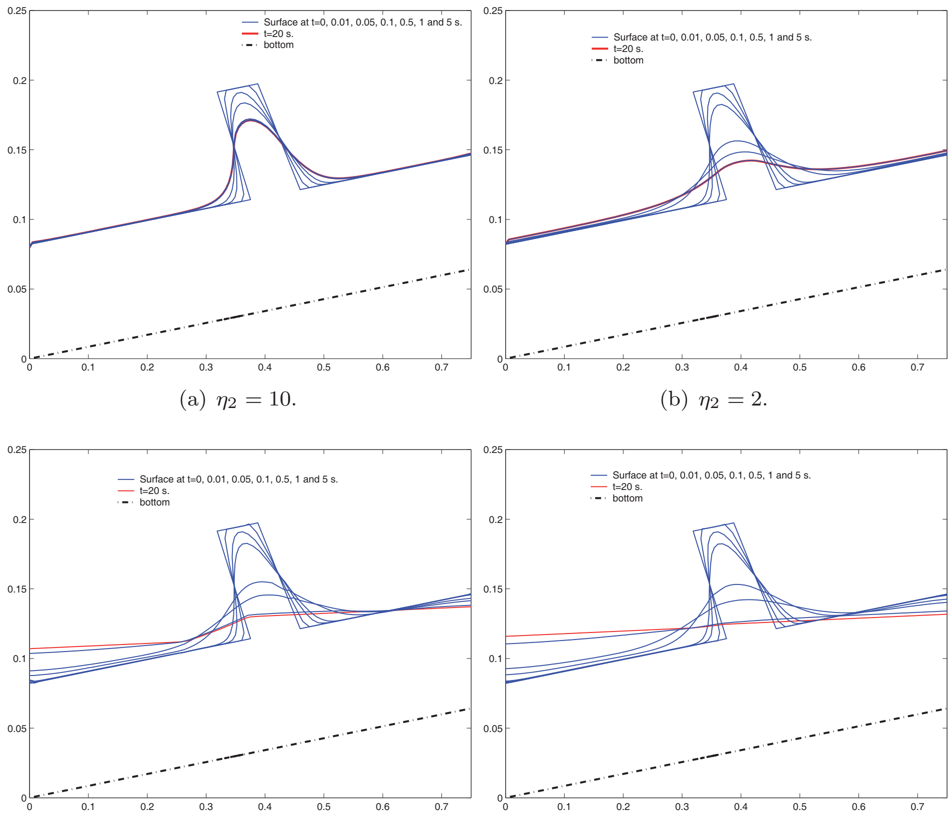

(a) t=0.1 s. (b) t=0.5 s.

(c) t=5 s. (d) t=20 s.

FIGURE 5. Test 1: Evolution of the material surface $\eta_2 = 10$. Comparison between the numerical result of the well-balanced scheme (continuous black line) and the non well-balanced schemes (Non-WB 1) (discontinuous red line) and (Non-WB 2) (dashed blue line)

In Figure 11 we present the evolution of the free surface for $t = 0.3$, 0.5, 1, 1.5 and 2 s. We observe that contrary to the two previous tests for the same value of $\eta_2$, in this case we do not obtain a stationary solution with a bump shape. There are several factors that induce the avalanche of the material in this test. First, the angle that defines the bottom is bigger in this case. Second, the height of the jump in $H$ at the initial condition. And finally, by condition (35) we observe that the length of the domain is also important. Even in the zone where the height of the material is constant, in $x \in [0, 9.5)$, the solution is not stationary.

In Figure 12 we compare the surface at $t = 1$ s. and $t = 2$ s. with the value of the Lagrange multiplier $\mu$. To picture both quantities in the same range we have




(a) $\eta_2 = 10$.  (b) $\eta_2 = 2$.

(c) $\eta_2 = 0.1$.  (d) $\eta_2 = 0.01$.

FIGURE 6. Test 1: Evolution of the free surface for $\eta_2 = 10$, 2, 0.1 and 0.01

multiplied $\mu$ by 0.3. We observe that the value of $\mu$ tends to follow the profile of the surface, in order to compensate the gradient of pressure. Nevertheless in this case, contrary to the Test 1, it is not sufficient to block the flow and an avalanche of the material occurs all over the domain.

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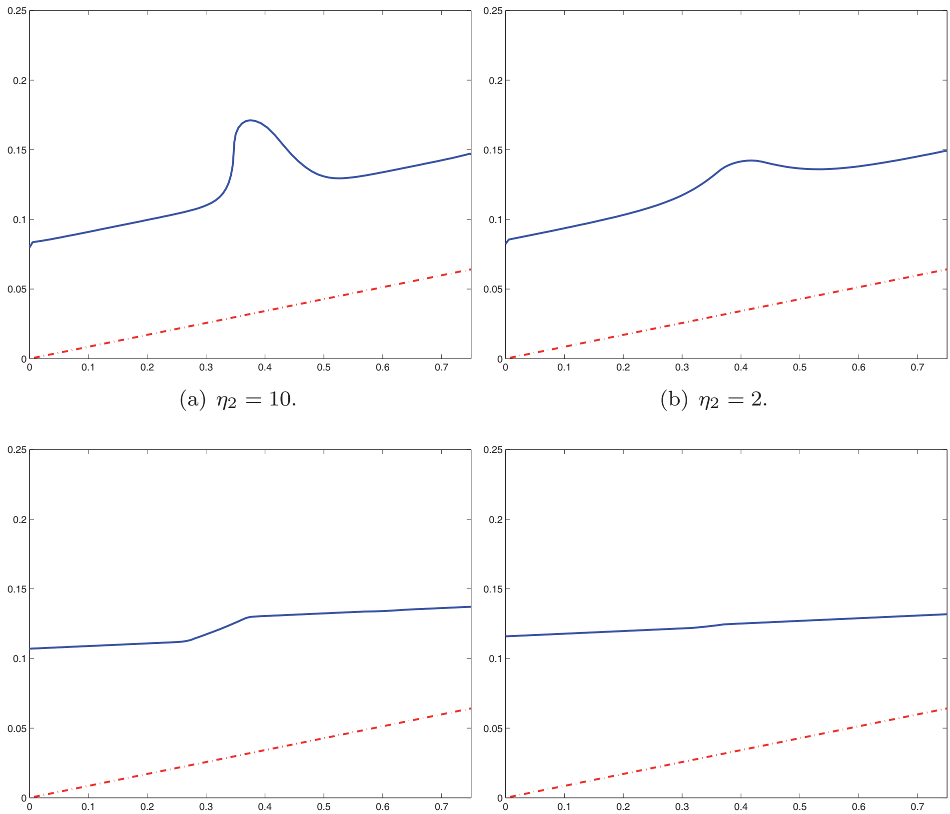


(a) $\eta_2 = 10$. (b) $\eta_2 = 2$.

(c) $\eta_2 = 0.1$. (d) $\eta_2 = 0.01$.

FIGURE 7. Test 1: Material surface of the stationary solutions obtained with $\eta_2 = 10$, 2, 0.1 and 0.01

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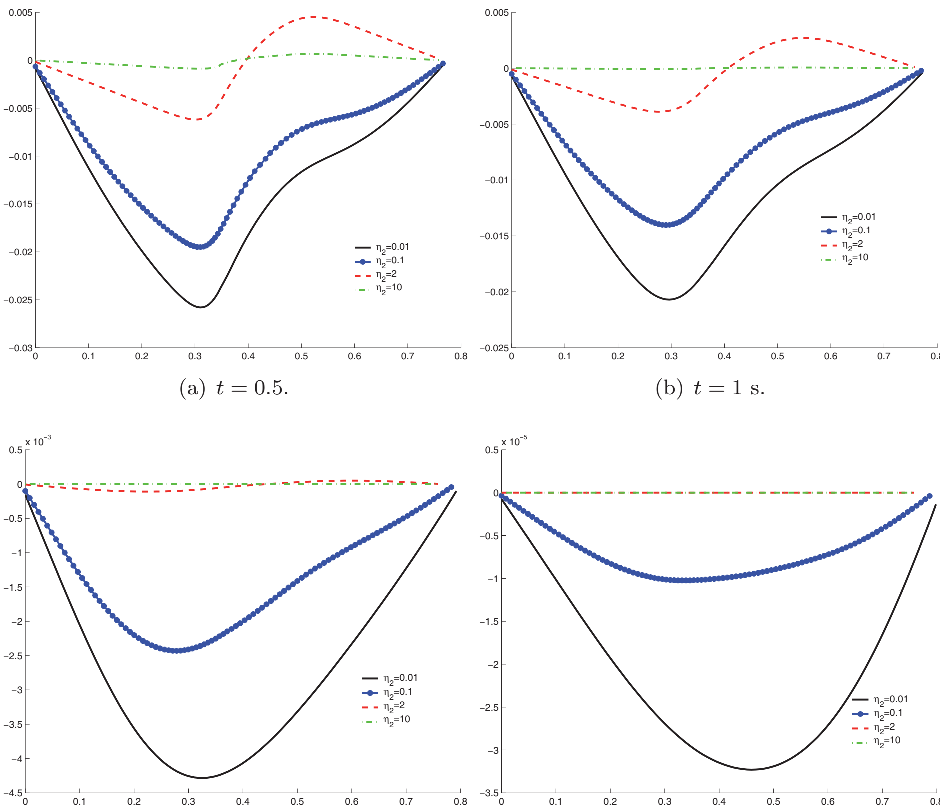


(a) $t = 0.5$.

(b) $t = 1$ s.

(c) $t = 5$ s.
(Range of the $y$-axe multiplied by $10^{-3}$)

(d) $t = 20$ s.
(Range of the $y$-axe multiplied by $10^{-5}$)

Figure 8. Test 1: Velocities obtained for $\eta_2 = 10$, 2, 0.1 and 0.01

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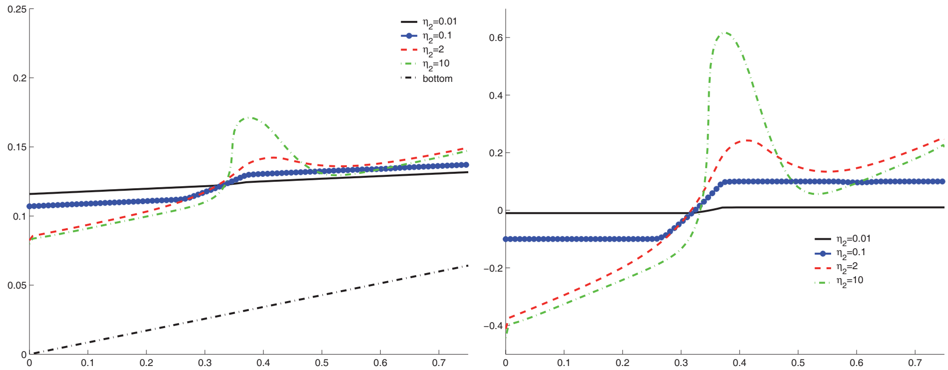
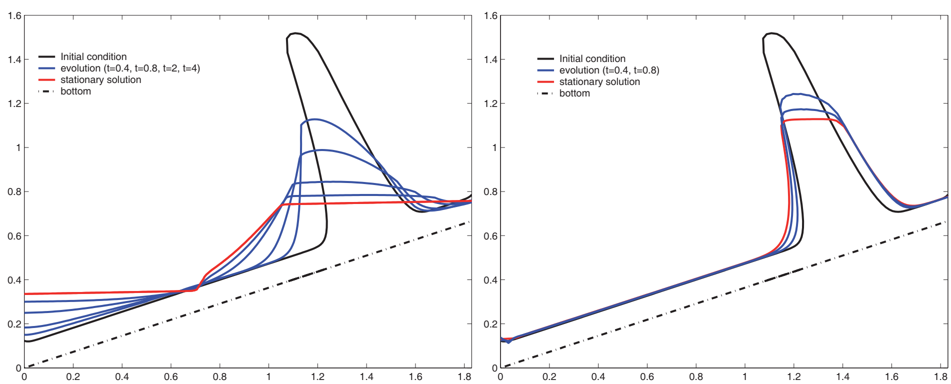



(a) Material surface. (b) $\mu$.

FIGURE 9. Test 1: Left: Material surface, Right: $\mu$, for the stationary solutions obtained with $\eta_2 = 10$, 2, 0.1 and 0.01




(a) $\eta_2 = 2, \theta = 20^o$. (b) $\eta_2 = 5, \theta = 20^o$.

FIGURE 10. Test 2: Free surface. Transition between two stationary solutions. Continuous black line: first stationary solution (used as initial condition for the second run, see text). Blue lines: evolution of the free surface. Red line: second stationary solution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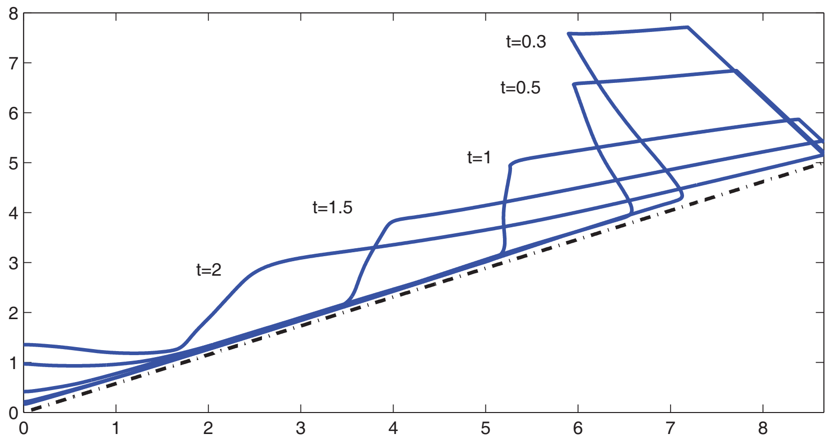


FIGURE 11. Test 3: avalanche. Surface evolution

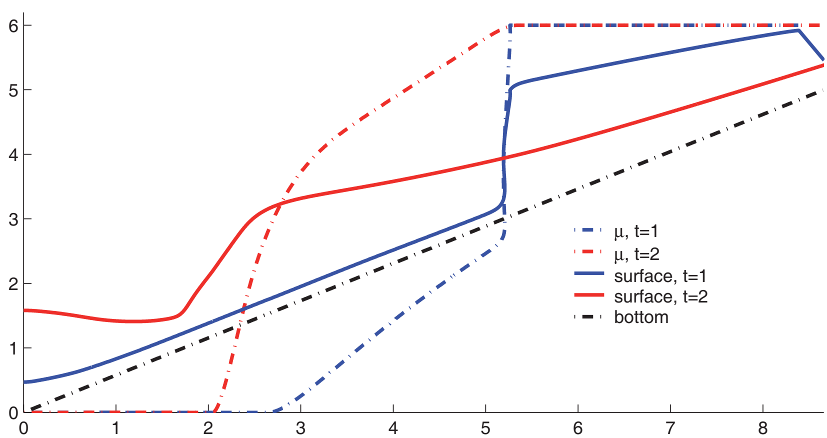


FIGURE 12. Test 3: Comparison of the free surface and the Lagrange multiplier

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# Oscillatory Limits with Changing Eigenvalues: A Formal Study

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*To the memory of Professor Alexander V. Kazhikhov: A mentor and a friend.*

**Abstract.** This paper deals with oscillatory limits with changing eigenvalues, more precisely with possibly crossing eigenvalues in space dimension greater than 1. The goal being to underline the various difficulties, to analyze them formally and present some related mathematical results obtained recently by the authors.

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## 1. Introduction

The aim of this paper is to study formally the limit $\varepsilon \to 0$ of dynamical systems of the form

$$\partial_t u^\varepsilon = \frac{1}{\varepsilon} A(S^\varepsilon) u^\varepsilon + Q(S^\varepsilon, u^\varepsilon), \tag{1}$$

$$\partial_t S^\varepsilon = F(S^\varepsilon, u^\varepsilon), \tag{2}$$

with initial data

$$u^\varepsilon(t=0) = u_0^\varepsilon, \quad S^\varepsilon(t=0) = S_0^\varepsilon, \tag{3}$$

where $A(S)$ is a linear skew-symmetric operator, and $Q(S,u)$, $F(S,u)$ are quadratic in $S$ and $u$.

The main novelty in our study is that we assume that $u^\varepsilon(t)$ takes its values in some Hilbert space $H_1$ and that $S^\varepsilon(t)$ takes its values in some Hilbert space $H_2$. Note that the asymptotic limit $\varepsilon \to 0$ of such a dynamical system has been justified (see [16]) in the finite-dimensional case, namely assuming that $S \mapsto A(S)$ is $\mathcal{C}^\infty$ from $\mathcal{O} \subset \mathbb{R}^n$ to the set of $N \times N$ real matrices, $F$ being an $n \times N$ matrix, which is $\mathcal{C}^\infty$ in $S \in \mathcal{O}$.

Note that because of the $1/\varepsilon$ factor in (1), it is in general difficult to prove existence of solutions $(u^\varepsilon, S^\varepsilon)$ on a time interval $[0, T]$ with $T$ independent of $\varepsilon$, and even more difficult to get uniform bounds. For instance for non-isentropic compressible Euler equations, this work has been done by G. MÉTIVIER and S. SCHOCHET in [16]–[17] in the whole space $\mathbb{R}^n$, in the periodic case $\mathbb{T}^n$. These results have been extended to exterior domains in [2]. Such existence issues are not addressed in this paper and we will assume the following:

*First assumption: There exist Hilbert spaces $H_1'$ and $H_2'$ compactly imbedded in $H_1$ and $H_2$ respectively, and solutions $(u^\varepsilon, S^\varepsilon)$ which are uniformly bounded in $L^\infty(0,T;H_1') \times L^\infty(0,T;H_2')$ for some $T > 0$ and for $0 < \varepsilon \leq 1$.*

The main problem is now to pass to the limit in $\varepsilon$, using compactness arguments. Part of compactness is ensured by compact embedding of $H_1'$ in $H_1$ and of $H_2'$ in $H_2$, however time compactness is lacking since, *a priori*, $\partial_t u^\varepsilon$ is not bounded. In fact, $\partial_t u^\varepsilon$ may be of order $1/\varepsilon$, which reveals a small time scale, of order $O(\varepsilon)$. The solution $u^\varepsilon$ may undergo very large variations on small times of order $O(\varepsilon)$. At these scales the dynamics is given by interaction between the time derivative and the skew-symmetric operator (the only two terms of order $O(1/\varepsilon)$ will be justified later).

The aim of this paper is to clarify the limit equation, to underline the various problems and to analyze them formally. No rigorous results are derived here, only a general framework of analysis, which remains to be justified, indicating how to generalize finite-dimensional results to our Hilbert setting. Throughout the paper, we present some related mathematical results obtained recently by the authors that could be helpful in justifying the various steps that will be presented in a forthcoming paper.

*Motivation.* This study is motivated by the low Mach number limit for non-isentropic flows

$$\partial_t \rho + u \cdot \nabla \rho + \rho \operatorname{div} u = 0, \tag{4}$$

$$\rho(\partial_t u + u \cdot \nabla u) + \nabla p = 0, \tag{5}$$

$$\partial_t S + u \cdot \nabla S = 0, \tag{6}$$

where $\rho$ is the density of the fluid, $u$ its velocity and $S$ its entropy. The pressure $p$ is implicitly given by the equation of state

$$\rho = R(p, S). \tag{7}$$

For instance, for an ideal gas,

$$\rho = p^{1/\gamma} e^{-S/\gamma}.$$

For many flows, the Mach number, ratio of the typical speed of the flow, divided by the speed of sound, is very small, and it is usual to consider it as a small parameter and to let it go to 0. After transformation of the equations into non-dimensional form and turning to $(p, u, S)$ as unknowns, this gives

$$A\partial_t p + u \cdot \nabla p + \mathrm{div} u = 0, \tag{8}$$

$$\rho(\partial_t u + u \cdot \nabla u) + \frac{1}{\varepsilon^2}\nabla p = 0, \tag{9}$$

$$\partial_t S + u \cdot \nabla S = 0, \tag{10}$$

where $\varepsilon$ is the Mach number and

$$A = \frac{1}{R}\frac{\partial R}{\partial p}.$$

Two problems arise: first, to prove existence of solutions to (8, 9, 10) uniformly bounded in $\varepsilon$, second, to study the limit of these solutions.

The first point has been solved by G. Métivier and S. Schochet, see [16], who, using refined energy estimates, proved existence of solutions $u^\varepsilon$ which remain bounded in $L^\infty(0, T; H^s(\Omega))$ for some $T > 0$, uniformly in $\varepsilon$.

The present work addresses the second point. Let us recall the classical formal analysis. The first step is to write that, as $\varepsilon$ goes to 0, $\nabla p$ must go to 0, hence $p$ goes to some constant $\bar{p}$. Following [17], we introduce $q$ defined by $p = \bar{p}e^{\varepsilon q}$ which gives

$$a(\partial_t q + u \cdot \nabla q) + \frac{1}{\varepsilon}\mathrm{div} u = 0, \tag{11}$$

$$r(\partial_t u + u \cdot \nabla u) + \frac{1}{\varepsilon}\nabla p = 0, \tag{12}$$

$$\partial_t S + u \cdot \nabla S = 0, \tag{13}$$

where $a$ and $r$ are two smooth functions of $S$ and $\varepsilon q$. As $\varepsilon$ goes to 0, Equation (8) degenerates into

$$\mathrm{div} u = 0, \tag{14}$$

and Equation (9) only gives that $\rho(\partial_t u + u \cdot \nabla u)$ is a gradient

$$\rho(\partial_t u + u \cdot \nabla u) + \nabla\pi = 0, \tag{15}$$

and (10) is unchanged

$$\partial_t S + u \cdot \nabla S = 0. \tag{16}$$

Thus we expect solutions of (8, 9, 10) to converge to solutions of (14, 15, 16). However such a claim can not be true in a strong sense, since the initial data of (8, 9, 10) have no reasons to satisfy $\mathrm{div}\, u = 0$ or $\nabla p = 0$.

Initial data $u^0, p^0, S^0$ will be called "well-prepared" if

$$\mathrm{div}\, u^0 = 0, \qquad \nabla p^0 = 0. \tag{17}$$

For well-prepared initial data, strong convergence is expected (and may be proved directly). For ill-prepared data however, no strong convergence can take place. Physically this is linked to the generation of strong sound waves, which propagate in the fluid with high speed ($1/\varepsilon$ by definition of the Mach number). These waves are described by the leading terms of (11, 12, 13), which lead to introduction of the operator

$$\mathcal{A} = \begin{pmatrix} 0 & a^{-1}\nabla\cdot \\ r^{-1}\nabla & 0 \end{pmatrix}. \tag{18}$$

*Oscillatory limits with constant coefficients.* The study of singular limits involving large time oscillations, in the periodic space framework, was pioneered by S. Schochet in [19] for the low Mach number of isentropic Euler equations. It was then used by B. Babin, A. Mahalov and B. Nikolaenko and E. Grenier for high rotation limit of incompressible Navier–Stokes equations, see [5], [14]. Readers interested in mathematical results around rotating fluids are referred to the book [12]. We also mention [13] for the asymptotics of the solutions of hyperbolic equations with a skew-symmetric perturbation.

For all these singular limits however, the underlying wave equation has constant coefficients, and therefore a constant spectrum. It has therefore global solutions, which are obtained by merely taking the Fourier transform of the equation, and the solutions are in particular bounded in Sobolev spaces, uniformly in time. To pass to the limit, the general method consists in conjugating the non-linear solutions by the wave equation. This gives the time compactness which was lacking and then enables us to pass to the limit in strong senses on the conjugated solutions.

*Oscillatory limits with non-constant coefficients.* In our present case, the eigenvalues and the spectrum of $\mathcal{A}$ will depend on the solution itself. This is the case through $r$ and $a$ in the low Mach number limit for the non-isentropic flows case (note however that time derivatives of $r$ and $a$ are bounded uniformly in $\varepsilon$) which depend on the entropy $S$. This leads to a highly complex problem, since eigenvalues may cross. This is the main difficulty since when two eigenvalues cross at time $t_0$, the limit of quadratic terms may depend not only on the limit but also on the full sequence (even on subsequences). Some examples with such behaviors are described in [16] pages 136–140. These examples indicate that if the spectrum of the wave operator does not have constant multiplicity, the limit is likely not only determined by the limit implying that there is no closed system of equations for the limit. The wave equation, in the low Mach number limit example, may also be written in this case

$$\varepsilon\partial_{tt}\psi - \operatorname{div}(S^{-1}\nabla\psi) = 0. \tag{19}$$

It is thus important to prove, after defining appropriate infinite-dimensional measures, that for almost all initial data, the limit flow does not meet double eigenvalues and crosses the resonance set transversally. Concerning the low Mach number limit problem, in the very special case of only one spatial dimension, the limit can be calculated both completely and justified, see [16]. In the multi-dimensional case, the formal calculation of the extra term in the limit, which once again involves the spectral decomposition of the fast operator, assumes that the spectrum of that fast operator is simple and non-resonant, see [9] for the viscous case and [16] for the inviscid one. For certain finite-dimensional truncations of the equations, those assumptions can be shown to be generic and to ensure convergence to the limit equations. This has been done in the nice paper [16]. Our result extends this paper to the real PDEs system proving, after defining appropriate infinite-dimensional measures, that for almost all initial data, the limit flow does not meet double eigenvalues and crosses the resonance set transversally. Difficulties occur since the flow is singular across the double eigenvalues set (in some sense not uniquely determined on it). Readers interested in an introduction to transversal mappings and flows are referred to the nice book [1].

Note that in all $R^d$, if the initial data decay sufficiently rapidly at infinity, then the fast waves still decay quickly, so that the limit satisfies the stratified incompressible Euler equations in which the entropy, and hence also the density, remain non-constant, see [17]. An extension of this result to exterior domains has recently been obtained in [3]. The reader interested in a review around a low Mach number limit are referred to [18] and [4].

## 2. Wave equation

Our asymptotic study is related to the so-called wave equation

$$\partial_t v(t) = \varepsilon^{-1} A(S(t)) v(t) \tag{20}$$

and more precisely is related to the eigenvalues and eigenfunctions behavior related to the associated wave operator.

### 2.1. A simple case: The matrix $A$ independent of $S(t)$

The equation is simply

$$\partial_t v(t) = \frac{1}{\varepsilon} A v(t), \tag{21}$$

where $A$ is a skew-symmetric operator. In order to solve (21), we introduce the eigenvalues $i\lambda_k$ and eigenvectors $\psi_k$ of $A$ and decompose the initial data $u_0^\varepsilon$ onto the Hilbert basis

$$u_0^\varepsilon = \sum_k \alpha_k \psi_k. \tag{22}$$

The solution is then

$$u^\varepsilon(t) = \sum_k \alpha_k \exp\Big(\frac{i\lambda_k t}{\varepsilon}\Big)\psi_k. \tag{23}$$

Let $\mathcal{L}^\varepsilon(t) u_0^\varepsilon$ denote this solution. We then introduce

$$v^\varepsilon(t) = \mathcal{L}^\varepsilon(-t) u^\varepsilon \tag{24}$$

which satisfies

$$\partial_t v^\varepsilon = \mathcal{L}^\varepsilon(-t) Q(S^\varepsilon)\Big(\mathcal{L}^\varepsilon(t) u^\varepsilon, \mathcal{L}^\varepsilon(t) u^\varepsilon\Big). \tag{25}$$

If $\mathcal{L}^\varepsilon(t)$ is uniformly bounded from $H_1'$ to $H_1'$, then $v^\varepsilon$, lie $u^\varepsilon$, is bounded uniformly in $L^\infty(0, T, H_1')$. Using (25) we may get bounds on $\partial_t v^\varepsilon$, uniform in $\varepsilon$. Classical compactness arguments then allow us to pass to the limit in $v^\varepsilon$ and also in (24). This approach has been pioneered by S. Schochet for isentropic compressible

Euler equations. However this approach fails in the case of non-isentropic fluids, since the operator $A$ depends on the solution $S^\varepsilon(t)$ itself.

### 2.2. General case

In this case, eigenvalues and eigenvectors depend on $S$ and therefore on time. We may always introduce $\mathcal{L}^\varepsilon(t_0,t)v_0$, a solution of (20) which equals $v_0$ at time $t_0$, however the explicit expression (23) is no longer valid.

Let $S_1 \in H_2'$. Let $i\lambda_k(S_1)$ be the eigenvalues of $A(S_1)$, counted with multiplicity. If the eigenvalue $\lambda_k(S_1)$ is simple, then we may define locally a smooth unit eigenvector $\psi_k(S_1)$. If two eigenvalues $\lambda_j$ and $\lambda_k$ collide at $S_1$, locally the corresponding eigenvectors $\psi_j$ and $\psi_k$ are singular at $S_1$, and only $\mathrm{Span}(\psi_j,\psi_k)$ is smooth. Let now $S(t) \in L^\infty(0,T,H_2')$ be given.

*Simple eigenvalues.* If all the eigenvalues of $\mathcal{A}(S(t))$ are simple, we introduce the corresponding eigenvectors $\psi_k(S(t))$ and the decomposition of $v^0$ on the (orthonormal) basis

$$v_0 = \sum_k \alpha_k^\varepsilon(t_0)\psi_k(S(t_0)).$$

Let us look for solutions of (20) formed by eigenvectors $\psi_k(S(t))$ and the decomposition of $v$ on the (orthonormal) basis

$$v^\varepsilon(t) = \sum_k \alpha_k^\varepsilon(t)\psi_k(S(t)).$$

Equation (20) then gives

$$\sum_k \partial_t\alpha_k^\varepsilon(t)\psi_k(S(t)) + \sum_k \alpha_k^\varepsilon(t)\nabla\psi_k(S(t)).S'(t) = \sum_k i\lambda_k(S(t))\alpha_k^\varepsilon(t)\psi_k(S(t)).$$

Let us then introduce $\beta_k^\varepsilon(t)$ defined by

$$\beta_k^\varepsilon = \exp\Big(-i\int_{t_0}^t \lambda_k(S(\tau))d\tau\Big)\alpha_k^\varepsilon.$$

Then

$$\partial_t\beta_k^\varepsilon = -\Big(\sum_k \alpha_k^\varepsilon(t)\nabla\psi_k(S(t)).S'(t) \quad | \quad \psi_k\Big).$$

By stationary phase arguments, $\partial_t\beta_k^\varepsilon$ goes to 0 as $\varepsilon$ goes to 0, a well-known fact (adiabatic limit).

Let

$$\Lambda_k^\varepsilon(t) = \int_{t_0}^t \lambda_k(S(\tau))d\tau.$$

All these computations fail if some eigenvalue of $A(S(t))$ is double or multiple.

*Crossing eigenvalues.* When eigenvalues cross, energy may be exchanged between the corresponding modes and no closed equation can be given on the limit. For the current analysis it is therefore crucial to avoid multiple eigenvalues. Let

$$\Sigma_{j,k} = \{S \quad | \quad \lambda_j(S) = \lambda_k(S)\}.$$

### 2.3. Approximate resolvent

We now define an approximate resolvent $\mathcal{L}^{\varepsilon}_{\text{app}}(t_1,t_2)$ of the wave equation by, if

$$v_1 = \sum_k \alpha_k \psi_k(S(t_1)).$$

Then

$$\mathcal{L}^{\varepsilon}_{\text{app}}(t_1,t_2)v_1 = \sum_k \alpha_k \exp\Big(-i\int_{t_1}^{t_2} \lambda_k(S(\tau))d\tau\Big)\psi_k(t_2). \tag{26}$$

Let

$$\mathcal{L}^{\varepsilon}_{\text{app}}(t) = \mathcal{L}(0,t).$$

Note that $w^{\varepsilon}(t) = \mathcal{L}^{\varepsilon}_{\text{app}}(t)v_0$ is not a solution of the wave equation, but of

$$\partial_t w^{\varepsilon}(t) = \varepsilon^{-1}A(S(t))w^{\varepsilon}(t) + \mathcal{E}^{\varepsilon}(t)w^{\varepsilon}(t) \tag{27}$$

where the error term $\mathcal{E}^{\varepsilon}(t)$ is a linear operator, defined by

$$\mathcal{E}^{\varepsilon}(t)\psi_k(t) = \nabla\psi_k(S(t)).S'(t). \tag{28}$$

Note that $\mathcal{E}^{\varepsilon}$ depends on $S(t)$, on $S'(t)$ and is well defined away from the sets $\Sigma_{j,k}$.

## 3. Limit equation

### 3.1. Derivation of the limit equation

If $S^{\varepsilon}(t)$ does not cross any of the $\Sigma_{j,k}$, then we introduce

$$v^{\varepsilon}(t) = \mathcal{L}^{\varepsilon}_{\text{app}}(t,0)u^{\varepsilon}(t). \tag{29}$$

Then $v^{\varepsilon}$ also satisfies

$$\mathcal{L}^{\varepsilon}_{\text{app}}(t)\partial_t v^{\varepsilon} = Q(S^{\varepsilon})\Big(\mathcal{L}^{\varepsilon}_{\text{app}}(t)v^{\varepsilon}, \mathcal{L}^{\varepsilon}_{\text{app}}(t)v^{\varepsilon}\Big) + \mathcal{E}^{\varepsilon}(t)\mathcal{L}^{\varepsilon}_{\text{app}}(t)v^{\varepsilon}(t). \tag{30}$$

We will assume

*Second assumption: $Q$ is continuous from $H_1' \times H_1'$ to some negative Sobolev space $H^{-s_0}$.*

Note that this second assumption is very weak and generally very easy to check. The main problem now is that $\mathcal{L}^{\varepsilon}$ depends on the solution. If $S^{\varepsilon}$ remains away from the various $\Sigma_{j,k}$, then we expect $\mathcal{L}^{\varepsilon}$ to be bounded from $L^{\infty}(0,T,H_1')$ into itself, uniformly in $\varepsilon$. This claim will be explained in detail below.

With this claim, $\partial_t v^{\varepsilon}$ is bounded and therefore, up to the extraction of a subsequence, $v^{\varepsilon}$ converges to some function $v \in L^{\infty}(0,T,H_1)$. It remains to pass the limit in the right-hand side of (30).

For this we decompose $v^{\varepsilon}(t)$,

$$v^{\varepsilon}(t) = \sum_j \beta_j^{\varepsilon}(t)\psi_j^{\varepsilon}(t),$$

and use bilinearity of $Q$ to get

$$\mathcal{L}^\varepsilon_{\rm app}(t)\partial_t v^\varepsilon = \sum_{j,k} \beta^\varepsilon_j(t)\beta^\varepsilon_k(t) e^{-i\Lambda^\varepsilon_j(t)-i\Lambda^\varepsilon_k(t)} Q(S^\varepsilon)\Big(\psi^\varepsilon_j(t),\psi^\varepsilon_k(t)\Big) \qquad (31)$$
$$+ \sum_j \beta^\varepsilon_j(t) e^{-i\Lambda_j(t)} \mathcal{E}^\varepsilon(t)\psi^\varepsilon_j(t).$$

This leads to

$$\partial_t v^\varepsilon = \sum_{j,k,l} \beta^\varepsilon_j(t)\beta^\varepsilon_k(t) e^{i\Lambda^\varepsilon_l(t)-i\Lambda^\varepsilon_j(t)-i\Lambda^\varepsilon_k(t)} \Big(Q(S^\varepsilon)\Big(\psi^\varepsilon_j(t),\psi^\varepsilon_k(t)\Big)|\psi_l(t)\Big)\psi_l(t) \qquad (32)$$
$$+ \sum_{j,l} \beta^\varepsilon_j(t) e^{i\Lambda^\varepsilon_l(t)-i\Lambda^\varepsilon_j(t)} \Big(\mathcal{E}^\varepsilon(t)\psi^\varepsilon_j(t)|\psi^\varepsilon_l(t)\Big)\psi^\varepsilon_l(t).$$

Note that if we choose $\psi_j$ to be orthonormal in $L^2$, the last sum reduces to $j \neq k$.

Next let us define the resonant set, namely the set

$$\Sigma_{j,k,l} = \{S \quad | \quad \lambda_j(S) + \lambda_k(S) = \lambda_l(S)\}.$$

We have now to pass to the limit in (32). If we remain away from $\Sigma_{j,l}$ (see Section 6 and 7 for indications) the last term vanishes by a stationary phase argument. If we show that $\Sigma_{j,k,l}$ is a smooth manifold of codimension 1 and that for almost any initial data, the limit flow crosses it transversally (see Section 6), multiplying by $\psi_k(t)$ and using the orthonormality, we get

$$\partial_t \beta_l = \sum_{j,k,l\,;\,\lambda_j(S)+\lambda_k(S)=\lambda_k(S)} \beta_j(t)\beta_k(t)\Big(Q(S))\Big(\psi_j(t),\psi_k(t)\Big)|\psi_l(t)\Big). \qquad (33)$$

### 3.2. Structure of the limit equation

It is important to note that (33) is smooth outside the $\Sigma_{j,k}$ where it is singular, since $\psi_j$ and $\psi_k$ are singular. As in the finite-dimensional case, see [16]–[15], we do not expect to have results valid for any initial data, since in this case the initial data, or the limit, may cross some $\Sigma_{j,k}$. In this case energy exchange may take place between the various eigenvectors, or the initial data or the limit may cross $\Sigma_{j,k,l}$ in a tangential way, leading to a non-expected contribution to the limit behavior which depends on the way one passes to the limit. Thus, we have to prove that for almost any initial data, the limit flow avoids $\Sigma_{j,k}$.

## 4. Almost everywhere results

Since we search for "almost everywhere" results, this implies putting a measure on the functional spaces. Note however that this measure need neither to be invariant under the limit flow, as in recent works around supercritical wave equations (see [10]–[11]), nor to have a particular link to the limit equation or even to the underlying functional space.

To handle measures is just a way to study a group of solutions instead of a single solution. A single solution may have an exceptional behavior whereas a large number of solutions will follow the average behavior for almost all of them.

A first way to introduce measures is to choose some $N \geq 1$, to define an application $\Phi$ from the unit ball $B_1$ of $\mathbb{R}^N$ to $H^s$ or any functional space we want and to choose a measure $\mu$ on $B_1$. We may then consider the set of initial conditions $\Phi(B_1)$ which is naturally parameterized by $(B_1, \mu)$. We then have to study the position of $\Phi(B_1)$ and its image by the limit flow with respect to the resonant sets and to $\Sigma_{j,k}$. These latest sets being of codimension 1 or 2, they can contain $\Phi(B_1)$. Hence it is better to embed infinite-dimensional measures in our functional space.

This can be done in an easy way as described below, by introducing Besov spaces. Namely, let

$$B_s = B_s^{2,\infty}.$$

There are many different ways to define measures on $B_s(\Omega)$. If $\Omega = \mathbb{T}^d$ for instance, we introduce $(\psi_k)_{k\in\mathbb{N}}$, an orthonormal basis of eigenvectors of the Laplace operator. Any $u \in B_s$ may be decomposed,

$$u = \sum_k \alpha_k \psi_k,$$

and the $B_s$ norm is equivalent to $\sup_k |k|^s |\alpha_k|$. We will define the projector $P_N$ by

$$P_N u = \sum_{k\leq N} \alpha_k \psi_k.$$

Let $\mu$ be the Lebesgue measure on $\mathbb{R}$. We define

$$\mu_{s,R} = \otimes_k \Big( \frac{|k|^s \mu}{2R} \Big).$$

Let $B_s(R)$ be the ball of center 0 and radius $R$. Then

$$\mu_{s,R}(B_s(R)) = 1.$$

We also define

$$\mu_{s,R,N} = \otimes_{0\leq k\leq N} \Big( \frac{|k|^s \mu}{2R} \Big)$$

which is a measure on $\mathbb{R}^N$. Note that such kind of measures have been recently used in [6], [7] to get measure type estimates related to a crossing or resonant eigenvalues set.

## 5. Study of $\Sigma_{j,k}$

The aim of this section is to show formally that, under some general assumptions, $\Sigma_{j,k}$ is of codimension 2. This result will be important in the sequel to prove that for almost any initial data, the limit flow avoids this codimension 2 set.

Let $\mathcal{L}(x)$ denote a linear operator and $(\lambda_j(x))_{j\geq 0}$ its spectrum. To prove that $\Sigma$ is of codimension 2 we introduce

$$\phi_{j,k}(x) = \lambda_j(x) - \lambda_k(x).$$

First we have to check that $\lambda_j$ and $\lambda_k$ are Lipschitz continuous on $B_s$ and $B_\sigma$. Next we expand $\phi_{j,k}$. For this let us consider a small perturbation $\varepsilon\tilde{x}$ to $x$ and let $\tilde{\psi}$ and $\tilde{\lambda}$ be an eigenvector and an eigenvalue of $\mathcal{L}(x+\varepsilon\tilde{x})$. Let $\lambda_1$ and $\lambda_2$ be two eigenvalues and $\psi_1, \psi_2$ be a corresponding orthonormal basis of eigenvectors. We look for $\tilde{\psi}$ under the form

$$\tilde{\psi} = \hat{\psi} + \varepsilon\hat{\phi}$$

where $\hat{\psi} \in \mathrm{Span}(\psi_1, \psi_2)$ and $\phi$ is orthogonal to this vector space. Let $\Pi$ be the projection on $\mathrm{Span}(\psi_1, \psi_2)$. We have

$$\mathcal{L}(x+\varepsilon\tilde{x})(\hat{\psi}+\varepsilon\hat{\phi}) = \tilde{\lambda}(\hat{\psi}+\varepsilon\hat{\phi}).$$

Thus applying $\Pi$,

$$\Pi\mathcal{L}(x)\hat{\psi} - \tilde{\lambda}\hat{\psi} + \varepsilon\Pi\mathcal{L}(\tilde{x})\hat{\psi} = O(\varepsilon^2).$$

Hence $\tilde{\lambda}$ is an eigenvalue of the two-by-two array

$$\Pi\mathcal{L}(x) + \varepsilon\Pi\mathcal{L}(\tilde{x}) = \begin{pmatrix} \lambda_1 + \int \bar{\psi}_1\mathcal{L}(\tilde{x})\psi_1 & \int \bar{\psi}_2\mathcal{L}(\tilde{x})\psi_1 \\ \int \bar{\psi}_1\mathcal{L}(\tilde{x})\psi_2 & \lambda_2 + \int \bar{\psi}_2\mathcal{L}(\tilde{x})\psi_2 \end{pmatrix}.$$

Hence up to $O(\varepsilon^2)$ terms, $\tilde{\lambda}$ equals

$$\frac{\lambda_1 + \lambda_2 + L_1(\tilde{x}) \pm \sqrt{L_2(\tilde{x})^2 + 4|L_3(\tilde{x})|^2}}{2}$$

where

$$L_1(x) = \int \bar{\psi}_1\mathcal{L}(\tilde{x})\psi_1 + \int \bar{\psi}_2\mathcal{L}(\tilde{x})\psi_2,$$
$$L_2(x) = \int \bar{\psi}_1\mathcal{L}(\tilde{x})\psi_1 - \int \bar{\psi}_2\mathcal{L}(\tilde{x})\psi_2$$

and

$$L_3(x) = \int \bar{\psi}_1\mathcal{L}(\tilde{x})\psi_2,$$

expressions which are linear in $\tilde{x}$.

In particular,

$$\phi_{j,k}(x) = \sqrt{L_2(\tilde{x})^2 + 4|L_3(\tilde{x})|^2} + O(\varepsilon^2).$$

Now if $L_2$ and $L_3$ are not colinear, then locally $\phi_{j,k} = 0$ is of codimension 2.

## 6. Structure of a resonant set

Let

$$\phi_{j,k,l}(x) = \lambda_j(x) + \lambda_k(x) - \lambda_l(x),$$

which defines

$$\Sigma_{j,k,l} = \Big\{x \quad | \quad \phi_{j,k,l}(x) = 0\Big\}.$$

The normal to $\Sigma_{j,k,l}$ is

$$L_4(x) = \nabla_x \lambda_j + \nabla_x \lambda_k - \nabla_x \lambda_l$$

provided $L_4(x) \neq 0$. Hence provided $L_4(x) \neq 0$, $\Sigma_{j,k,l}$ is locally a manifold of codimension 1.

Resonances do not interfere in the limit equation provided they are transverse, namely provided

$$\nabla \phi_{j,k,l}(\Theta(t,x)).\partial_t \Theta(t,x) \neq 0$$

for any $x$ with $\Theta(t,x) \in \Sigma_{j,k,l}$. We introduce the distance $\phi(x)$ to the dangerous set

$$\phi(x) = \sqrt{\phi^2_{j,k,l}(x) + \Big(\nabla \phi_{j,k,l}(x).\partial_t \Theta(0,x)\Big)^2}.$$

Here we assume that the limit equation is autonomous. Note that

$$\phi_{j,k,l}(x + \varepsilon \tilde{x}) = \phi_{j,k,l}(x) + \varepsilon L_4(x)\tilde{x} + O(\varepsilon^2)$$

where

$$L_4(x) = \nabla \phi_{j,k,l}(x) = \nabla_x \lambda_j(x) + \nabla_x \lambda_k(x) - \nabla_x \lambda_l(x).$$

Next the differential of $\nabla \phi_{j,k,l}(x).\partial_t \Theta(0,x)$ is

$$L_5(x).\tilde{x} = \nabla^2 \phi_{j,k,l}(x).\tilde{x}.\partial_t \Theta(0,x) + \nabla \phi_{j,k,l}(x).\partial_t \nabla_x \Theta(0,x).\tilde{x}.$$

The first term takes into account the curvature of $\Sigma_{j,k,l}$ and the second one the spatial dependence of the flow $\Theta$.

If $L_4$ and $L_5$ are not colinear then locally $\phi = 0$ is of codimension 2. Hence for almost any initial data the limit flow does not meet double eigenvalues and crosses the resonant set transversally.

## 7. Formal justification of the limiting process

When the solution crosses

$$\Sigma_{j,k} = \{u \quad | \quad \lambda_j(u) = \lambda_k(u)\},$$

then the modes $j$ and $k$ may exchange energy, and this exchange may be arbitrary, and as a matter of fact the limit equation appears to be singular at $\Sigma_{j,k}$. This means that the limit of the quadratic term in Equation (33), namely

$$\lim_{\varepsilon \oslash 0} \sum_{j,l} \beta^\varepsilon_j e^{i(\Lambda^\varepsilon_l(t) - \Lambda^\varepsilon_j(t))} \Big(\mathcal{E}^\varepsilon(t)\psi^\varepsilon_j(t)|\psi^\varepsilon_l(t)\Big)\psi^\varepsilon_l(t),$$

is not likely determined by the limit $(v, S)$, implying that there is no closed system of equations for $(v, S, \Phi)$.

The limit solution must therefore avoid the double set $\Sigma_{j,k}$. As described in Section 5, the set $\Sigma_{j,k}$ is of codimension 2 in the $u$ space. Thus we have to prove that, for almost any initial data, the flow of the limit equation avoids $\Sigma_{j,k}$. The main difficulty of the proof lies in the fact that the limit flow is not Lipschitz, hence classical existence theorems do not apply. Indeed the limit flow is precisely singular on the resonant sets $\Sigma_{j,k}$. We also have to define what we mean by "for almost every initial data", a notion not so obvious since we are in infinite dimension. This study will be linked to the recent paper [8], written by the authors, where a singular ordinary differential equation, in the Hilbert setting, homogeneous of degree 0 near a codimension 2 set, is studied. More precisely, defining $H$ to be a Hilbert space, the authors study the dynamical system

$$\dot{\phi} = \phi\left(x, \frac{x_h}{|x_h|}\right)$$

where $\phi$ is a smooth function defined on $H \times S^1$ ($S^1$ being the unit circle). The Hilbert space $H$ is divided into two parts: a given plane $P$ identifying it to $\mathbb{C}$ and its orthogonal $P^{\perp}$ which plays the role of the codimension 2 singular set. We note $x_h = \Pi x$ and $x_v = x - \Pi x$. Under some structural assumptions on the smooth function, we prove that for almost all data there exists a unique global solution of this singular ODE. The main idea is to find sufficient properties to ensure global existence for almost all initial data that could be satisfied by the limit flow in the low Mach number limit procedure. Next let

$$\Sigma_{j,k,l} = \{u \quad | \quad \lambda_j(u) + \lambda_k(u) = \lambda_l(u)\}.$$

If $\mathcal{A}_j(u)$ is quadratic, then passing to the limit in these terms involves resonances, namely $\Sigma_{j,k,l}$. In the constant coefficient case, $\Sigma_{j,k,l}$ is either void or the whole space. But here, $\Sigma_{j,k,l}$ is a smooth manifold of codimension 1. Generically, the limit flow crosses it, but for almost any initial data, the limit flow crosses it transversally. Of course this must be proved.

Combining the two previous arguments we will get that for almost any initial data the limit flow avoids double eigenvalues and crosses resonances in a transverse way. With such properties we can pass to the limit $\varepsilon \to 0$ in Equation (33) and justify the formal analysis as done in Section 3.

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# Finite-dimensional Control for the Navier–Stokes Equations

Alexander Yu. Chebotarev

**Abstract.** The problem of partial controllability for the Navier–Stokes equations of viscous incompressible fluid is considered. The problem is to create in a given moment of time a velocity field with the null projection on the finite-dimensional subspace spanned by eigenfunctions of the Stokes operator. The control is selected from this subspace too. On the basis of estimates of the solution for the subdifferential Cauchy problem for a Navier–Stokes system, controllability of the flow is proven on the condition that the norm of the control is minimal.

**Mathematics Subject Classification (2000).** 35Q30, 49J20.

**Keywords.** Navier-Stokes equations, finite-dimensional control, subdifferential inclusion, feedback control

## 1. Introduction

The motion of a homogeneous viscous incompressible fluid in the bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\Gamma$ is described by the evolution Navier–Stokes equations (NSE),

$$y' - \nu\Delta y + (y \cdot \nabla)y + \nabla p = f, \quad \operatorname{div} y = 0. \tag{1}$$

Here $y = \{y_i\}_1^3$ is the velocity field, $p$ is the pressure, $f$ is the external force density, $\nu = \text{const} > 0$ is the coefficient of kinematic viscosity. Add to the equations (1) initial conditions

$$y|_{t=0} = y_0(x), \tag{2}$$

and conditions on the boundary $\Gamma$ of the flow region

$$y = 0. \tag{3}$$

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The initial-boundary problem (1)–(3) may be rewritten in an ordinary way as the Cauchy problem for an equation with operator coefficients [1]. We write

$$H = \{v \in L^2(\Omega),\ \operatorname{div} v = 0,\ v \cdot n|_\Gamma = 0\},\ V = \{v \in W_2^1(\Omega),\ \operatorname{div} v = 0,\ v|_\Gamma = 0\}.$$

Here and later we denote Sobolev spaces by $W_p^l(\Omega)$ and a unit vector of outer normal for the boundary $\Gamma$ by $n$.

Note that $V \subset H = H' \subset V'$ and these embeddings are dense and compact. We denote norms in spaces $V$ and $H$ correspondingly by $\|\cdot\|$, $|\cdot|$; the duality between $V'$ and $V$ (or scalar product in $H$) by $(\cdot,\cdot)$. Thus we write

$$(u, v) = \int\limits_\Omega (u \cdot v) dx,\ ((u, v)) = (\nabla u_i, \nabla v_i),\ \|u\|^2 = ((u, u)).$$

If $X$ is a Banach space, we will denote the space of $L^p$ functions defined on $(0, T)$ with values in $X$ by $L^p(0, T; X)$.

We define mappings $A : V \to V'$, $B : V \times V \to V'$ using equalities

$$(Ay, z) = \nu((y, z)),\ \ (B(y_1, y_2), z) = ((y_1 \cdot \nabla)y_2, z)$$

that hold for all $y, y_1, y_2, z$ in space $V$, $B[y] = B(y, y)$. Operators $A$ and $B$ satisfy conditions

$$(Ay, y) = \nu\|y\|^2,\ (Ay, z) = (Az, y),\ (B(y, z), z) = 0\ \forall y, z \in V. \tag{4}$$

Let $D(A) = \{y \in V : Ay \in H\}$. For the operator $B(y, z)$ the following estimates hold [1]:

$$((B(y_1, y_2), y_3) \le C|y_1|^{1/4} \cdot \|y_1\|^{3/4} \cdot \|y_2\| \cdot |y_3|^{1/4} \cdot \|y_3\|^{3/4},\ y_1, y_2, y_3 \in V, \tag{5}$$

$$(B(y_1, y_2), y_3) \le C\|y_1\| \cdot \|y_2\|^{1/2} \cdot |Ay_2|^{1/2} \cdot |y_3|,\ \ y_1 \in V,\ y_2 \in D(A),\ y_3 \in H. \tag{6}$$

Here constant $C > 0$ depends only on $\Omega, \nu$.

The eigenfunctions of $A$,

$$Aw_j = \lambda_j w_j,\ j = 1, 2, \dots\ \ (w_i, w_j) = \delta_{ij},$$

form the basis of the spaces $H$ and $V$, $\lambda_j \sim Cj^{2/3}$ as $j \to \infty$.

The problem (1)–(3) is reduced to the following Cauchy problem for an equation with operator coefficients:

$$y' + Ay + B[y] = f(t),\ t \in (0, T),\ y(0) = y_0. \tag{7}$$

Here $y \in L^\infty(0, T; H) \cap L^2(0, T; V)$, $y' \in L^1(0, T; V')$, $y_0 \in H$.

The statement of the control problem that will be considered below is concerned with the following interesting property of the solution to the problem (7). It is well known [2] that if initial velocity field $y_0$ is 'fast oscillating' (that is $y_0 = \sum_{j=m+1}^\infty y_{0j} w_j$, where $m$ is sufficiently large) then on an arbitrary time interval, a unique smooth solution of the three-dimensional problem (1)–(3) exists. Then the question arises: is it possible to build in any time interval a velocity field of given structure by selecting control $f(t)$ from a finite-dimensional subspace?

Denote the subspace spanned by the first $m$ eigenfunctions of $A$ by $H_m = \operatorname{span}\{w_1, \dots, w_m\}$. Let $P = P_m :\ H \mapsto H_m$ be the projection operator on $H_m$.

**Problem 1.** Given $y_0 \in H$ find $f \in U = L^2(0,T;H_m)$ and corresponding solution $y$ to the problem (7) that satisfies condition $Py|_{t=T} = 0$.

The solution $\{f, y\}$ of Problem 1 will be called an admissible pair.

**Problem 2.** Find an admissible pair $\{f, y\}$ such that

$$\|f\|_U^2 = \int_0^T |f(t)|^2 dt \to \inf .$$

The last problem is to finding a normal solution to Problem 1.

The main difficulty in investigation of these problems is proof of existence of a solution to Problem 1 that is founded on consideration of the feedback control [3], [4].

## 2. Cauchy problem for the subdifferential inclusion

Let $\Phi(y) = \rho|Py|, \;\; y \in V, \;\; \rho > 0$. Functional $\Phi$ is convex and lower semicontinuous; its subdifferential is

$$\partial\Phi(y) = \rho P \operatorname{sign} Py.$$

Here

$$\operatorname{sign} w = \begin{cases} w/|w|, & \text{if } w \neq 0, \\ \xi, \; |\xi| \leq 1 & \text{if } w = 0. \end{cases}$$

Note that if $\partial\Phi(y)$ is the subdifferential of $\Phi$, then for all $\chi \in \partial\Phi(y)$ it holds that $|\chi| \leq \rho$.

Consider the evolutional variational inequality of Navier–Stokes type

$$(y' + Ay + B[y], z - y) + \Phi(z) - \Phi(y) \geq 0 \;\; \forall\, z \in V, \; y(0) = y_0. \tag{8}$$

In [5], [6] existence of a weak solution of (8) is proven. This solution, subject to the structure of the functional $\Phi$, has properties

$$y \in L^\infty(0,T;H) \cap L^2(0,T;V), \;\; y' \in L^1(0,T;V'), \; y(0) = y_0.$$

And for any $z \in V$ it holds that

$$(y' + Ay + B[y] + \chi, z) = 0,$$

in the sense of distributions on $(0,T)$.

Let us assume $z = 0$ in (8), integrate over time and get an energy inequality, from which nonlocal estimates follow.

$$|y(t)| \leq C, \; \int_0^T \|y(\tau)\|^2 d\tau \leq C, \; \rho \int_0^T |Py(\tau)| d\tau \leq C. \tag{9}$$

Here and later we denote by $C$ constants depending only on basic data, in particular on $|y_0|$, $\nu$, $\Omega$.

Let us derive a priori estimates of the solution of inequality (8) that will guarantee existence and uniqueness of the strong solution on some interval $(0,T_*)$, where $T_*$ does not depend on the parameter $\rho$.

The strong solution of (8) on interval $(0,T)$ is the function $y \in L^\infty(0,T;V)$, $y' \in L^2(0,T;V)$ such that

$$-(y' + Ay + B[y]) \in \partial\Phi(y) \ \text{ a.e. on } (0,T). \tag{10}$$

In [5], [6] existence of the strong solution is proven by deriving a priori estimates for Galerkin approximations $y_k$ that are defined from system

$$(y_k' + Ay_k + B[y_k] + \nabla\Phi_\varepsilon(y_k), w_j) = 0, \ j = 1, \ldots, k; \ y_k(0) = P_k y_0. \tag{11}$$

Here $\{w_j\}$ is the orthonormal in $H$ basis of $V$, consisting of eigenfunctions of operator $A$. By $\Phi_\varepsilon$ we denote regularization of the functional $\Phi$ [9],

$$\Phi_\varepsilon(y) = \inf\{\frac{\|y-z\|^2}{2\varepsilon} + \Phi(z); \ z \in V\}, \ y \in V, \ \varepsilon > 0.$$

For this functional $\Phi$ we obtain

$$\Phi_\varepsilon(y) = \rho \begin{cases} \frac{1}{2\varepsilon}|Py|^2, & \text{if } |Py| \le \varepsilon, \\ |Py| - \frac{\varepsilon}{2}, & \text{else.} \end{cases}$$

Correspondingly,

$$\nabla\Phi_\varepsilon(y) = \rho \begin{cases} \frac{1}{\varepsilon}Py, & \text{if } |Py| \le \varepsilon, \\ \frac{1}{|Py|}Py, & \text{else.} \end{cases} \tag{12}$$

Note that

$$(\nabla\Phi_\varepsilon(y_k), Ay_k) \ge 0,$$

as $(Py_k, Ay_k) = \nu\|y_m\|^2, \ k > m$. Using the structure of the gradient (12) we obtain from (11) inequality

$$(y_k' + Ay_k + B[y_k], Ay_k) \le 0. \tag{13}$$

Using inequality (6) and Young inequality we estimate in (13) a term with the quadratic operator $\mathcal{B}[y_k]$.

$$-(\mathcal{B}[y_k], Ay_k) \le C\|y_k\|^{3/2}|Ay_k|^{3/2} \le C\left(\frac{3\alpha^{4/3}}{4}|Ay_k|^2 + \frac{1}{\alpha^4}\|y_k\|^6\right).$$

Selecting sufficiently small $\alpha > 0$ get from (13) inequality

$$\frac{d}{dt}\|y_k\|^2 + |Ay_k|^2 \le C_1\|y_k\|^6, \tag{14}$$

where constant $C_1$ depends only on $\Omega$ and $\nu$. It follows from differential inequality (14) that sequence $\{y_k\}$ is bounded in $L^\infty(0,T_*;V)$ and $Ay_k$ is bounded in $L^2(0,T_*;H)$ under condition

$$T_* < \frac{1}{2C_1\|y_0\|^2}. \tag{15}$$

We emphasize that $T_*$ does not depend on parameter $\rho$. Having these estimates we can pass to the limit in system (11) as $k \to \infty$ and $\varepsilon \to 0$. In the limit we get existence on a sufficiently small time interval, not depending on $\rho$, of the strong solution of (8). Uniqueness may be proved by standard means.

**Theorem 1.** *Suppose $y_0 \in V$. There exist $T_* > 0$, not depending on $\rho$, such that on $(0, T_*)$ there exists a unique strong solution $y$ of variational inequality (8), at which estimates (9) and inequality*

$$\|y(t)\| \le K$$

*holds where $K$ does not depend on $\rho$.*

## 3. Solvability of the control problem

The proof of solvability of Problem 1 is founded on the following result.

**Theorem 2.** *Suppose $y_0 \in H$. There exist such $\rho > 0$ depending on $\Omega$, $\nu$, $|y_0|$, such that the weak solution of (8) satisfying $Py|_{t=T} = 0$ exists.*

*Proof.* Let $y$ be the weak solution of (8) satisfying estimates (9). Write $\psi = Py$ and show that $\psi \in L^\infty(0,T;V)$, $A\psi \in L^2(0,T;H)$, $\psi' \in L^1(0,T;H)$ and

$$\psi' + A\psi + PB[y] = -\rho P \operatorname{sign} \psi \quad \text{a.e. on } (0,T). \tag{16}$$

From estimates (9) holding on any interval $(0,T)$ it follows that

$$\|\psi\|^2 \le \frac{\lambda_m}{\nu}|\psi|^2 \le \frac{\lambda_m}{\nu}|y|^2 \le C_m.$$

Here and a later constant $C_m$ depends only on $\Omega, \nu, m$. Similarly, $|A\psi|^2 \le C_m$ and $|\rho P \operatorname{sign} \psi| \le \rho$. From (9) and inequality

$$(PB[y], z) = (B[y], Pz) \le C\|y\|^2 \sqrt{\frac{\lambda_m}{\nu}}|z| \tag{17}$$

it follows that $PB[y] \in L^1(0,T;H)$. Therefore $\psi' \in L^1(0,T;H)$ and equation (16) holds.

Multiplying (16) by $\psi$ and taking into consideration estimate (17), we get

$$\frac{1}{2}\frac{d|\psi|^2}{dt} + \nu\|\psi\|^2 + \rho|\psi| \le C_m\|y\|^2|\psi|. \tag{18}$$

Integrating differential inequality (18) gives the estimate

$$|\psi(t)| \le |\psi(0)| + C_m \int_0^T \|y(\tau)\|^2 d\tau - \rho t,$$

from which follows that $\psi(T) = 0$ if $\rho > (|y_0| + C_m|y_0|^2/2\nu)/T$. $\square$

**Theorem 3.** *Suppose $y_0 \in H$. Then Problem 1 has at least one solution.*

The proof of Theorem 3 follows from Theorem 2 when we select for $f$ a control with feedback

$$f = -\rho P \operatorname{sign} Py,$$

where $y$ is the solution of variational inequality (8). Note also that $|f(t)| \le \rho$.

The question about solvability of Problem 2 is concerned with the weak closure of the set of admissible pairs that is the set of solutions of Problem 1. From (7) the energy inequality follows [1]:

$$\frac{1}{2}|y(t)|^2 + \nu \int_0^t \|y(\tau)\|^2 d\tau \leq \frac{1}{2}|y_0|^2 + \int_0^t |f(\tau)| \cdot |y(\tau)| d\tau. \tag{19}$$

It is easy to check that the set of admissible pairs is weakly closed in the space $U \times L^2(0,T;V)$. On account of weak lower semicontinuity of norm $\|\cdot\|_U$ solvability, of Problem 2 follows from nonemptiness of the set of solutions of Problem 1.

**Theorem 4.** *Let $y_0 \in H$. Then there exists a solution of Problem* 2.

## 4. About existence of the strong solution of NSE for all time

Consider again variational inequality (8). As was proven in Theorem 1, on a sufficiently small time interval $(0, T_*)$ there exists a unique strong solution, and $T_*$ does not depend on $\rho$. On the other hand, as follows from estimate (18) if $\rho$ is sufficiently large, then $Py(T_*) = 0$ holds. Reasoning by analogy with [2] (taking into account that $(Ay, P\mathrm{sign}\, Py) \geq 0$) we get inequality

$$\frac{1}{2}\frac{d}{dt}\|y\|^2 + |Ay|^2 \leq C|A^{1/4}y||Ay|^2, \tag{20}$$

where constant $C$ depends only on $\Omega$ and $\nu$. Then, if $y(T_*)$ is concentrated on high frequencies (that is $P_m y(T_*) = 0$), we obtain $\|y(T_*)\| \geq \lambda_m^{1/4}|A^{1/4}y(T_*)|$. Therefore, as $m$ are large, from (20) follows the estimate $\|y(t)\|$ that guarantees existence and smoothness of the global solution on an arbitrary interval $(0,T)$. The analogous statement holds if $f(t) = 0$ as $t > T_*$.

**Theorem 5.** *Suppose $y_0 \in V$. Then for arbitrarily small $T_* > 0$ there exist $\rho > 0$ and $m$ depending on $\Omega, \nu, \|y_0\|$, that a unique strong solution of problem* (7) *with*

$$f(t) = \begin{cases} -\rho P_m \mathrm{sign}\, P_m y, & \text{if } t \in (0, T_*), \\ 0, & \text{if } t \in (T_*, T). \end{cases}$$

*exists on $(0,T)$. Problem* (7) *is the three-dimensional Navier–Stokes system with no-slip boundary conditions.*

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# On the Sharp Vanishing Viscosity Limit of Viscous Incompressible Fluid Flows

H. Beirão da Veiga

**Abstract.** We consider the classical problem of the convergence of local-in-time regular solutions of the Navier-Stokes equations to a solution of the Euler equations, as the viscosity $\nu$ goes to zero. Initial data are given in an $H^k(\Omega)$ space, where $k > 1 + \frac{n}{2}$. Solutions are continuous in time, with values in the initial-data's space. We look for convergence of the solutions $v$ of the Navier-Stokes equations to the solution $w$ of the Euler equations in the space $C([0, T]; H^k)$. This convergence result, in the strong topology, is due to T. Kato, see [8]. We show here a very elementary proof. We assume, together with the convergence of $\nu$ to zero, the convergence of the initial data in the $H^k$ norm.

**Mathematics Subject Classification (2000).** 35Q30,76D05, 76D09, 76D99.

**Keywords.** Navier-Stokes equations; slip boundary conditions; inviscid limit; Euler equations.

## 1. Introduction

Our main concern is showing that (1.4) holds, where $v_\nu$ and $w$ are the solutions to the systems (1.1) and (1.2) respectively. See Theorem 1.2 and Corollary 1.2 below. This result was essentially proved, many years ago, by Kato, see [8], by appealing to a completely different method, based on rather general theorems on abstract equations. A nice proof was given recently by Masmoudi, see [12]. The proof followed here is borrowed from reference [3], where a substantially more difficult problem is considered (we take this occasion to quote our recent review [4], where an introduction to our methods to prove sharp singular limit results is given). We also refer to Ebin and Marsden, cf. [6], where the limit of zero viscosity is considered in $H^s$, for $s > 5 + \frac{n}{2}$. See [6], Section 15.4, p. 152.

In considering problems like vanishing viscosity limits, incompressible limits, dependence on initial data, etc., the results are called here *sharp* if convergence is shown in $C([0, T]; X)$, where $X$ is the initial data's space. As remarked by

T. Kato in reference [9] this is the more difficult part in a theory dealing with nonlinear equations of evolution. Note that sufficiently strong a priori estimates, independent of $\nu$, for the solutions to the Navier-Stokes equations immediately lead to non-sharp convergence results, by appealing to suitable compactness theorems and to the uniqueness of the strong solution to the Euler equations. For instance, by assuming that the initial data $a_\nu$ are bounded in $H^s$, for some $s > k$, (1.4) follows easily. Many non-sharp vanishing viscosity limit results are known in the literature. Classical, specific references, are [7] and [13]. A simpler approach is given in [5].

In the sequel $k_0$ denotes the smallest integer such that $k_0 \geq n/2$ and $k$ is a fixed integer satisfying $k \geq k_0 + 1$. The canonical norm in $H^k$ is denoted by $\|\cdot\|_k$. The norm in $L^2$ is simply denoted by $\|\cdot\|$.

We set

$$H^k_\sigma(\Omega) =: \left\{ u \in H^k(\Omega) : \nabla \cdot u = 0 \right\} .$$

We denote by $\|\cdot\|_{l,T}$ the standard norm in $C([0,T]; H^l)$ and by $[\,\cdot\,]_{l,T}$ that in $L^2(0,T; H^l)$.

In the sequel $\Omega = [0,1]^n$ is the $n$-dimensional torus, $n \geq 2$. Obvious modifications in the proofs allow one to assume that $\Omega = \mathbb{R}^n$. The motion of a viscous, incompressible, fluid is described by the system

$$\begin{cases} \partial_t v_\nu + (v_\nu \cdot \nabla) v_\nu + \nabla p_\nu = \nu \Delta v_\nu \quad \text{in} \quad Q_T , \\ \nabla \cdot v_\nu = 0 \quad \text{in} \quad Q_T , \\ v_\nu(0) = a_\nu(x) , \end{cases} \tag{1.1}$$

where $\nabla \cdot a_\nu = 0$ in $\Omega$, and $\nu \in \mathbb{R}^+_0$, the set of nonnegative reals. We also consider the "limit problem"

$$\begin{cases} \partial_t w + (w \cdot \nabla) w + \nabla \pi = \overline{\nu} \Delta w \quad \text{in} \quad Q_T , \\ \nabla \cdot w = 0 \quad \text{in} \quad Q_T , \\ w(0) = b , \end{cases} \tag{1.2}$$

where $\nabla \cdot b = 0$. Note that in the more interesting case, namely $\overline{\nu} = 0$, we are dealing with the Euler equation for non-viscous fluids

$$\begin{cases} \partial_t w + (w \cdot \nabla) w + \nabla \pi = 0 \quad \text{in} \quad Q_T , \\ \nabla \cdot w = 0 \quad \text{in} \quad Q_T , \\ w(0) = b(x) . \end{cases} \tag{1.3}$$

We are interested in showing that

$$\lim \|v_\nu - w\|_{C([0,T]; H^k)} = 0 , \tag{1.4}$$

as $(a_\nu, \nu) \to (b, \overline{\nu})$ in $H^k \times \mathbb{R}^+_0$.

We recall the following well-known existence and regularity theorem for local-in-time smooth solutions of (1.1). For the reader's convenience, in the next section we give a sketch of the proof.

**Theorem 1.1.** *Assume that*

$$\|a_\nu\|_{k_0+1} \leq c_1 \tag{1.5}$$

*and*

$$\|a_\nu\|_k \leq c_2\,. \tag{1.6}$$

*Then there is a positive constant $T$ depending only on $c_1$ such that the problem* (1.1) *has a unique solution in* $[0, T]$. *Moreover,*

$$\|v_\nu\|^2_{k,\,T} + \nu\,[\nabla\, v_\nu]^2_{k,\,T} \leq C\,, \tag{1.7}$$

*and*

$$\|\partial_t\, v_\nu\|^2_{k-2,\,T} + \nu\,[\nabla\,\partial_t\, v_\nu\,]^2_{k-2,\,T} \leq C\,. \tag{1.8}$$

Constants $C$ may depend on $k$ and $n$, on an arbitrarily fixed upper bound for the values $\nu$, and on $c_1$ and $c_2$. For convenience we do not show the explicit dependence of the various constants $C$ on $c_1$ and $c_2$.

Due to (1.11) below, the reader may assume that the initial data $a_\nu$ satisfy the constraint $\|a_\nu\|_k \leq \|b\|_k + 1$, so that $T$ and the constants $C$ that appear in equations (1.7) and (1.8) are fixed once and for all.

**Corollary 1.1.** *Under the assumption* (1.11) *one has*

$$v_\nu \rightharpoonup w \quad in \;\; L^\infty(0,\,T;\,H^k)\text{-}weak^* \quad and\ in \;\; C(0,\,T; H^{k-\,\epsilon})\,, \tag{1.9}$$

*for $\epsilon > 0$ small enough. Moreover,*

$$\partial_t\, v_\nu \rightharpoonup \partial_t\, w \quad in \;\; L^\infty(0,\,T;\,H^{k-\,2})\text{-}weak^* \quad and\ in \;\; C(0,\,T; H^{k-\,2-\,\epsilon})\,. \tag{1.10}$$

Corollary 1.1 follows immediately from the uniform estimates (1.7), (1.8), by appealing to well-known compact embedding theorems. These theorems guarantee that we may pass to the limit in equation (1.1), as $\nu \to 0$. The uniqueness of the strong solution $w$ of equation (1.2) is used in order to show that all the sequences $v_\nu$ converge to the same limit $w$.

The following is the main result here, especially when $\overline{\nu} = 0$.

**Theorem 1.2.** *Let $\overline{\nu} \geq 0$ and $a_\nu,\, b \in H^k_\sigma(\Omega)$. Assume that*

$$\lim_{\nu \to \overline{\nu}} \|a_\nu - b\|_k = 0\,. \tag{1.11}$$

*Then*

$$\lim_{\nu \to \overline{\nu}} \big(\|v_\nu - w\|^2_{k,\,T} + \overline{\nu}\,[v_\nu - w]^2_{k+\,1,\,T}\big) = 0\,. \tag{1.12}$$

*In particular,* (1.4) *holds.*

**Corollary 1.2.** *Under the assumptions of the above theorem one has*

$$\lim_{\nu \to \overline{\nu}} \big(\|\partial_t\, v_\nu - \partial_t\, w\|^2_{k-\,2,\,T} + \|\nabla\, p_\nu - \nabla\,\pi\|^2_{k-\,1,\,T} + \overline{\nu}\,[\partial_t\, v_\nu - \partial_t\, w]^2_{k-\,1,\,T}\big) = 0\,. \tag{1.13}$$

**Remark 1.1.** *Under the sole assumptions of Theorem* 1.2 *the equation*

$$\lim_{\nu \to \overline{\nu}} \left( \|\partial_t v_\nu - \partial_t w\|^2_{k-1,T} + \overline{\nu}\, [\partial_t v_\nu - \partial_t w]^2_{k,T} \right) = 0 \tag{1.14}$$

*is false in general. Obviously it holds under stronger regularity assumptions on the initial data, and for* $t > 0$.

## 2. Preliminaries

For the reader's convenience, we give in this section a sketch of the proof of equations (1.7) and (1.8). Here the parameter $\nu$ is fixed. Hence we denote $v_\nu$ simply by $v$ and $\partial_t v_\nu$ by $v_t$.

We start by some useful results.

For convenience, we denote integrals $\int_\Omega f(x)\, dx$ simply by $\int f(x)$, or even by $\int f$. If $D^\alpha$ denotes partial differentiation, $\alpha = (\alpha_1, \dots, \alpha_n)$, we set

$$\widetilde{D}^\alpha \{f\, g\} = D^\alpha (f\, g) - f\, D^\alpha g$$

and $|D^m f|^2 = \sum_{|\alpha| = m} |D^\alpha f|^2$. In the sequel we appeal to the following three results.

**Lemma 2.1.** *Let* $|\alpha| \leq l$. *Then*

$$\|\widetilde{D}^\alpha \{f\, g\}\| \leq c \left( |D f|_\infty \|g\|_{l-1} + |g|_\infty \|D f\|_{l-1} \right). \tag{2.1}$$

For a proof see [10], Lemma A.1.

**Lemma 2.2.** *For* $0 \leq |\alpha| \leq m \leq k$,

$$\|D^\alpha (f\, g)\| \leq c\, \|f\|_m \|g\|_{k-1} + c\, \delta^m_k\, |f|_\infty \|g\|_k\,. \tag{2.2}$$

See [3], equation (3.4).

**Lemma 2.3.** *Let* $k > 1 + n/2$ *and* $1 \leq l \leq k$. *If* $|\alpha| \leq l$ *then*

$$\|\widetilde{D}^\alpha \{f\, g\}\| \leq c\, \|D f\|_{k-1} \|g\|_{l-1}\,. \tag{2.3}$$

For a proof see [1] Appendix A, Corollary A.4.

By applying the operator $D^\alpha$ to both sides of (1.1), by multiplying by $D^\alpha v$, and by integrating in $\Omega$, we show that

$$\frac{1}{2} \frac{d}{dt} \|D^\alpha v\|^2 + \int \widetilde{D}^\alpha \{(v \cdot \nabla)\, v\} \cdot D^\alpha v + \nu\, \|\nabla D^\alpha v\|^2 = 0\,. \tag{2.4}$$

Then we add the above equations, side by side, for $0 \leq |\alpha| \leq m$. By taking into account (2.1), and also $|\cdot|_\infty \leq c\, \|\cdot\|_{k_0}$, it readily follows that

$$\frac{1}{2} \frac{d}{dt} \|v\|^2_m + \nu\, \|\nabla v\|^2_m \leq c\, \|v\|_{k_0+1} \|v\|^2_m\,. \tag{2.5}$$

By setting $m = k_0 + 1$, well-known methods lead to (1.7) for $k = k_0$, (with dependence of $T$ only on $c_1$). The estimate (1.7) for $k = k_0$, together with (2.5) written for $m = k$, shows (1.7) for $k$.

**Lemma 2.4.** *Assume that* (1.5) *and* (1.6) *hold. Let $l$ be an integer satisfying* $0 \le l \le k-2$. *Then there is a constant* $C$ *such that*

$$\|v_t\|^2_{l,\,T} + \nu\,[\nabla\, v_t]^2_{l,\,T} \le C\,. \tag{2.6}$$

*In particular* (1.8) *holds.*

*Proof.* From (1.1) it follows that

$$\partial_{tt}\, v + (v\cdot\nabla)\, v_t + (v_t\cdot\nabla)\, v + \nabla\, p_t = \nu\,\Delta\, v_t\,. \tag{2.7}$$

Next apply $D^\alpha$, $|\,\alpha\,| \le l$, to both sides of the above equation, multiply by $D^\alpha\, v_t$ and integrate over $\Omega$. This gives

$$\frac{1}{2}\frac{d}{dt}\|D^\alpha v_t\|^2 + \int \widetilde{D}^\alpha\{(v\cdot\nabla)v_t\}\cdot D^\alpha v_t + \int D^\alpha[(v_t\cdot\nabla)v]\cdot D^\alpha v_t + \nu\|\nabla D^\alpha v_t\|^2 = 0. \tag{2.8}$$

By using (2.3) and (2.2) we show that

$$\frac{1}{2}\,\frac{d}{dt}\,\|D^\alpha\, v_t\|^2 + \nu\,\|\nabla\, D^\alpha\, v_t\,\|^2 \le c\,\|D\,v\|_{k-1}\,\|\,v_t\|_l\,\|D^\alpha\, v_t\|\,.$$

Hence, for $|\,\alpha\,| \le l$,

$$\frac{1}{2}\,\frac{d}{dt}\,\|D^\alpha\, v_t\|^2 + \nu\,\|\nabla\, D^\alpha\, v_t\,\|^2 \le C\,\|v_t\|^2_l, \tag{2.9}$$

and a well-known argument leads to (2.6). Note that, by applying the divergence operator to both sides of the first equation (1.1), we show that $\|\nabla\, p\,\|_{k-2,\,T} \le C$. In particular, it readily follows that $\|\,v_t(0)\|_{k-2} \le C$. □

## 3. Proof of Theorem 1.2

In the sequel we appeal to Fourier series

$$\phi(x) = \sum_{\xi} \widehat{\phi}(\xi)\, e^{2\,\pi\, i\,\xi\cdot\, x}\,,$$

$$\widehat{\phi}(\xi) = \int_\Omega e^{-\,2\,\pi\, i\,\xi\cdot\, x}\,\phi(x)\,dx\,.$$

The $\xi_i$'s are nonnegative integers, and $\xi = (\xi_1, \ldots, \xi_n)$. The Euclidian norm of $\xi$ is denoted by $|\xi|$. For each nonnegative real $s$ one has

$$\|\phi\|^2_s = \sum_{\xi} (1 + |\xi|^2)^s\, |\widehat{\phi}(\xi)|^2\,.$$

Given $\delta \in\, ]0,\, 1]$, we define linear operators

$$T^\delta\,\phi = \sum_{|\xi|\le 1/\delta} \widehat{\phi}(\xi)\, e^{2\,\pi\, i\,\xi\cdot\, x}\,, \tag{3.1}$$

where $\phi$ is a scalar or a vector field, and set

$$a^\delta_\nu = T^\delta\, a_\nu\,, \quad b^\delta = T^\delta\, b\,. \tag{3.2}$$

Since $T^\delta$ commutes with the divergence operator, $a^\delta_\nu$ and $b^\delta$ are divergence free. Clearly, for each nonnegative real $s$, $T^\delta$ is a bounded linear operator. In particular, $|||\, T^\delta\, |||_{s,\,s} \leq 1$ where, in general, we denote by $|||\cdot|||_{s,\,r}$ the canonical norm in the space of bounded linear operators from $H^s$ to $H^r$. So, $a^\delta_\nu$ satisfies the assumptions (1.5), (1.6) with the same constants $c_1$ and $c_2$.

Also note that

$$|||T^\delta\,|||_{s,\,m} \leq (2/\,\delta)^{m-\,s}\,, \quad |||T^\delta - I\,|||_{m,\,s} \leq \delta^{m-\,s}\,, \tag{3.3}$$

if $0 \leq s \leq m$, where $s$ and $m$ are nonnegative integers. In particular

$$\|a^\delta_\nu\,\|_{k_0+1} \leq c_1\,, \quad \|a^\delta_\nu\|_{k+1} \leq \frac{2\,c_2}{\delta}\,. \tag{3.4}$$

and

$$\|a^\delta_\nu - b^\delta\|_{k+1} \leq \frac{2}{\delta}\,\|a_\nu - b\|_k\,. \tag{3.5}$$

Note that

$$a^\delta_\nu \to b^\delta \quad \text{in}\, H^{k+1} \qquad \text{if} \quad a_\nu \to b \quad \text{in}\, H^k\,.$$

The following system plays here a very central role:

$$\begin{cases} \partial_t\, v^\delta_\nu + (v^\delta_\nu \cdot \nabla)\, v^\delta_\nu + \nabla\, p^\delta_\nu = \nu\, \Delta\, v^\delta_\nu \quad \text{in} \quad Q_T\,, \\ \nabla \cdot v^\delta_\nu = 0 \quad \text{in} \quad Q_T\,, \\ v^\delta_\nu(0) = a^\delta_\nu\,. \end{cases} \tag{3.6}$$

We also consider the (inviscid, if $\overline{\nu} = 0$) counterpart of the system (3.6), namely

$$\begin{cases} \partial_t\, w^\delta + (w^\delta \cdot \nabla)\, w^\delta + \nabla\, \pi^\delta = \overline{\nu}\, \Delta\, w^\delta \quad \text{in} \quad Q_T\,, \\ \nabla \cdot w^\delta = 0 \quad \text{in} \quad Q_T\,, \\ w^\delta(0) = b^\delta\,. \end{cases} \tag{3.7}$$

From Corollary 1.1, with $k$ replaced by $k+1$, applied to the solutions $v^\delta$ and $w^\delta$ of the above problems, and also by taking into account (3.4) and (3.5), one shows the following result.

**Proposition 3.1.** *Under the assumptions of Theorem* 1.2 *one has*

$$\lim_{\nu \to \overline{\nu}} \big(\|v^\delta_\nu - w^\delta\|^2_{k,\,T} + \overline{\nu}\,[v^\delta_\nu - w^\delta]^2_{k+1,\,T}\big) = 0\,, \tag{3.8}$$

*for each fixed* $\delta > 0$.

The following estimate will be useful in the sequel:

$$\|a^\delta_\nu - a_\nu\|^2_k \leq 2\,\|b - a_\nu\|^2_k + 2 \sum_{|\xi| > 1/\,\delta} (1 + |\xi|^2)^k\, |\widehat{b}(\xi)|^2\,. \tag{3.9}$$

The proof is left to the reader.

In the sequel we denote by $\delta^m_k$ the Kronecker symbol and set

$$\overline{v} = v^\delta_\nu - v_\nu\,, \quad \overline{p} = p^\delta_\nu - p_\nu\,.$$

Clearly, $\overline{v}$ and $\overline{p}$ depend on $\delta$ and $\nu$.

Our next step is to prove the following result.

**Theorem 3.1.** *Let* $0 \leq m \leq k$*. Then, for each* $\delta > 0$*,*

$$\frac{1}{2}\,\frac{d}{d\,t}\,\|\overline{v}\|_m^2 + \nu\,\|\nabla\,\overline{v}\|_m^2 \leq C\,\|\overline{v}\|_m^2 + c\,\delta_k^m\,\|v_\nu^\delta\|_{k+1}\,|\overline{v}|_\infty\,\|\overline{v}\|_m\,. \tag{3.10}$$

*Proof.* In the calculations that follow the reader should take into account that the quantities $\|v_\nu\|_{k,\,T}$, $\|v_\nu^\delta\|_{k,\,T}$, $\nu\,[\,v_\nu\,]_{k+1,\,T}$ and $\nu\,[\,v_\nu^\delta\,]_{k+1,\,T}$ are uniformly bounded by constants $C$.

By taking the termwise difference between the equations (3.6) and (1.1) we find that

$$\overline{v}_t + (v_\nu\cdot\nabla)\,\overline{v} + \nabla\,\overline{p} = -(\overline{v}\cdot\nabla)\,v_\nu^\delta + \nu\,\Delta\,\overline{v}\,. \tag{3.11}$$

Apply $D^\alpha$ to (3.11), multiply by $D^\alpha\,\overline{v}$ and integrate on $\Omega$. Using previous estimates and formulae (in particular (2.3) and (2.2)), straightforward manipulations show that

$$\begin{aligned}&\tfrac{1}{2}\,\tfrac{d}{dt}\,\|D^\alpha\,\overline{v}\|^2 + \nu\,\|\nabla\,D^\alpha\,\overline{v}\,\|^2\\ &\leq C\,\|\overline{v}\|_m^2 + c\,\delta_k^m\,\|v_\nu^\delta\|_{k+1}\,|\overline{v}|_\infty\,\|\overline{v}\|_m\,.\end{aligned} \tag{3.12}$$

Equation (3.10) follows. □

Next, fix a real $\beta_0$ such that $0 < \beta_0 < k_0 - (n/2)$. Clearly, $0 < \beta_0 < 1$. Since $k_0 - \beta_0 > n/2$, one has $|\cdot|_\infty \leq c\,\|\cdot\|_{k_0-\beta_0}$. Well-known interpolation results for $L^2$-Sobolev spaces show that

$$|\cdot|_\infty \leq c\,\|\cdot\|_{k_0-1}^{\beta_0}\,\|\cdot\|_{k_0}^{1-\beta_0}\,. \tag{3.13}$$

**Theorem 3.2.** *For each* $\delta > 0$*,*

$$|\overline{v}|_{\infty,\,T} \leq C\,\delta^{2\,(k-k_0+\beta_0)}\,. \tag{3.14}$$

*Proof.* Let $0 \leq m \leq k-1$. From (1.7) one has $\nu\,[\nabla\,v_\nu^\delta]_{k,\,T}^2 \leq C$. Hence, by appealing to (3.10), it follows that

$$\|\overline{v}(t)\|_m^2 \leq C\,\|\overline{v}(0)\|_m^2\,, \quad \forall\,t \in [0,\,T]\,.$$

So,

$$\|\overline{v}\|_{m,\,T}^2 \leq C\,\|a_\nu^\delta - a_\nu\|_m^2\,.$$

By appealing to this inequality for $m = k_0$ and $m = k_0 - 1$, and by taking into account (3.13), we show that

$$|\overline{v}|_{\infty,\,T}^2 \leq C\,\|a_\nu^\delta - a_\nu\|_{k_0-1}^{2\,\beta_0}\,\|a_\nu^\delta - a_\nu\|_{k_0}^{2\,(1-\beta_0)}\,. \tag{3.15}$$

By using $(3.3)_2$ for $m = k$ and $m = k_0 - 1$, we obtain

$$\|a_\nu^\delta - a_\nu\|_{k_0-1}^2 \leq \delta^{2\,(k-k_0+1)}\,\|a_\nu\|_k^2\,. \tag{3.16}$$

Again by $(3.3)_2$, one has

$$\|a_\nu^\delta - a_\nu\|_{k_0}^2 \leq \delta^{2\,(k-k_0)}\,\|a_\nu\|_k^2\,. \tag{3.17}$$

The estimates (3.15), (3.16) and (3.17) lead to (3.14). □

**Corollary 3.1.** *One has, for each* $\delta \in ]\,0,\,1\,]$,

$$|\overline{v}|_{\infty,\,T}\,\|\,v_\nu^\delta\|_{k+\,1,\,T} \leq\; C\,\delta^{k-\,k_0-\,1+\,\beta_0}\,. \tag{3.18}$$

*Proof.* By applying the estimate (1.7) to the solution $v_\nu^\delta$, with $k$ replaced by $k+1$, and by appealing to $(3.3)_1$ for $m=\;k+\,1$ and $s=k$, it follows that

$$\|\,v_\nu^\delta\|^2_{k+\,1,\,T} \leq\; C/\,\delta^2\,. \tag{3.19}$$

This estimate together with (3.14) shows (3.18). □

**Theorem 3.3.** *For each* $\delta \in ]\,0,\,1\,]$,

$$\|\overline{v}\|^2_{k,\,T} +\,\nu\,\int_0^T\,\|\nabla\,\overline{v}(t)\|^2_k dt \leq\; C\,(\,\|a_\nu^\delta -\,a_\nu\|^2_k +\,\delta^{2\,\beta_0})\,. \tag{3.20}$$

*Proof.* From equation (3.10) for $m=\;k$, together with (3.18), we get

$$\frac{1}{2}\,\frac{d}{d\,t}\,\|\overline{v}(t)\|^2_k +\,\nu\,\|\nabla\,\overline{v}\|^2_k \leq C\,\|\overline{v}\|^2_k +\,C\,\|\overline{v}\|_k\,\delta^{\beta_0}\,. \tag{3.21}$$

Standard techniques yield

$$\|\overline{v}\|_{k,\,T} \leq\; e^{C\,T}\,(\|\overline{v}(0)\|_k + \delta^{\beta_0}\,)\,. \tag{3.22}$$

Equation (3.20) follows easily. Note that $e^{C\,T}$ is a constant of type $C$. □

**Proof of Theorem** 1.2

Define

$$|\|\,u\|\,|^2 =:\; \|u\|^2_{k,\,T} +\,\overline{\nu}\,[\,\nabla\,u\,]^2_{k,\,T}\,.$$

Let $\epsilon >\; 0$ be fixed. From (3.20) and (3.9) it follows that

$$|\|\overline{v}\|\|^2_{k,\,T} \leq\; C\,\Big(\,\|b -\,a_\nu\|^2_k +\,\sum_{|\xi|>\,1/\delta}\,(\,1 +\,|\xi|^2\,)^k\,|\widehat{b}(\xi)|^2 \;+\,\delta^{2\,\beta_0} \;+\,|\nu -\,\overline{\nu}|\,\Big)\,.$$

In particular,

$$\|\overline{v}\|^2_{k,\,T} \leq\; C\,(\,\|b -\,a_\nu\|^2_k +\,\widehat{h}(\delta) +\,|\nu -\,\overline{\nu}|\,)\,, \tag{3.23}$$

where $\widehat{h}(\delta)$ depends only on $\delta$ ($b$ and $k$ are fixed), and satisfies

$$\lim_{\delta\to\,0}\,\widehat{h}(\delta) =\; 0\,.$$

We fix (once and for all) $\delta =\; \delta(\epsilon)$ such that $C\,\widehat{h}(\delta) \leq\; \epsilon/\,3$. It follows that

$$|\|v_\nu^\delta -\,v_\nu\,|\|^2_{k,\,T} <\; \frac{\epsilon}{3} +\,C\,(\,\|b -\,a_\nu\|^2_k +\,|\nu -\,\overline{\nu}|\,)\,. \tag{3.24}$$

The same argument applied to the particular case in which $(a_\nu,\,\nu) =\; (b,\,\overline{\nu})$ shows that

$$|\|w^\delta -\,w\,|\|^2_{k,\,T} \leq\; \frac{\epsilon}{3}\,. \tag{3.25}$$

On the other hand, Proposition 3.1 shows that there is $\lambda =\; \lambda(\,\delta(\epsilon),\,\epsilon\,)$ for which

$$|\|v_\nu^\delta -\,w^\delta|\|^2_{k,\,T} \leq\; \epsilon\,, \tag{3.26}$$

if $|\nu -\,\overline{\nu}| <\; \lambda\,.$

In short, from (3.24), (3.25) and (3.26) it follows that given $\epsilon > 0$ there is a $\lambda = \lambda(\epsilon)$ such that

$$|||v_\nu - w|||^2_{k,\,T} \leq \epsilon\,, \tag{3.27}$$

if $|\nu - \overline{\nu}| < \lambda$. This proves (1.12).

**Proof of Corollary** 1.2

*Proof.* One has

$$\begin{aligned}&\partial_t\,(v_\nu - w) + (v_\nu \cdot \nabla)\,(v_\nu - w) + \big(\,(v_\nu - w)\cdot \nabla\,\big)\,w + \nabla\,(p_\nu - \pi)\\ &\qquad = \nu\,\Delta\,(v_\nu - w)\,, \quad \text{in} \quad Q_T\,.\end{aligned} \tag{3.28}$$

In particular, by applying the divergence operator to both sides of (3.28), one gets

$$-\,\Delta\,(p_\nu - \pi) = \nabla\cdot\,\big\{(v_\nu\cdot\nabla)\,(v_\nu - w) + \big(\,(v_\nu - w)\cdot\nabla\,\big)\,w\,\big\}\,.$$

It readily follows, by appealing to previous estimates, that

$$\|\,(v_\nu\cdot\nabla)\,(v_\nu - w) + \big(\,(v_\nu - w)\cdot\nabla\,\big)\,w\,\|_{k-1,\,T} \leq C\,\|v_\nu - w\|_{k,\,T}\,.$$

The pressure-estimate in equation (1.13) follows from classical regularity results for solutions to elliptic equations $-\,\Delta\,u = f$, together with (1.12).

Now, the time-derivative estimates in equation (1.13) follow from (3.28). Note that more elaborate manipulations lead to better results concerning the convergence of $\partial_t\,v_\nu$ to $\partial_t\,w$, but not to (1.14). □

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# Small Péclet Number Approximation as a Singular Limit of the Full Navier-Stokes-Fourier System with Radiation

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*Dedicated to the memory of Alexander Kazhikov*

**Abstract.** We study a singular limit for the scaled Navier-Stokes-Fourier system, where the Mach, Froude, and Péclet numbers tend to zero. As a limit problem, we recover a model proposed by Chandrasekhar as a simple alternative to the Oberbeck-Boussinesq system applicable in stellar radiative zones.

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## 1. Introduction

Investigations in astrophysical fluid dynamics are hampered by both theoretical and observational problems. The vast range of different scales extending in the case of stars from the stellar radius to $10^2$ m or even less entirely prevents a complex numerical solution. Progress in this field therefore calls for a combination of physical intuition with rigorous analysis of highly simplified mathematical models.

Understanding the flow dynamics in stellar radiative zones represents a major challenge of the current theory of stellar evolution. Under these circumstances, the fluid is a plasma characterized by the following specific features (see Lignières [20]):

- A strong *radiative transport* predominates the molecular one. This is due to the extremely hot and energetic radiation fields pervading the plasma. The Prandtl and Péclet numbers are therefore expected to be vanishingly small.

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- Strong *stratification effects* because of the enormous gravitational potential of gaseous celestial bodies determine many of the properties of the fluid in the large.
- The convective motions are much slower than the speed of sound yielding a small the Mach number. The fluid is therefore almost *incompressible*, for which the density variations can be simulated via the anelastic approximation (see also Gough [16], Gilman and Glatzmaier [15]).

### 1.1. Primitive equations

A suitable but still highly simplified mathematical model of astrophysical fluids is represented by the Navier-Stokes-Fourier system (see Battaner [3]). Adopting the Cartesian coordinates $x = (x_1, x_2, x_3)$, where $(x_1, x_2)$ denotes the horizontal direction and $x_3$ is the depth variable pointing downward parallel to the gravity $g\mathbf{j}$, $\mathbf{j} = (0, 0, 1)$, we assume that the time evolution of the fluid density $\varrho = \varrho(t, x)$, the absolute temperature $\vartheta = \vartheta(t, x)$, and the velocity field $\mathbf{u} = \mathbf{u}(t, x)$ is governed by the following system of equations (balance of mass, momentum, and entropy):

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S} - \varrho g \mathbf{j}, \tag{1.2}$$

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma, \tag{1.3}$$

where the pressure $p = p(\varrho, \vartheta)$ and the specific entropy $s = s(\varrho, \vartheta)$ are given numerical functions of the state variables $\varrho$, $\vartheta$ satisfying Gibbs' equation

$$\vartheta D s(\varrho, \vartheta) = D e(\varrho, \vartheta) + p(\varrho, \vartheta) D\left(\frac{1}{\varrho}\right), \tag{1.4}$$

with $e = e(\varrho, \vartheta)$ denoting the specific internal energy. The symbol $\mathbb{S}$ denotes the viscous stress tensor, $\mathbf{q}$ is the heat flux, $g$ is the gravitational constant, while $\sigma$ is the entropy production rate to be specified below.

For the sake of simplicity, periodic boundary conditions are imposed on the horizontal variables $(x_1, x_2)$ meaning the underlying spatial domain $\Omega$ is given through

$$\Omega = \mathcal{T}^2 \times (0, 1), \text{ with } \mathcal{T}^2 = ([0,1]|_{\{0,1\}})^2 \text{ being a two-dimensional torus.} \tag{1.5}$$

Note this is still a physically relevant approximation, with the $x_3-$direction parallel to the gravitational force.

In addition, the complete slip boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0 \tag{1.6}$$

are adopted for the velocity field, while the heat flux satisfies

$$\mathbf{q} \cdot \mathbf{n}|_{\{x_3 = 0\}} = 0, \tag{1.7}$$

together with the "radiative" boundary condition on the upper part of the boundary

$$\mathbf{q} \cdot \mathbf{n} = \eta(\vartheta - \overline{\vartheta})|_{\{x_3 = 1\}}, \tag{1.8}$$

where $\overline{\vartheta} > 0$ is a given reference temperature. Note that (1.8) has a significant stabilizing effect driving the system to the state of reference temperature $\overline{\vartheta}$ for large time.

Accordingly, the total energy balance takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\Big[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e + \varrho g x_3\Big]\mathrm{d}x = \int_{\{x_3=1\}} \eta(\overline{\vartheta} - \vartheta)\ \mathrm{d}S_x. \tag{1.9}$$

### 1.2. Constitutive relations

The constitutive relations considered in the present paper are motivated by the characteristic properties of the astrophysical fluids mentioned above. In particular, the pressure as well as the heat transport are substantially enhanced by the effect of thermal radiation.

Accordingly, we shall assume that the pressure $p$ is related to $\varrho$, $\vartheta$ through a general state equation of the form

$$p(\varrho,\vartheta) = \underbrace{p^A(\varrho,\vartheta)}_{\text{atomic pressure}} + \underbrace{p^R(\varrho,\vartheta)}_{\text{radiation pressure}},\ p^A = \vartheta^{\frac{5}{2}} P\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big),\ p^R = \frac{a}{3}\vartheta^4,\ a > 0. \tag{1.10}$$

Similarly, in agreement with (1.4),

$$e(\varrho,\vartheta) = e^A(\varrho,\vartheta) + e^R(\varrho,\vartheta),\ e^A = \frac{3}{2}\frac{\vartheta^{\frac{5}{2}}}{\varrho} P\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big),\ e^R = a\frac{\vartheta^4}{\varrho}, \tag{1.11}$$

and

$$s(\varrho,\vartheta) = s^A(\varrho,\vartheta) + s^R(\varrho,\vartheta),\ s^A = S\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big),\ s^R\frac{4}{3}a\frac{\vartheta^3}{\varrho}, \tag{1.12}$$

where

$$S'(Y) = -\frac{3}{2}\frac{\frac{5}{3}P(Y) - P'(Y)Y}{Y^2} \text{ for any } Y > 0. \tag{1.13}$$

Note that the atomic pressure and associated specific internal energy are interrelated through

$$p_A = \frac{2}{3}\varrho e^A, \tag{1.14}$$

which is a universal relation to be satisfied by any *monoatomic* gas (see Eliezer et al. [10]). As a matter of fact, it is a routine matter that (1.14) is compatible with Gibbs' relation (1.4) if and only if $p^A$ takes the specific form introduced in (1.10). The reader interested in more information on the physical background of (1.10–1.13) may consult the monographs by Müller and Ruggeri [24] or Oxenius [25].

The viscous stress tensor $\mathbb{S}$ will satisfy Newton's rheological law

$$\mathbb{S} = \mu(\vartheta)\Big(\nabla_x\mathbf{u} + \nabla_x^{\perp}\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\Big), \tag{1.15}$$

where we have deliberately omitted the bulk viscosity component, assumed zero for plasmas.

The heat flux $\mathbf{q}$ is given through Fourier's law

$$\mathbf{q} = -\kappa(\vartheta)\nabla_x\vartheta, \tag{1.16}$$

where, in accordance with the basic idea pursued here, the heat conductivity is substantially enhanced by radiation, specifically,

$$\kappa(\vartheta) = \kappa_0(\vartheta) + d\vartheta^3, \ d > 0 \tag{1.17}$$

(see [25]).

In the context of variational solutions considered in this paper, the entropy production rate $\sigma$ is understood as a non-negative quantity (a measure) satisfying

$$\sigma \geq \frac{1}{\vartheta}\Big(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\Big) = \frac{\mu(\vartheta)}{2\vartheta}\Big|\nabla_x\mathbf{u} + \nabla_x^\perp\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\Big|^2 + \frac{\kappa(\vartheta)}{\vartheta^2}|\nabla_x\vartheta|^2. \tag{1.18}$$

It is easy to check that (1.18) reduces to equality as soon as all quantities solving (1.1–1.7) are smooth (see [13]).

### 1.3. Scaling

The main goal of the present paper is to introduce a small parameter $\varepsilon > 0$ in order to rescale system (1.1–1.9) in accordance with the characteristic properties of the astrophysical plasma discussed above, and to identify the limit problem when $\varepsilon$ tends to zero.

As a matter of fact, there are different possibilities of scaling giving rise to the same underlying system of dimensionless equations. Keeping in mind the characteristic features of the physical system, we suppose:

- the characteristic temperature is proportional to $\varepsilon^{-\frac{2}{3}}$;
- the characteristic time is large of order $\varepsilon^{-\frac{2}{3}}$;
- the characteristic velocity is low of order $\varepsilon^{\frac{2}{3}}$ so that the characteristic length is of order 1;
- the characteristic values of the viscosity coefficient $\mu$ as well as of the heat conductivity $\kappa_0$ are small of order $\varepsilon^{\frac{2}{3}}$, meaning that $\varepsilon^{-\frac{2}{3}}\mu(\varepsilon^{-\frac{2}{3}}\vartheta) \approx \mu(\vartheta)$ and $\varepsilon^{-\frac{2}{3}}\kappa_0(\varepsilon^{-\frac{2}{3}}\vartheta) \approx \kappa(\vartheta)$;
- the characteristic value of $\eta$ entering into the boundary condition for the heat flux is $\varepsilon^{-\frac{1}{3}}$, meaning that $\varepsilon^{\frac{1}{3}}\eta(\varepsilon^{-\frac{2}{3}}\vartheta) \approx \eta(\vartheta)$;
- the gravitational constant $g$ is of order $\varepsilon^{-\frac{2}{3}}$;
- $a \approx \varepsilon^3$, $d \approx \varepsilon^{\frac{2}{3}}$.

Thus system (1.1–1.9), rewritten in terms of the above characteristic values of the physical quantities, reads as follows:

$$\partial_t\varrho + \mathrm{div}_x(\varrho\mathbf{u}) = 0, \tag{1.19}$$

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \frac{1}{\varepsilon^2}\nabla_x p_\varepsilon(\varrho,\vartheta) = \mathrm{div}_x\mathbb{S} - \frac{1}{\varepsilon^2}\varrho\mathbf{j}, \tag{1.20}$$

$$\partial_t(\varrho s_\varepsilon(\varrho,\vartheta)) + \mathrm{div}_x(\varrho s_\varepsilon(\varrho,\vartheta)\mathbf{u}) + \mathrm{div}_x\Big(\frac{\mathbf{q}_\varepsilon}{\vartheta}\Big) = \sigma_\varepsilon, \tag{1.21}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\Big[\frac{\varepsilon^2}{2}\varrho|\mathbf{u}|^2 + \varrho e_\varepsilon(\varrho,\vartheta) + \varrho x_3\Big]\mathrm{d}x = \int_{\{x_3=1\}}\eta\frac{\overline{\vartheta}-\vartheta}{\varepsilon}\,\mathrm{d}S_x, \tag{1.22}$$

with the thermodynamic functions

$$p_\varepsilon(\varrho,\vartheta) = \frac{\vartheta^{\frac{5}{2}}}{\varepsilon} P\Big(\varepsilon \frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big) + \frac{\varepsilon}{3}\vartheta^4, \tag{1.23}$$

$$e_\varepsilon(\varrho,\vartheta) = \frac{3}{2}\frac{\vartheta^{\frac{5}{2}}}{\varepsilon\varrho} P\Big(\varepsilon \frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big) + \varepsilon\vartheta^4, \tag{1.24}$$

$$s_\varepsilon(\varrho,\vartheta) = S\Big(\varepsilon \frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big) - S(\varepsilon) + \varepsilon\frac{4}{3}\frac{\vartheta^3}{\varrho}, \tag{1.25}$$

and the heat flux

$$\mathbf{q}_\varepsilon = -\Big(\kappa_0(\vartheta) + \frac{1}{\varepsilon^2}\vartheta^3\Big)\nabla_x\vartheta. \tag{1.26}$$

Accordingly, the entropy production rate takes the form

$$\begin{aligned}\sigma_\varepsilon &\geq \frac{1}{\vartheta}\Big(\varepsilon^2\mathbb{S}:\nabla_x\mathbf{u} - \frac{\mathbf{q}_\varepsilon\cdot\nabla_x\vartheta}{\vartheta}\Big) \qquad (1.27)\\ &= \varepsilon^2\frac{\mu(\vartheta)}{2\vartheta}\Big|\nabla_x\mathbf{u} + \nabla_x^\perp\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\Big|^2 + \frac{\kappa_0(\vartheta)}{\vartheta^2}|\nabla_x\vartheta|^2 + \frac{1}{\varepsilon^2}\vartheta|\nabla_x\vartheta|^2.\end{aligned}$$

System (1.19–1.27) is supplemented with the boundary conditions (1.6–1.7), condition (1.8) being replaced by

$$\mathbf{q}_\varepsilon\cdot\mathbf{n} = \eta(\vartheta)\frac{\vartheta-\overline{\vartheta}}{\varepsilon}\Big|_{\{x_3=1\}}. \tag{1.28}$$

### 1.4. Singular limit

The scaling in (1.20), (1.26) corresponds to the situation when the Mach, Froude, and Péclet numbers are simultaneously proportional to a small parameter $\varepsilon$ (cf. the survey paper by Klein et al. [19]). Our main aim is to perform the asymptotic limit in (1.19–1.28) for $\varepsilon \to 0$ and to identify the limit (target) problem. More specifically, we show (see Theorem 3.1 below) that for a family $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ of solutions to the complete Navier-Stokes-Fourier system (1.19–1.28)

$$\mathbf{u}_\varepsilon \to \mathbf{U},\ \varrho_\varepsilon \to \tilde{\varrho},\ \vartheta_\varepsilon \to \overline{\vartheta},\ \text{and}\ \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon^2} \to \Theta \text{ as } \varepsilon \to 0,$$

where $\mathbf{U}$, $\tilde{\varrho}$, $\overline{\vartheta}$, $\Theta$ solve the problem:

$$\partial_t(\tilde{\varrho}\mathbf{U}) + \mathrm{div}_x(\tilde{\varrho}\mathbf{U}\times\mathbf{U}) + \tilde{\varrho}\nabla_x\Pi = \mu(\overline{\vartheta})(\Delta\mathbf{U} + \frac{1}{3}\nabla_x\mathrm{div}_x\mathbf{U}) - \frac{\tilde{\varrho}}{\overline{\vartheta}}\Theta\mathbf{j}, \tag{1.29}$$

$$\beta\tilde{\varrho}U_3 + \overline{\vartheta}^3\Delta\Theta = 0, \tag{1.30}$$

$$\mathrm{div}_x(\tilde{\varrho}\mathbf{U}) = 0. \tag{1.31}$$

System (1.29–1.31) was introduced by Chandrasekhar [6] as a simple alternative to the Oberbeck-Boussinesq system in the case when both the Froude and Prandtl numbers approach zero. More recently, Ligniéres [20] used the same idea in order to describe the flow dynamics in stellar radiative zones. Here $\mathbf{U}$ denotes the fluid velocity, $\Theta$ is the temperature, and $\beta$ is a positive constant. The density $\tilde{\varrho} = \tilde{\varrho}(x_3)$ obeys the hydrostatic balance equation

$$\nabla_x p(\tilde{\varrho}, \overline{\vartheta}) = -\tilde{\varrho}\mathbf{j}, \tag{1.32}$$

where $\overline{\vartheta} > 0$ plays the role of a reference temperature. Equation (1.31) represents the so-called *anelastic* constraint replacing the more common incompressibility condition $\mathrm{div}_x \mathbf{U} = 0$.

Similarly to (1.6), the velocity field $\mathbf{U}$ obeys the complete slip boundary conditions

$$\mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{T}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0, \tag{1.33}$$

with the viscous stress tensor

$$\mathbb{T} = \mu(\overline{\vartheta})\Big(\nabla_x \mathbf{U} + \nabla_x \mathbf{U}^{\perp} - \frac{2}{3}\mathrm{div}_x \mathbf{U}\Big),$$

while the temperature $\Theta$ satisfies the homogeneous Neumann boundary conditions

$$\nabla_x \Theta \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{1.34}$$

Our approach leans on the concept of weak solutions to the complete Navier-Stokes-Fourier system introduced in [11] and further developed in [13]. Similarly to the results by Bresch et al. [5], Desjardins et al. [9], Lions and Masmoudi [21] (for more references see the survey paper by Masmoudi [22]) devoted to the barotropic Navier-Stokes system, our theory is based on the uniform bounds available in the framework of weak solutions defined on an arbitrarily large time interval $(0, T)$. Note that there is an alternative approach proposed in the pioneering paper by Klainerman and Majda [18] followed by Danchin [7], [8], Hoff [17], Schochet [27], [26], among others, which is based on uniform estimates in Sobolev spaces of higher order imposing severe restrictions on the length of the associated existence time interval $(0, T)$. The most relevant results for the complete Navier-Stokes-Fourier system in this direction were obtained quite recently by Alazard [1], [2].

The paper is organized as follows. To begin with, we review the standard material concerning the proper definition, existence, and basic properties of the weak solutions to the full Navier-Stokes-Fourier system (see Section 2). Main results are formulated in Section 3. In Section 4, we establish the so-called dissipation inequality and deduce uniform estimates for the family of solutions $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ independent of $\varepsilon \to 0$. To this end, we introduce a modified Helmholtz free energy functional in the form

$$H_{\overline{\vartheta}}(\varrho, \vartheta) = \varrho e(\varrho, \vartheta) - \overline{\vartheta}\varrho s(\varrho, \vartheta). \tag{1.35}$$

Preliminary results on convergence towards the target system are established in Section 5. The bulk of the paper consists in analysis of the acoustic equation governing the time evolution of the gradient component of the velocity field (see Section 6). Although the underlying ideas are reasonably intuitive, the analysis becomes rather technical. This is mainly because the wave speed varies with the vertical coordinate as a consequence of the strong stratification in the fluid. Note that the relevant mathematical theory was developed by Wilcox [28] and subsequently used in [12] in order to study the low Mach-Froude number limit for isentropic fluids. Similar results were obtained independently by Masmoudi [23]. In this context, condition (1.31) is called an anelastic constraint, appearing in most of the simplified models of strongly stratified fluids. The proof of the main result is completed in Section 7, where we establish convergence of the temperature. Since the Péclet number is of the same order as the Mach number, the limit temperature $\Theta$ is recovered, being proportional to the $\varepsilon^2-$term in the (formal) asymptotic expansion of $\vartheta$ with respect to $\varepsilon$.

## 2. Preliminaries

We shall say that a trio $\{\varrho, \mathbf{u}, \vartheta\}$ is a weak solution of problem (1.1–1.9) on a time interval $(0,T)$ satisfying the initial conditions

$$\varrho(0,\cdot) = \varrho_0,\ \mathbf{u}(0,\cdot) = \mathbf{u}_0,\ \vartheta(0,\cdot) = \vartheta_0 \tag{2.1}$$

if

$$\left\{ \begin{array}{c} \varrho \geq 0,\ \varrho \in L^\infty(0,T;L^{\frac{5}{3}}(\Omega)), \\ \mathbf{u} \in L^2(0,T;W^{1,2}(\Omega;R^3)),\ \mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0, \\ \vartheta > 0 \text{ a.a. in } (0,T)\times\Omega,\ \vartheta \in L^\infty(0,T;L^4(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega)), \end{array} \right\} \tag{2.2}$$

and the following integral identities are satisfied:

$$\int_0^T \int_\Omega \Big[\varrho B(\varrho)\partial_t\varphi + \varrho B(\varrho)\mathbf{u}\cdot\nabla_x\varphi - b(\varrho)\mathrm{div}_x\mathbf{u}\varphi\Big]\mathrm{d}x\ \mathrm{d}t = -\int_\Omega \varrho_0 B(\varrho_0)\varphi(0,\cdot)\ \mathrm{d}x \tag{2.3}$$

for any test function $\varphi \in \mathcal{D}([0,T)\times\overline{\Omega})$, and any $b$,

$$b \in L^\infty \cap C[0,\infty),\ B(\varrho) = B(1) + \int_1^\varrho \frac{b(z)}{z}\ \mathrm{d}z; \tag{2.4}$$

$$\begin{aligned} &\int_0^T \int_\Omega \Big[\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\varphi + p(\varrho,\vartheta)\mathrm{div}_x\varphi\Big]\mathrm{d}x\ \mathrm{d}t \\ &= \int_0^T \int_\Omega \Big[\mathbb{S} : \nabla_x\varphi + \varrho g\varphi_3\Big]\mathrm{d}x\ \mathrm{d}t - \int_\Omega \varrho_0\mathbf{u}_0\varphi(0,\cdot)\ \mathrm{d}x, \end{aligned} \tag{2.5}$$

for any test function $\varphi \in \mathcal{D}([0,T);\mathcal{D}(\overline{\Omega};R^3))$, $\varphi\cdot\mathbf{n}|_{\partial\Omega} = 0$, where

$$\mathbb{S} = \mu\Big(\nabla_x\mathbf{u} + \nabla_x^\perp\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\Big); \tag{2.6}$$

$$\begin{aligned} &\int_0^T \int_\Omega \Big[\varrho s(\varrho,\vartheta)\partial_t\varphi + \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla_x\varphi + \frac{\mathbf{q}}{\vartheta}\cdot\nabla_x\varphi\Big]\mathrm{d}x\ \mathrm{d}t \\ &+ \int_0^T \int_\Omega \frac{1}{\vartheta}\Big[\mathbb{S} : \nabla_x\mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\Big]\varphi\mathrm{d}x\ \mathrm{d}t - \int_0^T \int_{\{x_3=1\}} \frac{\eta}{\vartheta}(\vartheta - \overline{\vartheta})\ \mathrm{d}S_x\ \mathrm{d}t \\ &\leq -\int_\Omega \varrho_0 s(\varrho_0,\vartheta_0)\varphi(0,\cdot)\ \mathrm{d}x \end{aligned} \tag{2.7}$$

for any $\varphi \in \mathcal{D}([0,T);\mathcal{D}(\overline{\Omega}))$, $\varphi \geq 0$, where

$$\mathbf{q} = -\kappa(\vartheta)\nabla_x \vartheta; \tag{2.8}$$

$$\int_\Omega \Big[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) + \varrho g x_3\Big](\tau)\ \mathrm{d}x \tag{2.9}$$
$$= \int_\Omega \Big[\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0,\vartheta_0) + \varrho_0 x g_3\Big]\mathrm{d}x + \int_0^\tau \int_{\{x_3=1\}} \eta(\overline{\vartheta} - \vartheta)\ \mathrm{d}S_x\ \mathrm{d}t$$

for a.a. $\tau \in (0,T)$.

The *existence* of the weak solutions in the sense of the above definition for any finite energy initial data $\{\varrho_0, \mathbf{u}_0, \vartheta_0\}$ can be established similarly to [13] provided the constitutive equations are similar to those introduced in Section 1.3.

## 3. Main results

Having introduced the necessary preliminary material we are in a position to state our main result.

**Theorem 3.1.** *Let* $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ *be a family of weak solutions to the rescaled system* (1.19–1.22), *supplemented with the boundary conditions* (1.6), (1.7), (1.28), *in the sense specified in Section* 2, *where*

- *the quantities* $p_\varepsilon$, $e_\varepsilon$, $s_\varepsilon$ *are given through* (1.23–1.25), *and, in addition,*

$$P(Y) = \beta Y,\ \ \beta > 0 \text{ for all } 0 \leq Y \leq \overline{Y}, \tag{3.1}$$

$$\lim_{Y\to\infty} \frac{P(Y)}{Y^{\frac{5}{3}}} = P_\infty > 0, \tag{3.2}$$

*and the thermodynamics stability hypothesis*

$$P'(Y) > 0,\ \ -\infty < \inf_{y>0} yS'(y) \leq YS'(Y) < 0 \tag{3.3}$$

*holds for all* $Y > 0$;

- *the viscous stress tensor* $\mathbb{S}$ *is given by* (1.15), *the heat flux* $\mathbf{q}_\varepsilon$ *satisfies* (1.26), *with*

$$0 < \underline{\mu}(1+\vartheta) \leq \mu(\vartheta) \leq \overline{\mu}(1+\vartheta), \tag{3.4}$$

$$0 < \underline{\kappa} \leq \kappa_0(\vartheta) \leq \overline{\kappa}(1+\vartheta) \tag{3.5}$$

*and the coefficient* $\eta$ *in equation* (1.28) *verifies*

$$0 < \underline{\eta}\vartheta \leq \eta(\vartheta) \leq \overline{\eta}\vartheta \tag{3.6}$$

*for all* $\vartheta > 0$, *where* $\underline{\mu}$, $\overline{\mu}$, $\underline{\kappa}$, $\overline{\kappa}$, $\underline{\eta}$, $\overline{\eta}$ *are positive constants;*

- *the initial data satisfy*

$$\varrho_0 = \varrho_{0,\varepsilon} = \tilde{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon},\ \mathbf{u}_0 = \mathbf{u}_{0,\varepsilon},\ \vartheta_0 = \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon}, \tag{3.7}$$

*with*

$$\{\varrho^{(1)}_{\varepsilon,0}\}_{\varepsilon>0},\ \{|\mathbf{u}_{\varepsilon,0}|\}_{\varepsilon>0},\ \{\vartheta^{(1)}_{\varepsilon,0}\}_{\varepsilon>0} \textit{ bounded in } L^\infty(\Omega), \tag{3.8}$$

$$\int_\Omega \varrho^{(1)}_{\varepsilon,0}\ \mathrm{d}x = 0 \textit{ for all } \varepsilon \to 0, \tag{3.9}$$

*where $\overline{\vartheta}$ is a positive constant and $\tilde{\varrho}$, $\overline{\vartheta}$ solve the hydrostatic balance equation*

$$\nabla_x p_\varepsilon(\tilde{\varrho}, \overline{\vartheta}) = -\tilde{\varrho}\mathbf{j} \textit{ in } \Omega. \tag{3.10}$$

*Then, at least for a suitable subsequence,*

$$\varrho_\varepsilon \to \tilde{\varrho} \textit{ in } L^\infty(0,T;L^{\frac{5}{3}}(\Omega)) \cap C([0,T];L^q(\Omega)) \textit{ for all } 1 \le q < \frac{5}{3},$$
$$\mathbf{u}_\varepsilon \to \mathbf{U} \textit{ weakly in } L^2(0,T;W^{1,2}(\Omega;R^3)),$$
$$\vartheta_\varepsilon \to \overline{\vartheta} \textit{ in } L^2(0,T;W^{1,2}(\Omega)),$$

*and*

$$\nabla_x \Big( \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon^2} \Big) \to \nabla_x \Theta \textit{ weakly in } L^1((0,T) \times \Omega; R^3),$$

*where $\tilde{\varrho}$, $\mathbf{U}$, $\overline{\vartheta}$, $\Theta$ represent a weak solution to problem* (1.29–1.34)*, supplemented with the initial condition*

$$\mathbf{U}(0,\cdot) = \text{weak } \lim_{\varepsilon\to 0} \mathbf{H}_{\tilde{\varrho}}[\tilde{\varrho}\mathbf{u}_{0,\varepsilon}], \tag{3.11}$$

*where the symbol $\mathbf{H}_{\tilde{\varrho}}$ denotes a "weighted" Helmholtz projection introduced in* (6.12)*,* (6.13)*.*

Hypothesis (3.1) asserts that the fluid behaves like a perfect gas governed by the state equation $p = \beta\varrho\vartheta$ provided the argument $\varrho/\vartheta^{3/2}$ is bounded by $\overline{Y}$, while relation (3.2) characterizes a mixture of monoatomic gases, where at least one constituent is a Fermi gas, in the degenerate area of large densities and/or low temperature (see Müller and Ruggeri [24] for more details on the physical background).

The thermodynamics stability hypothesis (3.4) states that both compressibility $\partial_\varrho p$, and the specific heat at constant volume $\partial_\vartheta e$, are strictly positive. In particular, it follows that problem (1.19–1.22) admits a unique *static* solution $\tilde{\varrho}$,

$$\nabla_x p_\varepsilon(\tilde{\varrho}, \overline{\vartheta}) = -\tilde{\varrho}\mathbf{j} \text{ in } \Omega,\ \int_\Omega \tilde{\varrho}\ \mathrm{d}x = M,\ \tilde{\varrho} \ge 0, \tag{3.12}$$

for any given $\overline{\vartheta} > 0$, $M \ge 0$. Note that, in accordance with hypothesis (3.1),

$$\tilde{\varrho} = c(\overline{\vartheta}, M) \exp\Big( -\frac{x_3}{\beta\overline{\vartheta}} \Big)$$

as soon as $\varepsilon > 0$ is small enough. Thus, without loss of generality, we shall assume $\beta = 1$, $\overline{\vartheta} = 1$, and $M > 0$ have been chosen in such a way that

$$\tilde{\varrho} = \exp(-x_3). \tag{3.13}$$

It is worth noting that the thermodynamics stability hypothesis (3.3) implies *linear* stability of the static solution $\tilde{\varrho}$, $\overline{\vartheta}$ of system (1.19–1.22) (see Bechtel et al. [4]).

The rest of the paper is devoted to the proof of Theorem 3.1. As already pointed out, there are two fundamental issues to be addressed: (i) stability, or uniform estimates, of the family $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$, (ii) possible time oscillations of the acoustic waves, meaning the gradient part in the Helmholtz decomposition of the velocity fields $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$. As we will see in the next section, the thermodynamics stability hypothesis (3.3), together with Gibbs' relation (1.4), are sufficient for the family $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ to be bounded in the function spaces that are basically determined by *a priori* estimates in the framework of finite-energy weak solutions.

To simplify notation, it seems convenient to decompose each quantity $h_\varepsilon = h_\varepsilon(t,x)$ as

$$h = [h]_{\rm ess} + [h]_{\rm res},$$

with the "essential" part $[h]_{\rm ess}$ given as

$$[h]_{\rm ess} = h \ {\rm char}\{(t,x) \mid \underline{\varrho} < \varrho_\varepsilon(t,x) < \overline{\varrho}, \ \overline{\vartheta}/2 < \vartheta_\varepsilon(t,x) < 2\overline{\vartheta}\}, \tag{3.14}$$

where

$$0 < 2\underline{\varrho} \le \inf_x \tilde{\varrho}(x) \le \sup_x \tilde{\varrho}(x) \le \overline{\varrho}/2. \tag{3.15}$$

Accordingly, the "residual" part is determined as

$$[h]_{\rm res} = h - [h]_{\rm ess}. \tag{3.16}$$

Clearly, both the residual and essential parts depend on $\varepsilon$.

## 4. Uniform estimates

### 4.1. Total mass conservation

As the densities $\varrho_\varepsilon$ satisfy (2.3), we have $\varrho_\varepsilon \in C_{\rm weak}([0,T]; L^{\frac{5}{3}}(\Omega))$, and, taking $b \equiv 0$ we get, in accordance with hypotheses (3.8), (3.9),

$$M_0 = \int_\Omega \tilde{\varrho} \ {\rm d}x = \int_\Omega \varrho_\varepsilon(t,\cdot) \ {\rm d}x \text{ for all } t \in [0,T]. \tag{4.1}$$

In other words, the total mass $M_0$ of the fluid is a constant of motion.

### 4.2. Total dissipation balance

The entropy balance formulated through (1.22) holds in the weak sense specified in (2.7). In particular, in view of the standard Riesz representation theorem, the entropy production rate $\sigma_\varepsilon$ can be interpreted as a non-negative measure $\sigma_\varepsilon \in \mathcal{M}^+([0,T] \times \overline{\Omega})$ satisfying

$$\int_\Omega \varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon)(\tau,\cdot) \ {\rm d}x - \sigma_\varepsilon[[0,\tau] \times \overline{\Omega}] + \int_0^\tau \int_{\{x_3=1\}} \frac{\eta(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \ {\rm d}S_x \ {\rm d}t$$
$$= \int_\Omega \varrho_{0,\varepsilon} s_\varepsilon(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) \ {\rm d}x \text{ for a.a. } \tau \in (0,T), \tag{4.2}$$

where, in accordance with (1.27),

$$\sigma_\varepsilon \geq \varepsilon^2 \frac{\mu(\vartheta_\varepsilon)}{2\vartheta_\varepsilon}\Big|\nabla_x \mathbf{u}_\varepsilon + \nabla_x^\perp \mathbf{u}_\varepsilon - \frac{2}{3}\mathrm{div}_x \mathbf{u}_\varepsilon \mathbb{I}\Big|^2 + \frac{\kappa_0(\vartheta_\varepsilon)}{\vartheta_\varepsilon^2}|\nabla_x \vartheta_\varepsilon|^2 + \frac{1}{\varepsilon^2}\vartheta_\varepsilon|\nabla_x\vartheta_\varepsilon|^2. \tag{4.3}$$

Combining (4.2) with the total energy equality (1.22) we arrive at the total dissipation balance in the form

$$\begin{aligned}
&\int_\Omega \Big[\frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2}H^\varepsilon_{\overline{\vartheta}}(\varrho_\varepsilon,\vartheta_\varepsilon) + \frac{1}{\varepsilon^2}\varrho_\varepsilon x_3\Big](\tau,\cdot)\ \mathrm{d}x \\
&\quad + \frac{\overline{\vartheta}}{\varepsilon^2}\sigma_\varepsilon[[0,\tau]\times\overline{\Omega}] + \int_0^\tau \int_{\{x_3=1\}} \frac{\eta(\vartheta_\varepsilon)}{\vartheta_\varepsilon}\frac{(\vartheta_\varepsilon - \overline{\vartheta})^2}{\varepsilon^3}\ \mathrm{d}S_x\ \mathrm{d}t \\
&= \int_\Omega \Big[\frac{1}{2}\varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^2}H^\varepsilon_{\overline{\vartheta}}(\varrho_{\varepsilon,0},\vartheta_{\varepsilon,0}) + \frac{1}{\varepsilon^2}\varrho_{0,\varepsilon}x_3\Big]\mathrm{d}x \text{ for a.a. } \tau\in(0,T),
\end{aligned} \tag{4.4}$$

where we have set

$$H^\varepsilon_{\overline{\vartheta}}(\varrho,\vartheta) = \varrho e_\varepsilon(\varrho,\vartheta) - \overline{\vartheta}\varrho s_\varepsilon(\varrho,\vartheta), \tag{4.5}$$

with $e_\varepsilon$, $s_\varepsilon$ being introduced in (1.24), (1.25).

Our next goal is to establish certain coercivity properties of the mapping

$$[\varrho,\vartheta] \mapsto H^\varepsilon_{\overline{\vartheta}}(\varrho,\vartheta) = \varrho e_\varepsilon(\varrho,\vartheta) - \overline{\vartheta}\varrho s_\varepsilon(\varrho,\vartheta).$$

To this end, write

$$H^\varepsilon_{\overline{\vartheta}}(\varrho,\vartheta) = H^\varepsilon_{\overline{\vartheta}}(\varrho,\overline{\vartheta}) + H^\varepsilon_{\overline{\vartheta}}(\varrho,\vartheta) - H^\varepsilon_{\overline{\vartheta}}(\varrho,\overline{\vartheta}), \tag{4.6}$$

where, by virtue of Gibbs' relation (1.4),

$$\frac{\partial^2 H^\varepsilon_{\overline{\vartheta}}(\varrho,\overline{\vartheta})}{\partial\varrho^2} = \frac{1}{\varrho}\frac{\partial p_\varepsilon(\varrho,\overline{\vartheta})}{\partial\varrho}, \tag{4.7}$$

in other words,

$$\frac{\partial H^\varepsilon_{\overline{\vartheta}}(\varrho,\overline{\vartheta})}{\partial\varrho} = \int_1^\varrho \frac{1}{z}\frac{\partial p_\varepsilon(z,\overline{\vartheta})}{\partial\varrho}\ \mathrm{d}z + c(\overline{\vartheta}).$$

Consequently, seeing that $\tilde{\varrho}$ solves the static problem (3.12) we get

$$\frac{\partial H^\varepsilon_{\overline{\vartheta}}(\tilde{\varrho},\overline{\vartheta})}{\partial\varrho} = -x_3 + c(\tilde{\varrho},\overline{\vartheta});$$

whence

$$\begin{aligned}
H^\varepsilon_{\overline{\vartheta}}(\varrho_\varepsilon,\vartheta_\varepsilon) + \varrho_\varepsilon x_3 &= H^\varepsilon_{\overline{\vartheta}}(\varrho_\varepsilon,\vartheta_\varepsilon) - H^\varepsilon_{\overline{\vartheta}}(\varrho_\varepsilon,\overline{\vartheta}) + H^\varepsilon_{\overline{\vartheta}}(\varrho_\varepsilon,\overline{\vartheta}) \\
&\quad - (\varrho_\varepsilon - \tilde{\varrho})\frac{\partial H^\varepsilon_{\overline{\vartheta}}(\tilde{\varrho},\overline{\vartheta})}{\partial\varrho} + c(\tilde{\varrho},\overline{\vartheta})\varrho_\varepsilon - \tilde{\varrho}\frac{\partial H^\varepsilon_{\overline{\vartheta}}(\tilde{\varrho},\overline{\vartheta})}{\partial\varrho},
\end{aligned}$$

while

$$\begin{aligned}
H^\varepsilon_{\overline{\vartheta}}(\varrho_{\varepsilon,0},\vartheta_{\varepsilon,0}) + \varrho_{\varepsilon,0}x_3 &= H^\varepsilon_{\overline{\vartheta}}(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon}) - H^\varepsilon_{\overline{\vartheta}}(\varrho_{0,\varepsilon},\overline{\vartheta}) + H^\varepsilon_{\overline{\vartheta}}(\varrho_{0,\varepsilon},\overline{\vartheta}) \\
&\quad - (\varrho_{0,\varepsilon} - \tilde{\varrho})\frac{\partial H^\varepsilon_{\overline{\vartheta}}(\tilde{\varrho},\overline{\vartheta})}{\partial\varrho} + c(\tilde{\varrho},\overline{\vartheta})\varrho_{0,\varepsilon} - \tilde{\varrho}\frac{\partial H^\varepsilon_{\overline{\vartheta}}(\tilde{\varrho},\overline{\vartheta})}{\partial\varrho}.
\end{aligned}$$

Thus the dissipation balance (4.4) can be rewritten in the form

$$\frac{1}{2}\int_\Omega \frac{1}{2}\varrho_\varepsilon |\mathbf{u}_\varepsilon|^2(\tau,\cdot)\ \mathrm{d}x + \int_\Omega \Big[\frac{H^\varepsilon_{\overline{\vartheta}}(\varrho_\varepsilon,\vartheta_\varepsilon) - H^\varepsilon_{\overline{\vartheta}}(\varrho_\varepsilon,\overline{\vartheta})}{\varepsilon^2}\Big](\tau,\cdot)\ \mathrm{d}x \tag{4.8}$$
$$+ \frac{1}{\varepsilon^2}\int_\Omega \Big[H^\varepsilon_{\overline{\vartheta}}(\varrho_\varepsilon,\overline{\vartheta}) - (\varrho_\varepsilon - \tilde{\varrho})\frac{\partial H^\varepsilon_{\overline{\vartheta}}(\tilde{\varrho},\overline{\vartheta})}{\partial \varrho} - H^\varepsilon_{\overline{\vartheta}}(\tilde{\varrho},\overline{\vartheta})\Big](\tau,\cdot)\ \mathrm{d}x$$
$$+ \frac{\overline{\vartheta}}{\varepsilon^2}\sigma_\varepsilon[[0,\tau]\times\overline{\Omega}] + \int_0^\tau \int_{\{x_3=1\}} \frac{1}{\vartheta_\varepsilon}\frac{(\vartheta_\varepsilon - \overline{\vartheta})^2}{\varepsilon^3}\ \mathrm{d}S_x\ \mathrm{d}t$$
$$= \int_\Omega \frac{1}{2}\varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}|^2\ \mathrm{d}x + \int_\Omega \Big[\frac{H^\varepsilon_{\overline{\vartheta}}(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon}) - H^\varepsilon_{\overline{\vartheta}}(\varrho_{0,\varepsilon},\overline{\vartheta})}{\varepsilon^2}\Big]\mathrm{d}x$$
$$+ \frac{1}{\varepsilon^2}\int_\Omega \Big[H^\varepsilon_{\overline{\vartheta}}(\varrho_{0,\varepsilon},\overline{\vartheta}) - (\varrho_{0,\varepsilon} - \tilde{\varrho})\frac{\partial H^\varepsilon_{\overline{\vartheta}}(\tilde{\varrho},\overline{\vartheta})}{\partial \varrho} - H^\varepsilon_{\overline{\vartheta}}(\tilde{\varrho},\overline{\vartheta})\Big]\ \mathrm{d}x$$

for a.a. $\tau \in (0,T)$.

Now, as a direct consequence of (1.4), we have

$$\frac{\partial H^\varepsilon_{\overline{\vartheta}}(\varrho,\vartheta)}{\partial \vartheta} = \varrho(\vartheta - \overline{\vartheta})\frac{\partial s_\varepsilon(\varrho,\vartheta)}{\partial \vartheta}, \tag{4.9}$$

which, together with (4.7) and the thermodynamics stability hypothesis (3.3), yields

$$\left\{\begin{array}{c} \underline{\Lambda}(\vartheta - \overline{\vartheta})^2 \le \Big[\frac{H^\varepsilon_{\overline{\vartheta}}(\varrho,\vartheta) - H^\varepsilon_{\overline{\vartheta}}(\varrho,\overline{\vartheta})}{\varepsilon^2}\Big] \le \overline{\Lambda}(\vartheta - \overline{\vartheta})^2 \\ \text{provided} \\ 0 < \underline{\varrho} \le \varrho \le \overline{\varrho},\ \overline{\vartheta}/2 \le \vartheta \le 2\overline{\vartheta}, \end{array}\right\} \tag{4.10}$$

and

$$\left\{\begin{array}{c} \underline{\Lambda}(\varrho - \tilde{\varrho})^2 \le \Big[H^\varepsilon_{\overline{\vartheta}}(\varrho,\overline{\vartheta}) - (\varrho - \tilde{\varrho})\frac{\partial H^\varepsilon_{\overline{\vartheta}}(\tilde{\varrho},\overline{\vartheta})}{\partial \varrho} - H^\varepsilon_{\overline{\vartheta}}(\tilde{\varrho},\overline{\vartheta})\Big] \le \overline{\Lambda}(\varrho - \tilde{\varrho})^2 \\ \text{as soon as} \\ 0 < \underline{\varrho} \le \varrho \le \overline{\varrho} \end{array}\right\} \tag{4.11}$$

for positive constants $\underline{\Lambda}$, $\overline{\Lambda}$, with $\underline{\varrho}$, $\overline{\varrho}$ introduced in (3.15). It follows from (1.23–1.25), and, notably (3.1), that the quantity $H^\varepsilon_{\overline{\vartheta}}$ is in fact independent of $\varepsilon$ provided $\varrho$, $\vartheta$ satisfy (4.11) and $\varepsilon > 0$ is small enough. In particular, in agreement with hypotheses (3.8), (3.9), we deduce that the expression on the right-hand side of (4.8) is bounded uniformly for $\varepsilon \to 0$.

Now relations (1.23), (4.7) give rise to

$$\frac{\partial^2 H^\varepsilon_{\overline{\vartheta}}(\varrho,\overline{\vartheta})}{\partial \varrho^2} = \frac{\overline{\vartheta}}{\varrho}P'\Big(\varepsilon\frac{\varrho}{\overline{\vartheta}^{\frac{3}{2}}}\Big) \ge \frac{c(\overline{\vartheta})}{\varrho}, \tag{4.12}$$

where we have used hypotheses (3.1), (3.2). Thus the dissipation inequality (4.8) yields

$$\operatorname*{ess\,sup}_{t\in(0,T)} \left\| \left[ \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right]_{\rm ess} \right\|_{L^2(\Omega)} \le c \tag{4.13}$$

and

$$\operatorname*{ess\,sup}_{t\in(0,T)} \int_\Omega |[\varrho_\varepsilon \log(\varrho_\varepsilon)]_{\rm res}| \ {\rm d}x \le \varepsilon^2 c, \tag{4.14}$$

where the "essential" and "residual" components have been introduced in (3.14), (3.16).

Similarly,

$$\operatorname*{ess\,sup}_{t\in(0,T)} \left\| \left[ \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right]_{\rm ess} \right\|_{L^2(\Omega)} \le c, \tag{4.15}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \int_\Omega \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 \ {\rm d}x \le c, \tag{4.16}$$

and, as a consequence of the presence of "radiation" terms in (1.23–1.25),

$$\operatorname*{ess\,sup}_{t\in(0,T)} \int_\Omega |[\vartheta_\varepsilon]^4_{\rm res}| \ {\rm d}x \le \varepsilon c. \tag{4.17}$$

In addition, using the coercivity properties of $H^\varepsilon_{\overline{\vartheta}}$ stated in (4.7), (4.9–4.11) we deduce that the measure of the "residual" set is small of order $\varepsilon^2$, more specifically,

$$\operatorname*{ess\,sup}_{t\in(0,T)} \int_\Omega [1]_{\rm res} \ {\rm d}x \le c\varepsilon^2. \tag{4.18}$$

Furthermore, we have

$$\int_0^T \int_{\{x_3=1\}} \left| \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right|^2 \ {\rm d}S_x \ {\rm d}t \le c\varepsilon, \tag{4.19}$$

and

$$\sigma_\varepsilon[[0,T]\times\overline{\Omega}] \le c\varepsilon^2, \tag{4.20}$$

where, in view of (4.3) and hypotheses (3.4), (3.5), the latter bound gives rise to

$$\int_0^T \int_\Omega |\nabla_x \mathbf{u}_\varepsilon + \nabla_x^\perp \mathbf{u}_\varepsilon - \frac{2}{3}{\rm div}_x \mathbf{u}_\varepsilon \mathbb{I}|^2 \ {\rm d}x \ {\rm d}t \le c, \tag{4.21}$$

$$\int_0^T \int_\Omega \vartheta_\varepsilon \left| \nabla_x \Big( \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon^2} \Big) \right|^2 \ {\rm d}x \ {\rm d}t \le c, \tag{4.22}$$

and

$$\int_0^T \int_\Omega \left| \nabla_x \Big( \frac{\log(\vartheta_\varepsilon) - \log(\overline{\vartheta})}{\varepsilon} \Big) \right|^2 \ {\rm d}x \ {\rm d}t \le c. \tag{4.23}$$

Estimate (4.21), combined with (4.13), (4.15), and Korn's and Poincaré's inequalities, yields

$$\{\mathbf{u}_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^2(0,T;W^{1,2}(\Omega;R^3)). \tag{4.24}$$

By the same token, estimates (4.15), (4.18), (4.22), (4.23) give rise to

$$\left\{ \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right\}_{\varepsilon>0} \text{ bounded in } L^2(0,T;W^{1,2}(\Omega)), \tag{4.25}$$

$$\left\{ \frac{\log(\vartheta_\varepsilon) - \log(\overline{\vartheta})}{\varepsilon} \right\}_{\varepsilon>0} \text{ bounded in } L^2(0,T;W^{1,2}(\Omega)). \tag{4.26}$$

In order to derive uniform estimates on the residual components, we need the following assertion.

**Lemma 4.1.** *Let*

$$H_{\overline{\vartheta}}(\varrho,\vartheta) = \varrho e(\varrho,\vartheta) - \overline{\vartheta}\varrho s(\varrho,\vartheta), \ \overline{\vartheta} > 0,$$

*where the functions* $e$, $s$ *obey Gibb's equation* (1.4), *together with the thermodynamics stability hypotheses*

$$\frac{\partial p(\varrho,\vartheta)}{\partial \varrho} > 0, \ \frac{\partial e(\varrho,\vartheta)}{\partial \vartheta} > 0.$$

*Then*

$$H_{\overline{\vartheta}}(\varrho,\vartheta) \geq \frac{1}{4}\Big(\varrho e(\varrho,\vartheta) + \overline{\vartheta}\varrho|s(\varrho,\vartheta)|\Big) - \left|(\varrho - \tilde{\varrho})\frac{\partial H_{2\overline{\vartheta}}}{\partial \varrho}(\tilde{\varrho},2\overline{\vartheta}) + H_{2\overline{\vartheta}}(\tilde{\varrho},2\overline{\vartheta})\right|$$

*for all positive* $\varrho$, $\vartheta$ *and any* $\tilde{\varrho} > 0$.

*Proof.* As the result obviously holds for $s(\varrho,\vartheta) \leq 0$, we focus on the case $s(\varrho,\vartheta) > 0$. By virtue of (4.7), (4.9), we get

$$H_{2\overline{\vartheta}}(\varrho,\vartheta) \geq (\varrho - \tilde{\varrho})\frac{\partial H_{2\overline{\vartheta}}}{\partial \varrho}(\tilde{\varrho},2\overline{\vartheta}) + H_{2\overline{\vartheta}}(\tilde{\varrho},2\overline{\vartheta});$$

whence

$$\begin{aligned} H_{\overline{\vartheta}}(\varrho,\vartheta) &= \frac{1}{2}\varrho e(\varrho,\vartheta) + \frac{1}{2}H_{2\overline{\vartheta}}(\varrho,\vartheta) \\ &\geq \frac{1}{2}\varrho e(\varrho,\vartheta) + \frac{1}{2}\Big((\varrho - \tilde{\varrho})\frac{\partial H_{2\overline{\vartheta}}}{\partial \varrho}(\tilde{\varrho},2\overline{\vartheta}) + H_{2\overline{\vartheta}}(\tilde{\varrho},2\overline{\vartheta})\Big), \end{aligned}$$

and, similarly,

$$\begin{aligned} H_{\overline{\vartheta}}(\varrho,\vartheta) &= \overline{\vartheta}\varrho s(\varrho,\vartheta) + H_{2\overline{\vartheta}}(\varrho,\vartheta) \\ &\geq \overline{\vartheta}\varrho s(\varrho,\vartheta) + (\varrho - \tilde{\varrho})\frac{\partial H_{2\overline{\vartheta}}}{\partial \varrho}(\tilde{\varrho},2\overline{\vartheta}) + H_{2\overline{\vartheta}}(\tilde{\varrho},2\overline{\vartheta}). \end{aligned}$$

Summing up the last two inequalities we obtain the desired conclusion. □

At this stage, estimates (4.8), (4.14) can be combined with the conclusion of Lemma 4.1 in order to deduce that

$$\operatorname{ess}\sup_{t\in(0,T)} \int_\Omega [\varrho_\varepsilon e_\varepsilon(\varrho_\varepsilon,\vartheta_\varepsilon) + \varrho_\varepsilon|s_\varepsilon(\varrho_\varepsilon,\vartheta_\varepsilon)|]_{\mathrm{res}} \,\mathrm{d}x \,\mathrm{d}t \leq c\varepsilon^2. \tag{4.27}$$

Thus, finally, hypotheses (3.1), (3.2), with (1.24), imply

$$\operatorname{ess}\sup_{t\in(0,T)} \int_\Omega [\varrho_\varepsilon]_{\mathrm{res}}^{\frac{5}{3}} \,\mathrm{d}x \leq c\varepsilon^{\frac{4}{3}}, \tag{4.28}$$

and

$$\operatorname*{ess\ sup}_{t\in(0,T)} \int_\Omega [\varrho_\varepsilon\vartheta_\varepsilon]_{\rm res}\ {\rm d}x \le c\varepsilon^2. \tag{4.29}$$

**4.3. Pressure estimates**

In order to control the quantity $p_\varepsilon - \varrho_\varepsilon\vartheta_\varepsilon$ in the limit for $\varepsilon \to 0$, we evoke the technique developed in [14] in order to obtain an additional piece of information concerning integrability of the pressure.

The principal idea is to consider the test functions

$$\vec\varphi(t,x) = \psi(t)\nabla_x\Delta_n^{-1}\Big[b(\varrho_\varepsilon) - \frac{1}{|\Omega|}\int_\Omega b(\varrho_\varepsilon)\ {\rm d}x\Big],\ \psi\in\mathcal{D}(0,T),$$

in the variational formulation of the momentum equation (2.5). Here the symbol $\Delta_n^{-1}$ stands for the inverse of the Laplace operator on the domain $\Omega$ supplemented with the homogeneous Neumann boundary conditions on $\partial\Omega$.

After a bit tedious but straightforward manipulation, we obtain

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\int_0^T \psi\int_\Omega p_\varepsilon(\varrho_\varepsilon,\vartheta_\varepsilon)b(\varrho_\varepsilon)\ {\rm d}x\ {\rm d}t\\
&= \frac{1}{\varepsilon^2|\Omega|}\int_0^T\psi\int_\Omega p_\varepsilon(\varrho_\varepsilon,\vartheta_\varepsilon)\ {\rm d}x\int_\Omega b(\varrho_\varepsilon)\ {\rm d}x\ {\rm d}t\\
&\quad + \frac{1}{\varepsilon^2}\int_0^T\psi\int_\Omega \varrho_\varepsilon\mathbf{j}\cdot\nabla_x\Delta_n^{-1}\Big[b(\varrho_\varepsilon) - \frac{1}{|\Omega|}\int_\Omega b(\varrho_\varepsilon)\ {\rm d}x\Big]\ {\rm d}x\ {\rm d}t + I_\varepsilon,
\end{aligned}\tag{4.30}$$

where

$$\begin{aligned}
I_\varepsilon = &\int_0^T\int_\Omega\Big(\mathbb{S}_\varepsilon : \nabla_x\vec\varphi - \varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon : \nabla_x\vec\varphi\ {\rm d}x\Big){\rm d}t\\
&- \int_0^T \partial_t\psi\int_\Omega\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla_x\Delta_n^{-1}\Big[b(\varrho_\varepsilon) - \frac{1}{|\Omega|}\int_\Omega b(\varrho_\varepsilon)\ {\rm d}x\Big]\ {\rm d}x\ {\rm d}t\\
&+ \int_0^T\psi\int_\Omega \varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla_x\Delta_n^{-1}{\rm div}_x(b(\varrho_\varepsilon)\mathbf{u}_\varepsilon)\ {\rm d}x\ {\rm d}t\\
&+ \int_0^T\psi\int_\Omega \varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla_x\Delta_n^{-1}\Big[(\varrho_\varepsilon b'(\varrho_\varepsilon) - b(\varrho_\varepsilon)){\rm div}_x\mathbf{u}_\varepsilon\\
&\qquad\qquad - \frac{1}{|\Omega|}\int_\Omega (b(\varrho_\varepsilon) - b'(\varrho_\varepsilon)\varrho_\varepsilon){\rm div}_x\mathbf{u}_\varepsilon\ {\rm d}x\Big]\ {\rm d}x\ {\rm d}t.
\end{aligned}$$

Here we have used the renormalized continuity equation (2.3) in order to compute

$$\partial_t b(\varrho_\varepsilon) = -{\rm div}(b(\varrho_\varepsilon)\mathbf{u}_\varepsilon) - (\varrho_\varepsilon b'(\varrho_\varepsilon) - b(\varrho_\varepsilon)){\rm div}_x\mathbf{u}_\varepsilon \quad \text{in } \mathcal{D}'((0,T)\times\overline{\Omega}). \tag{4.31}$$

Taking the uniform estimates established in the preceding section into account we can show, exactly as in [14], that all integrals contained in $I_\varepsilon$ are bounded uniformly for $\varepsilon\to 0$ as soon as

$$|b(\varrho)| + |\varrho b'(\varrho)| \le c\varrho^\gamma \text{ for } \gamma > 0 \text{ small enough.} \tag{4.32}$$

In order to comply with (4.32), let us take $b \in C^\infty[0,\infty)$ such that

$$b(\varrho) = \begin{cases} 0 \text{ for } 0 \le \varrho \le 2\overline{\varrho}, \\ \in [0, \varrho^\gamma] \text{ for } 2\overline{\varrho} < \varrho \le 3\overline{\varrho}, \\ \varrho^\gamma \text{ if } \varrho > 3\overline{\varrho}, \end{cases} \tag{4.33}$$

with $\gamma > 0$ sufficiently small to be specified below. In particular, is easy to check that

$$b(\varrho_\varepsilon) = b([\varrho_\varepsilon]_{\rm res});$$

whence, in accordance with (4.14),

$$\operatorname*{ess\ sup}_{t\in(0,T)} \int_\Omega b(\varrho_\varepsilon)\ {\rm d}x \le c\varepsilon^2 \tag{4.34}$$

as soon as $0 < \gamma < 1$.

Furthermore, since $\tilde\varrho$, $\overline{\vartheta}$ are interrelated through (3.10), we have

$$\begin{aligned} &\frac{1}{\varepsilon^2}\int_\Omega \varrho_\varepsilon \mathbf{j}\cdot\nabla_x\Delta_n^{-1}\Big[b(\varrho_\varepsilon) - \frac{1}{|\Omega|}\int_\Omega b(\varrho_\varepsilon)\ {\rm d}x\Big]\ {\rm d}x \\ &= \frac{1}{\varepsilon}\int_\Omega \Big[\frac{\varrho_\varepsilon-\tilde\varrho}{\varepsilon}\Big]_{\rm ess}\mathbf{j}\cdot\nabla_x\Delta_n^{-1}\Big[b(\varrho_\varepsilon) - \frac{1}{|\Omega|}\int_\Omega b(\varrho_\varepsilon)\ {\rm d}x\Big]\ {\rm d}x \\ &\quad + \int_\Omega \Big[\frac{\varrho_\varepsilon-\tilde\varrho}{\varepsilon^2}\Big]_{\rm res}\mathbf{j}\cdot\nabla_x\Delta_n^{-1}\Big[b(\varrho_\varepsilon) - \frac{1}{|\Omega|}\int_\Omega b(\varrho_\varepsilon)\ {\rm d}x\Big]\ {\rm d}x \\ &\quad - \frac{1}{\varepsilon^2}\int_\Omega \tilde\varrho\ \overline{\vartheta}\Big(b(\varrho_\varepsilon) - \frac{1}{|\Omega|}\int_\Omega b(\varrho_\varepsilon)\ {\rm d}x\Big)\ {\rm d}x. \end{aligned} \tag{4.35}$$

Using the standard $L^p$-elliptic estimates for $\Delta_n$ we deduce

$$\begin{aligned} &\frac{1}{\varepsilon}\Big|\int_\Omega \Big[\frac{\varrho_\varepsilon-\tilde\varrho}{\varepsilon}\Big]_{\rm ess}\mathbf{j}\cdot\nabla_x\Delta_n^{-1}\Big[b(\varrho_\varepsilon) - \frac{1}{|\Omega|}\int_\Omega b(\varrho_\varepsilon)\ {\rm d}x\Big]\ {\rm d}x\Big| \\ &\le \frac{c}{\varepsilon}\operatorname*{ess\ sup}_{t\in(0,T)}\Big\|\Big[\frac{\varrho_\varepsilon-\tilde\varrho}{\varepsilon}\Big]_{\rm ess}\Big\|_{L^2(\Omega)}\operatorname*{ess\ sup}_{t\in(0,T)}\Big\|b(\varrho_\varepsilon) - \frac{1}{|\Omega|}\int_\Omega b(\varrho_\varepsilon)\ {\rm d}x\Big\|_{L^{\frac{6}{5}}(\Omega)}, \end{aligned} \tag{4.36}$$

$$\begin{aligned} &\Big|\int_\Omega \Big[\frac{\varrho_\varepsilon-\tilde\varrho}{\varepsilon^2}\Big]_{\rm res}\mathbf{j}\cdot\nabla_x\Delta_n^{-1}\Big[b(\varrho_\varepsilon) - \frac{1}{|\Omega|}\int_\Omega b(\varrho_\varepsilon)\ {\rm d}x\Big]\ {\rm d}x\Big| \\ &\le \operatorname*{ess\ sup}_{t\in(0,T)}\Big\|\Big[\frac{\varrho_\varepsilon-\tilde\varrho}{\varepsilon^2}\Big]_{\rm res}\Big\|_{L^1(\Omega)}\operatorname*{ess\ sup}_{t\in(0,T)}\Big\|b(\varrho_\varepsilon) - \frac{1}{|\Omega|}\int_\Omega b(\varrho_\varepsilon)\ {\rm d}x\Big\|_{L^4(\Omega)}, \end{aligned} \tag{4.37}$$

while the last integral on the right-hand side of (4.35) is bounded because of (4.34).

In order to conclude, we observe that estimate (4.14) yields

$$\int_\Omega |b(\varrho_\varepsilon)|^{\frac{6}{5}}\ {\rm d}x \le \int_\Omega [\varrho_\varepsilon]_{\rm res}^{\frac{6\beta}{5}}\ {\rm d}x \le \int_\Omega [\varrho_\varepsilon\log(\varrho_\varepsilon)]_{\rm res}\ {\rm d}x \le c\varepsilon^2$$

as soon as $\beta \le 5/6$. Thus we obtain, combining estimates (4.30–4.37), that

$$\int_0^T\int_\Omega p_\varepsilon(\varrho_\varepsilon,\vartheta_\varepsilon)b(\varrho_\varepsilon)\ {\rm d}x\ {\rm d}t \le \varepsilon^2 c, \tag{4.38}$$

with $b$ given by (4.33) provided $\gamma \in (0, \frac{5}{6})$.

## 5. Convergence

It follows from estimates (4.13), (4.16), (4.24), and (4.27) that

$$\left\{ \begin{array}{c} \varrho_\varepsilon \to \tilde{\varrho} \text{ in } L^\infty(0,T;L^{\frac{5}{3}}(\Omega)), \\ \mathbf{u}_\varepsilon \to \mathbf{U} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega;R^3)), \\ \varrho_\varepsilon \mathbf{u}_\varepsilon \to \tilde{\varrho}\mathbf{U} \text{ weakly-}* \text{ in } L^\infty(0,T;L^{\frac{5}{4}}(\Omega;R^3)), \end{array} \right\} \tag{5.1}$$

passing to suitable subsequences as the case may be. Clearly, we recover the boundary conditions

$$\mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

and, taking $b = 0$ and letting $\varepsilon \to 0$ in the renormalized equation of continuity (2.3), we infer

$$\mathrm{div}_x(\tilde{\varrho}\mathbf{U}) = 0.$$

### 5.1. Pressure

Let us examine the pressure term $p_\varepsilon$ that can be written as

$$p_\varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon) = \beta\varrho_\varepsilon\vartheta_\varepsilon + \Big[\frac{\vartheta_\varepsilon^{\frac{5}{2}}}{\varepsilon} P\Big(\varepsilon\frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{\frac{3}{2}}}\Big) - \beta\varrho_\varepsilon\vartheta_\varepsilon\Big] + \frac{\varepsilon}{3}(\vartheta_\varepsilon^4 - \overline{\vartheta}^4).$$

Our aim is to show that $p_\varepsilon \approx \beta\varrho_\varepsilon\vartheta_\varepsilon$ in the asymptotic limit $\varepsilon \to 0$.

To begin with, the radiative component can be decomposed as

$$\vartheta_\varepsilon^4 - \overline{\vartheta}^4 = [\vartheta_\varepsilon^4 - \overline{\vartheta}^4]_{\mathrm{res}} + [\vartheta_\varepsilon^4 - \overline{\vartheta}^4]_{\mathrm{ess}},$$

where, by virtue of the uniform estimates (4.17), (4.25),

$$\int_0^T \int_\Omega |[\vartheta_\varepsilon^4 - \overline{\vartheta}^4]_{\mathrm{res}}| \, \mathrm{d}x \, \mathrm{d}t \le c \int_0^T \int_\Omega |\vartheta_\varepsilon - \overline{\vartheta}|([\vartheta_\varepsilon]^3_{\mathrm{res}} + [\overline{\vartheta}]^3_{\mathrm{res}})| \, \mathrm{d}x \, \mathrm{d}t$$
$$\le \|\vartheta_\varepsilon - \overline{\vartheta}\|_{L^2(0,T;L^4(\Omega))} \operatorname*{ess\,sup}_{t\in(0,T)} \Big( \|[\vartheta_\varepsilon]^3_{\mathrm{res}}\|_{L^{\frac{4}{3}}(\Omega)} + \|[\overline{\vartheta}]^3_{\mathrm{res}}\|_{L^{\frac{4}{3}}(\Omega)} \Big) \le c\varepsilon^{\frac{7}{4}}.$$

In order to control the essential component of the radiation pressure, we first recall Poincaré's inequality

$$\|\vartheta_\varepsilon^{\frac{3}{2}} - \overline{\vartheta}^{\frac{3}{2}}\|^2_{L^2((0,T)\times\Omega)} \le c\Big[\int_0^T \int_\Omega \vartheta_\varepsilon |\nabla_x \vartheta_\varepsilon|^2 \, \mathrm{d}x + \Big(\int_0^T \int_{\{x_3=1\}} |\vartheta_\varepsilon^{\frac{3}{2}} - \overline{\vartheta}^{\frac{3}{2}}| \, \mathrm{d}x \, \mathrm{d}t\Big)^2\Big], \tag{5.2}$$

where

$$\Big(\int_0^T \int_{\{x_3=1\}} |\vartheta_\varepsilon^{\frac{3}{2}} - \overline{\vartheta}^{\frac{3}{2}}| \, \mathrm{d}x \, \mathrm{d}t\Big)^2$$
$$\le \int_0^T \int_{\{x_3=1\}} |\vartheta_\varepsilon - \overline{\vartheta}|^2 \, \mathrm{d}x \, \mathrm{d}t \int_0^T \int_{\{x_3=1\}} (\vartheta_\varepsilon + \overline{\vartheta}) \mathrm{d}x \, \mathrm{d}t.$$

Thus we can use the uniform estimates (4.19), (4.22) in order to conclude that

$$\|[\vartheta_\varepsilon^4 - \overline{\vartheta}^4]_{\mathrm{ess}}\|_{L^2((0,T)\times\Omega)} \le c\|[\vartheta_\varepsilon^{\frac{3}{2}} - \overline{\vartheta}^{\frac{3}{2}}]_{\mathrm{ess}}\|_{L^2((0,T)\times\Omega)} \le c\varepsilon^{\frac{3}{2}}. \tag{5.3}$$

Accordingly, the radiation component of the pressure appearing in the rescaled momentum equation (1.20) tends to zero for $\varepsilon \to 0$, specifically,

$$\Big\| \frac{\vartheta_\varepsilon^4 - \overline{\vartheta}^4}{\varepsilon} \Big\|_{L^1((0,T)\times\Omega)} \leq c\varepsilon^{\frac{1}{2}}. \tag{5.4}$$

The next goal is to show that

$$\frac{1}{\varepsilon^2}\Big[ \frac{\vartheta_\varepsilon^{\frac{5}{2}}}{\varepsilon} P\Big( \varepsilon \frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{\frac{3}{2}}} \Big) - \beta \varrho_\varepsilon \vartheta_\varepsilon \Big] \to 0 \text{ in } L^1((0,T)\times\Omega) \text{ for } \varepsilon \to 0. \tag{5.5}$$

In order to see that, we first evoke hypothesis (3.1) obtaining

$$\begin{aligned} &\int_0^T \int_\Omega \Big| \frac{\vartheta_\varepsilon^{\frac{5}{2}}}{\varepsilon} P\Big( \varepsilon \frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{\frac{3}{2}}} \Big) - \beta \varrho_\varepsilon \vartheta_\varepsilon \Big| \, \mathrm{d}x \, \mathrm{d}t \\ &= \int_{\{\overline{Y}^{\frac{2}{3}} \vartheta_\varepsilon \leq \varepsilon^{\frac{2}{3}} \varrho_\varepsilon^{\frac{2}{3}}\}} \Big| \Big[ \frac{\vartheta_\varepsilon^{\frac{5}{2}}}{\varepsilon} P\Big( \varepsilon \frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{\frac{3}{2}}} \Big) - \beta \varrho_\varepsilon \vartheta_\varepsilon \Big]_{\mathrm{res}} \Big| \, \mathrm{d}x \, \mathrm{d}t \\ &= \int_{\{\overline{Y}^{\frac{2}{3}} \vartheta_\varepsilon \leq \varepsilon^{\frac{2}{3}} \varrho_\varepsilon^{\frac{2}{3}}, \ \varrho_\varepsilon \leq K\}} \Big| \Big[ \frac{\vartheta_\varepsilon^{\frac{5}{2}}}{\varepsilon} P\Big( \varepsilon \frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{\frac{3}{2}}} \Big) - \beta \varrho_\varepsilon \vartheta_\varepsilon \Big]_{\mathrm{res}} \Big| \, \mathrm{d}x \, \mathrm{d}t \\ &\quad + \int_{\{\overline{Y}^{\frac{2}{3}} \vartheta_\varepsilon \leq \varepsilon^{\frac{2}{3}} \varrho_\varepsilon^{\frac{2}{3}}, \ \varrho_\varepsilon > K\}} \Big| \Big[ \frac{\vartheta_\varepsilon^{\frac{5}{2}}}{\varepsilon} P\Big( \varepsilon \frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{\frac{3}{2}}} \Big) - \beta \varrho_\varepsilon \vartheta_\varepsilon \Big]_{\mathrm{res}} \Big| \, \mathrm{d}x \, \mathrm{d}t \end{aligned} \tag{5.6}$$

for any $K > 0$ provided $\varepsilon > 0$ is small enough.

It follows from hypotheses (3.1), (3.2), combined with estimate (4.18), that

$$\begin{aligned} \int_{\{\overline{Y}^{\frac{2}{3}} \vartheta_\varepsilon \leq \varepsilon^{\frac{2}{3}} \varrho_\varepsilon^{\frac{2}{3}}, \ \varrho_\varepsilon \leq K\}} \Big| \Big[ \frac{\vartheta_\varepsilon^{\frac{5}{2}}}{\varepsilon} P\Big( \varepsilon \frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{\frac{3}{2}}} \Big) - \varrho_\varepsilon \vartheta_\varepsilon \Big]_{\mathrm{res}} \Big| \, \mathrm{d}x \, \mathrm{d}t &\leq c_1(K) \varepsilon^{\frac{2}{3}} \int_0^T \int_\Omega 1_{\mathrm{res}} \, \mathrm{d}x \\ &\leq c_2(K) \varepsilon^{\frac{8}{3}}. \end{aligned} \tag{5.7}$$

On the other hand, by virtue of (4.38) and hypotheses (3.1), (3.2) , we have

$$\int_{\{\varrho_\varepsilon > K\}} \varepsilon^{\frac{2}{3}} \varrho_\varepsilon^{\frac{5}{3}+\gamma} \, \mathrm{d}x \, \mathrm{d}t \leq \int_{\{\varrho_\varepsilon > K\}} \frac{\vartheta_\varepsilon^{\frac{5}{2}}}{\varepsilon} P\Big( \varepsilon \frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{\frac{3}{2}}} \Big) \varrho_\varepsilon^\gamma \, \mathrm{d}x \, \mathrm{d}t \leq c\varepsilon^2,$$

and, consequently,

$$\begin{aligned} \int_{\{\overline{Y}^{\frac{2}{3}} \vartheta_\varepsilon \leq \varepsilon^{\frac{2}{3}} \varrho_\varepsilon^{\frac{2}{3}}, \ \varrho_\varepsilon > K\}} \Big| \Big[ \frac{\vartheta_\varepsilon^{\frac{5}{2}}}{\varepsilon} P\Big( \varepsilon \frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{\frac{3}{2}}} \Big) - \beta \varrho_\varepsilon \vartheta_\varepsilon \Big]_{\mathrm{res}} \Big| \, \mathrm{d}x \, \mathrm{d}t &\leq c_1 \int_{\{\varrho_\varepsilon > K\}} \varepsilon^{\frac{2}{3}} \varrho_\varepsilon^{\frac{5}{3}} \, \mathrm{d}x \, \mathrm{d}t \\ &\leq \frac{c_2 \varepsilon^2}{K^\gamma}. \end{aligned} \tag{5.8}$$

Combining (5.6), together with (5.7), (5.8), we get (5.5). In view of (5.4), (5.5), we conclude that

$$\frac{1}{\varepsilon^2} p_\varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon) = \frac{\beta}{\varepsilon^2} \varrho_\varepsilon \vartheta_\varepsilon + \chi_\varepsilon, \ \text{ where } \chi_\varepsilon \to 0 \text{ in } L^1((0,T)\times\Omega). \tag{5.9}$$

### 5.2. Driving term

Our next goal is to identify the driving force in (1.29). To this end, we rewrite the corresponding term in (1.20) as follows:

$$\begin{aligned}&\frac{1}{\varepsilon^2}\int_0^T\int_\Omega\Big[\beta\varrho_\varepsilon\vartheta_\varepsilon\mathrm{div}_x\varphi-\varrho_\varepsilon\varphi_3\Big]\mathrm{d}x\ \mathrm{d}t \qquad (5.10)\\ &\quad=\frac{\beta}{\varepsilon^2}\int_0^T\int_\Omega\overline{\vartheta}\frac{\varrho_\varepsilon}{\tilde\varrho}\mathrm{div}_x(\tilde\varrho\varphi)\mathrm{d}x\ \mathrm{d}t+\frac{\beta}{\varepsilon^2}\int_0^T\int_\Omega\varrho_\varepsilon(\vartheta_\varepsilon-\overline{\vartheta})\mathrm{div}_x\varphi\mathrm{d}x\ \mathrm{d}t,\end{aligned}$$

where

$$\begin{aligned}&\frac{\beta}{\varepsilon^2}\int_0^T\int_\Omega\varrho_\varepsilon(\vartheta_\varepsilon-\overline{\vartheta})\mathrm{div}_x\varphi\mathrm{d}x\ \mathrm{d}t\\ &\quad=\frac{\beta}{\varepsilon^2}\int_0^T\int_\Omega(\varrho_\varepsilon-\tilde\varrho)(\vartheta_\varepsilon-\overline{\vartheta})\mathrm{div}_x\varphi\mathrm{d}x\ \mathrm{d}t+\frac{\beta}{\varepsilon^2}\int_0^T\int_\Omega\tilde\varrho(\vartheta_\varepsilon-\overline{\vartheta})\mathrm{div}_x\varphi\mathrm{d}x\ \mathrm{d}t,\end{aligned}$$

and, furthermore,

$$\frac{1}{\varepsilon^2}(\varrho_\varepsilon-\tilde\varrho)(\vartheta_\varepsilon-\overline{\vartheta})=\Big[\frac{\varrho_\varepsilon-\tilde\varrho}{\varepsilon}\Big]_{\rm ess}\Big[\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}\Big]_{\rm ess}+\Big[\frac{\varrho_\varepsilon-\tilde\varrho}{\varepsilon}\Big]_{\rm res}\Big(\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}\Big).$$

By virtue of (4.13), (5.3), we obtain

$$\Big\|\Big[\frac{\varrho_\varepsilon-\tilde\varrho}{\varepsilon}\Big]_{\rm ess}\Big[\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}\Big]_{\rm ess}\Big\|_{L^1((0,T)\times\Omega)}\le c\sqrt{\varepsilon}\to 0, \qquad (5.11)$$

while, in agreement with (4.25) and the standard imbedding $W^{1,2}\hookrightarrow L^6$,

$$\Big\|\Big[\frac{\varrho_\varepsilon-\tilde\varrho}{\varepsilon}\Big]_{\rm res}\Big(\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}\Big)\Big\|_{L^1((0,T)\times\Omega)}\le c\Big\|\Big[\frac{\varrho_\varepsilon-\tilde\varrho}{\varepsilon}\Big]_{\rm res}\Big\|_{L^2(0,T;L^{\frac65}(\Omega))}. \qquad (5.12)$$

On the other hand, by interpolation,

$$\|r\|_{L^{\frac65}(\Omega)}\le\|r\|^{\frac{7}{12}}_{L^1(\Omega)}\|r\|^{\frac{5}{12}(\Omega)}_{L^{\frac53}};$$

whence, by virtue of the uniform estimates (4.14), (4.28),

$$\Big\|\Big[\frac{\varrho_\varepsilon-\tilde\varrho}{\varepsilon}\Big]_{\rm res}\Big\|_{L^infty(0,T;L^{\frac65}(\Omega))}\le c\sqrt{\varepsilon},$$

which, combined with (5.11), (5.12), yields

$$\frac{1}{\varepsilon^2}(\varrho_\varepsilon-\tilde\varrho)(\vartheta_\varepsilon-\overline{\vartheta})\to 0 \text{ in } L^1((0,T)\times\Omega). \qquad (5.13)$$

Finally,

$$\begin{aligned}&\frac{\beta}{\varepsilon^2}\int_0^T\int_\Omega\tilde\varrho(\vartheta_\varepsilon-\overline{\vartheta})\mathrm{div}_x\varphi\mathrm{d}x\ \mathrm{d}t\\ &\quad=\frac{\beta}{\varepsilon^2}\int_0^T\int_\Omega(\vartheta_\varepsilon-\overline{\vartheta})\mathrm{div}_x(\tilde\varrho\varphi)\mathrm{d}x\ \mathrm{d}t-\frac{\beta}{\varepsilon^2}\int_0^T\int_\Omega(\vartheta_\varepsilon-\overline{\vartheta})\nabla_x\tilde\varrho\cdot\varphi\mathrm{d}x\ \mathrm{d}t,\end{aligned}$$

where, furthermore,

$$\frac{\beta}{\varepsilon^2}\int_0^T\int_\Omega(\vartheta_\varepsilon-\overline{\vartheta})\nabla_x\tilde{\varrho}\cdot\varphi\mathrm{d}x\ \mathrm{d}t \tag{5.14}$$
$$=\frac{\beta}{\varepsilon^2}\frac{1}{\Omega}\int_0^T\Big(\int_\Omega\vartheta_\varepsilon-\overline{\vartheta}\ \mathrm{d}z\Big)\int_\Omega\nabla_x\tilde{\varrho}\cdot\varphi\ \mathrm{d}x\ \mathrm{d}t$$
$$+\frac{\beta}{\varepsilon^2}\int_0^T\int_\Omega\Big(\vartheta_\varepsilon-\frac{1}{|\Omega|}\int_\Omega\vartheta_\varepsilon\ \mathrm{d}z\Big)\nabla_x\tilde{\varrho}\cdot\varphi\mathrm{d}x\ \mathrm{d}t$$
$$=-\frac{\beta}{\varepsilon^2}\frac{1}{\Omega}\int_0^T\Big(\int_\Omega\vartheta_\varepsilon-\overline{\vartheta}\ \mathrm{d}z\Big)\int_\Omega\log(\tilde{\varrho})\mathrm{div}_x(\tilde{\varrho}\varphi)\ \mathrm{d}x\ \mathrm{d}t$$
$$+\frac{\beta}{\varepsilon^2}\int_0^T\int_\Omega\Big(\vartheta_\varepsilon-\frac{1}{|\Omega|}\int_\Omega\vartheta_\varepsilon\ \mathrm{d}z\Big)\nabla_x\tilde{\varrho}\cdot\varphi\mathrm{d}x\ \mathrm{d}t.$$

At this stage, our ultimate goal is to deduce uniform estimates on the quantity

$$\frac{1}{\varepsilon^2}\Big(\vartheta_\varepsilon-\frac{1}{|\Omega|}\int_\Omega\vartheta_\varepsilon\ \mathrm{d}z\Big)$$

appearing in the last integral in (5.14). To this end, we write

$$\sqrt{\overline{\vartheta}}\frac{\nabla_x(\vartheta_\varepsilon-\overline{\vartheta})}{\varepsilon^2}=\frac{\sqrt{\overline{\vartheta}}-\sqrt{\vartheta_\varepsilon}}{\varepsilon}\nabla_x\frac{(\vartheta_\varepsilon-\overline{\vartheta})}{\varepsilon}+\sqrt{\vartheta_\varepsilon}\nabla_x\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon^2},$$

where, by virtue of (4.17), (4.25),

$$\Big\{\frac{\sqrt{\vartheta_\varepsilon}-\sqrt{\overline{\vartheta}}}{\varepsilon}\Big\}_{\varepsilon>0}\ \text{is bounded in}\ L^\infty(0,T;L^1(\Omega))\cap L^2(0,T;L^6(\Omega)).$$

Consequently, in accordance with (4.22), (4.25),

$$\Big\{\frac{\nabla_x(\vartheta_\varepsilon-\overline{\vartheta})}{\varepsilon^2}\}_{\varepsilon>0}\ \text{is bounded in}\ L^q((0,T)\times\Omega;R^3)\ \text{for a certain}\ q>1,$$

therefore, by virtue of Poincaré's inequality,

$$\Big\|\vartheta_\varepsilon-\frac{1}{|\Omega|}\int_\Omega\vartheta_\varepsilon\ \mathrm{d}x\Big\|_{L^q(0,T;L^q(\Omega))}\le c\varepsilon^2\ \text{for a certain}\ q>1. \tag{5.15}$$

**5.3. Asymptotic limit in the equation of momentum**

Summing up the results established in Sections 5.1, 5.2 we are allowed to let $\varepsilon\to 0$ in the rescaled momentum equation (1.20) in order to obtain an integral identity

$$\int_0^T\int_\Omega\Big[\tilde{\varrho}\mathbf{U}\cdot\partial_t\varphi+\tilde{\varrho}\ \overline{\mathbf{u}\otimes\mathbf{u}}:\nabla_x\varphi\Big]\mathrm{d}x\ \mathrm{d}t \tag{5.16}$$
$$=\int_0^T\int_\Omega\Big[\mathbb{S}:\nabla_x\varphi+\frac{\tilde{\varrho}}{\overline{\vartheta}}\Theta\varphi_3\Big]\mathrm{d}x\ \mathrm{d}t-\int_\Omega\tilde{\varrho}\mathbf{u}_0\varphi(0,\cdot)\ \mathrm{d}x,$$

for any test function $\varphi\in\mathcal{D}([0,T);\mathcal{D}(\overline{\Omega};R^3))$, $\varphi\cdot\mathbf{n}|_{\partial\Omega}=0$, satisfying, in addition,

$$\mathrm{div}_x(\tilde{\varrho}\varphi)=0.$$

Here, we have denoted

$$\mathbb{S} = \mu(\overline{\vartheta})\Big(\nabla_x \mathbf{U} + \nabla_x^{\perp} \mathbf{U} - \frac{2}{3}\mathrm{div}_x \mathbf{U}\mathbb{I}\Big),$$

$$\overline{\mathbf{u} \times \mathbf{u}} = (\mathrm{weak} - L^q) \lim_{\varepsilon \to 0} \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon, \tag{5.17}$$

and, in accordance with (5.15),

$$\Theta = (\mathrm{weak} - L^q) \lim_{\varepsilon \to 0} \frac{\vartheta_\varepsilon - \frac{1}{|\Omega|}\int_\Omega \vartheta_\varepsilon \ \mathrm{d}x}{\varepsilon^2} \tag{5.18}$$

for a certain $q > 1$.

Note that the integral identity (5.16) represents a weak formulation of the momentum equation (1.29) in the target system as soon as we can show that the weak limit $\overline{\mathbf{u} \otimes \mathbf{u}}$ can be replaced by $\mathbf{U} \otimes \mathbf{U}$. This step will be fully justified in the following section.

## 6. Analysis of the acoustic waves

### 6.1. Acoustic equation

System (2.3), (2.5) can be written in the form

$$\int_0^T \int_\Omega \Big[\varepsilon \frac{\varrho_\varepsilon - \tilde{\varrho}}{\varepsilon \tilde{\varrho}} \partial_t \varphi + \tilde{\varrho} \frac{\varrho_\varepsilon \tilde{\mathbf{u}}_\varepsilon}{\tilde{\varrho}} \nabla_x \frac{\varphi}{\tilde{\varrho}}\Big] \mathrm{d}x \ \mathrm{d}t = - \int_\Omega \varepsilon \frac{\varrho_{0,\varepsilon} - \tilde{\varrho}}{\varepsilon \tilde{\varrho}} \varphi(0,\cdot) \ \mathrm{d}x \tag{6.1}$$

for any $\varphi \in \mathcal{D}([0,T) \times \overline{\Omega})$,

$$\int_0^T \int_\Omega \Big[\varepsilon \frac{\varrho_\varepsilon \mathbf{u}_\varepsilon}{\tilde{\varrho}} \cdot \partial_t \varphi + \beta \overline{\vartheta} \frac{\varrho_\varepsilon - \tilde{\varrho}}{\varepsilon \tilde{\varrho}} \mathrm{div}_x \varphi\Big] \mathrm{d}x \ \mathrm{d}t \tag{6.2}$$

$$= \int_\Omega \Big[\varepsilon h_\varepsilon \mathrm{div}_x \frac{\varphi}{\tilde{\varrho}} + \varepsilon \mathbb{G}_\varepsilon : \nabla_x \frac{\varphi}{\tilde{\varrho}} + \beta \tilde{\varrho} \frac{\overline{\vartheta} - \vartheta_\varepsilon}{\varepsilon} \mathrm{div}_x \frac{\varphi}{\tilde{\varrho}}\Big] \mathrm{d}x - \int_\Omega \varepsilon \frac{\varrho_{0,\varepsilon} \mathbf{u}_{0,\varepsilon}}{\tilde{\varrho}} \cdot \varphi(0,\cdot) \ \mathrm{d}x$$

for any $\varphi \in \mathcal{D}([0,T) \times \overline{\Omega}; R^3)$, $\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$, where we have written

$$h_\varepsilon = \frac{1}{\varepsilon^2}\Big(\beta \varrho_\varepsilon \vartheta_\varepsilon - p_\varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon)\Big) + \beta \frac{\tilde{\varrho} - \varrho_\varepsilon}{\varepsilon} \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon},$$

$$\mathbb{G}_\varepsilon = \mathbb{S}_\varepsilon - \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon.$$

By virtue of (5.9), (5.13),

$$h_\varepsilon \to 0 \text{ in } L^1((0,T) \times \Omega),$$

while

$$\{\mathbb{G}_\varepsilon\}_{\varepsilon > 0} \text{ is bounded in } L^q(0,T; L^q(\Omega; R^{3\times 3})) \text{ for a certain } q > 1.$$

In addition, estimate (5.3) gives rise to

$$\Big\| \Big[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\Big]_{\mathrm{ess}} \Big\|_{L^2((0,T)\times\Omega)} \le c\sqrt{\varepsilon},$$

while (4.18), (4.25) yield

$$\left\| \left[ \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right]_{\rm res} \right\|_{L^1((0,T)\times\Omega)} \le c\varepsilon.$$

Consequently, for the new variables

$$r_\varepsilon = \frac{\varrho_\varepsilon - \tilde{\varrho}}{\varepsilon \tilde{\varrho}}, \ \mathbf{Q}_\varepsilon = \frac{\varrho_\varepsilon \mathbf{u}_\varepsilon}{\tilde{\varrho}}, \tag{6.3}$$

system (6.1), (6.2) can be rewritten in a concise form:

$$\int_0^T \int_\Omega \Big[ \varepsilon r_\varepsilon \partial_t \varphi + \tilde{\varrho} \mathbf{Q}_\varepsilon \nabla_x \frac{\varphi}{\tilde{\varrho}} \Big] \mathrm{d}x \ \mathrm{d}t = - \int_\Omega \varepsilon r_\varepsilon(0,\cdot) \varphi(0,\cdot) \ \mathrm{d}x \tag{6.4}$$

for any $\varphi \in \mathcal{D}([0,T) \times \overline{\Omega})$,

$$\int_0^T \int_\Omega \Big[ \varepsilon \mathbf{Q}_\varepsilon \cdot \partial_t \varphi + \beta \overline{\vartheta} r_\varepsilon \mathrm{div}_x \varphi \Big] \mathrm{d}x \ \mathrm{d}t = \int_\Omega \Big[ \sqrt{\varepsilon} H_\varepsilon \mathrm{div}_x \frac{\varphi}{\tilde{\varrho}} \Big] \mathrm{d}x - \int_\Omega \varepsilon \mathbf{Q}_\varepsilon(0,\cdot) \cdot \varphi(0,\cdot) \ \mathrm{d}x \tag{6.5}$$

for any $\varphi \in \mathcal{D}([0,T) \times \overline{\Omega}; R^3)$, $\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$, where

$$\{H_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^1((0,T)\times\Omega). \tag{6.6}$$

Obviously, the integral identities (6.4), (6.5) represent a weak formulation of a wave equation governing the time evolution of the acoustic waves studied in the seminal paper by Schochet [26]. However, there are two fundamental new ingredients in the analysis of system (6.4), (6.5):

- the wave speed of the acoustic waves depends effectively on the vertical coordinate $x_3$,
- the perturbation represented through term $H_\varepsilon$ is "large" of order $\sqrt{\varepsilon}$ in comparison with the frequency of the acoustic waves, which is of order $\varepsilon^{-1}$.

### 6.2. Spectral analysis

Following Wilcox [28] we consider an eigenvalue problem associated to (6.4), (6.5):

$$\tilde{\varrho} \nabla_x \Big( \frac{\omega}{\tilde{\varrho}} \Big) = \lambda \mathbf{V}, \ \beta \overline{\vartheta} \mathrm{div}_x \mathbf{V} = \lambda \omega \text{ in } \Omega, \ \mathbf{V} \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{6.7}$$

or, equivalently,

$$-\mathrm{div}_x \Big[ \tilde{\varrho} \nabla_x \Big( \frac{\omega}{\tilde{\varrho}} \Big) \Big] = \Lambda \tilde{\varrho} \Big( \frac{\omega}{\tilde{\varrho}} \Big) \text{ in } \Omega, \tag{6.8}$$

supplemented with the Neumann boundary condition

$$\nabla_x \Big( \frac{\omega}{\tilde{\varrho}} \Big) \cdot \mathbf{n}|_{\partial\Omega} = 0, \text{ with } \lambda^2 = -\Lambda \beta \overline{\vartheta}. \tag{6.9}$$

Similarly to Chapter 3 in [28], it is a routine matter to check that problem (6.8), (6.9) admits a complete system of real eigenfunctions $\{\omega_{j,m}\}_{j=0,m=1}^{\infty,m_j}$, together with the associated real eigenvalues $\Lambda_{j,m}$ such that

$$\begin{gathered} m_0 = 1, \ \Lambda_{0,1} = 0, \ \omega_{0,1} = \tilde{\varrho}, \\ 0 < \Lambda_{1,1} = \cdots = \Lambda_{1,m_1} (= \Lambda_1) < \Lambda_{2,1} = \cdots = \Lambda_{2,m_2} (= \Lambda_2) < \cdots, \end{gathered} \tag{6.10}$$

where $m_j$ stands for the multiplicity of the eigenvalue $\Lambda_j$. The system of functions $\{\omega_{j,m}\}_{j=0,m=1}^{\infty,m_j}$ forms an orthonormal basis of the Hilbert space $L^2_{1/\tilde{\varrho}}(\Omega)$ endowed with the scalar product

$$\langle v, w\rangle_{L^2_{1/\tilde{\varrho}}(\Omega)} = \int_\Omega vw\ \frac{\mathrm{d}x}{\tilde{\varrho}}.$$

Consequently, all solutions of (6.7) can be written as

$$\begin{aligned} \lambda_j &= \mathrm{i}\sqrt{\beta\overline{\vartheta}\Lambda_j}, \\ \mathbf{V}_{j,m} &= \mathrm{i}\Big(\sqrt{\beta\overline{\vartheta}\Lambda_j}\Big)^{-1}\tilde{\varrho}\nabla_x\Big(\frac{\omega_{j,m}}{\tilde{\varrho}}\Big), \end{aligned} \tag{6.11}$$

for $j = 1, \dots, m$, and $m = 1, \dots, m_j$.

In addition, the eigenspace associated to the eigenvalue $\Lambda_{0,1} = 0$ coincides with the space of solenoidal (divergenceless) functions.

At this stage, we introduce a "weighted" Helmholtz projection

$$\mathbf{H}_{\tilde{\varrho}}[\mathbf{v}] = \mathbf{v} - \tilde{\varrho}\nabla_x\Psi,\ \mathbf{H}^{\perp}_{\tilde{\varrho}}[\mathbf{v}] = \tilde{\varrho}\nabla_x\Psi, \tag{6.12}$$

where $\Psi$ is the unique solution of the Neumann problem

$$\mathrm{div}_x(\tilde{\varrho}\nabla_x\Psi) = \mathrm{div}_x\mathbf{v} \text{ in } \Omega,\ \tilde{\varrho}\nabla_x\Psi\cdot\mathbf{n} = \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega},\ \int_\Omega \Psi\ \mathrm{d}x = 0. \tag{6.13}$$

As $\tilde{\varrho}$ is smooth and bounded below away from zero on $\Omega$, we can use the standard elliptic theory in order to show that $\mathbf{H}_{\tilde{\varrho}}$ is a bounded linear operator on the Sobolev space $W^{1,q}_n(\Omega; R^3)$, and on $L^q(\Omega; R^3)$, $1 < q < \infty$ provided, in the latter case, $\mathrm{div}_x\mathbf{v}$ is identified with a linear form on $W^{1,q}_n(\Omega; R^3)$. Here the subscript $n$ denotes the subspace of vector functions whose normal component vanishes on $\partial\Omega$.

Thus the weighted space $L^2_{1/\tilde{\varrho}}(\Omega; R^3)$ admits an orthogonal decomposition

$$L^2_{1/\tilde{\varrho}}(\Omega; R^3) = L^2_{\mathrm{div},1/\tilde{\varrho}}(\Omega; R^3) \oplus \overline{\mathrm{span}\{\mathrm{i}\mathbf{V}_{j,m}\}_{j=1,m=1}^{\infty,m_j}}, \tag{6.14}$$

where

$$L^2_{\mathrm{div},1/\tilde{\varrho}}(\Omega; R^3) = \{\mathbf{v} \in L^2_{1/\tilde{\varrho}}(\Omega; R^3)\ |\ \mathrm{div}_x\mathbf{v} = 0\}. \tag{6.15}$$

Moreover, it is easy to check that the corresponding projectors in the decomposition (6.14) are represented by $\mathbf{H}_{\tilde{\varrho}}$, $\mathbf{H}^{\perp}_{\tilde{\varrho}}$.

Finally, we take $\varphi = \psi(t)\omega_{j,m}$ in (6.4), and $\varphi = \psi(t)\mathbf{V}_{j,m}$ in (6.5) in order to obtain a system of equations:

$$\varepsilon\partial_t[r_\varepsilon]_{j,m} + \mathrm{i}\sqrt{\Lambda_j}[\mathbf{Q}_\varepsilon]_{j,m} = 0, \tag{6.16}$$

$$\varepsilon\partial_t[\mathbf{Q}_\varepsilon]_{j,m} + \mathrm{i}\sqrt{\Lambda_j}[r_\varepsilon] = \sqrt{\varepsilon}H^{j,m}_\varepsilon, \tag{6.17}$$

$j = 1, 2, \dots,\ m = 1, \dots, m_j$, where we have set

$$[r_\varepsilon]_{j,m} = \int_\Omega r_\varepsilon\omega_{j,m}\ \mathrm{d}x,\ [\mathbf{Q}_\varepsilon]_{j,m} = \int_\Omega \mathbf{Q}_\varepsilon\cdot\mathbf{V}_{j,m}\ \mathrm{d}x. \tag{6.18}$$

Here, in accordance with (6.6),

$$\{H^{j,m}_\varepsilon\}_{j=1}^{\infty} \text{ is bounded in } L^1(0, T). \tag{6.19}$$

### 6.3. Analysis of the convective term

Having established all the necessary preliminary material, we are ready to show that the term $\overline{\mathbf{u} \times \mathbf{u}}$ can be replaced by $\mathbf{U} \times \mathbf{U}$ in (5.16). In other words, we have to show that

$$\int_0^T \int_\Omega \varrho_\varepsilon [\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon] : \nabla_x \Big( \frac{\varphi}{\tilde{\varrho}} \Big) \mathrm{d}x \ \mathrm{d}t \to \int_0^T \int_\Omega \tilde{\varrho} \mathbf{U} \otimes \mathbf{U} : \nabla_x \Big( \frac{\varphi}{\tilde{\varrho}} \Big) \mathrm{d}x \ \mathrm{d}t \tag{6.20}$$

for any test function $\varphi$ such that

$$\varphi \in \mathcal{D}([0,T) \times \overline{\Omega}; R^3), \ \ \varphi \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \operatorname{div}_x \varphi = 0 \text{ in } \Omega. \tag{6.21}$$

To begin with, observe that

$$\mathbf{H}_{\tilde{\varrho}}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \to \mathbf{H}_{\tilde{\varrho}}[\tilde{\varrho} \mathbf{U}] = \tilde{\varrho} \mathbf{U} \text{ in } L^1(0,T; L^1(\Omega; R^3)). \tag{6.22}$$

Indeed the uniform estimates on the pressure as well as the driving term in the rescaled momentum equation (1.20) obtained in Sections 5.1, 5.2 imply that the mappings

$$t \in [0,T] \mapsto \int_\Omega \mathbf{H}_{\tilde{\varrho}}[\varrho_\varepsilon \mathbf{u}_\varepsilon(t)] \cdot \phi \ \mathrm{d}x = \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon(t) \cdot \mathbf{H}_{\tilde{\varrho}}[\tilde{\varrho}\phi] \frac{\mathrm{d}x}{\tilde{\varrho}}$$

are precompact in $C[0,T]$ as soon as $\phi \in C^\infty(\overline{\Omega}; R^3)$, $\phi \cdot \mathbf{n}|_{\partial\Omega} = 0$. Thus

$$\mathbf{H}_{\tilde{\varrho}}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \to \mathbf{H}_{\tilde{\varrho}}[\tilde{\varrho} \mathbf{U}] = \tilde{\varrho} \mathbf{U} \text{ in } C_{\text{weak}}([0,T]; L^{\frac{5}{4}}(\Omega; R^3)) \tag{6.23}$$

in agreement with (5.1).

Consequently, as $\mathbf{H}_{\tilde{\varrho}}$ and $\mathbf{H}^\perp_{\tilde{\varrho}}$ are orthogonal projections in the weighted space $L^2_{1/\tilde{\varrho}}$, we get

$$\begin{aligned} \int_0^T \Big( \int_\Omega \mathbf{H}_{\tilde{\varrho}}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \cdot \mathbf{H}_{\tilde{\varrho}}[\tilde{\varrho} \mathbf{u}_\varepsilon] \frac{\mathrm{d}x}{\tilde{\varrho}} \Big) \ \mathrm{d}t &= \int_0^T \int_\Omega \mathbf{H}_{\tilde{\varrho}}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \cdot \mathbf{u}_\varepsilon \ \mathrm{d}x \ \mathrm{d}t \\ \to \int_0^T \int_\Omega \mathbf{H}_{\tilde{\varrho}}[\tilde{\varrho} \mathbf{U}] \cdot \mathbf{U} \ \mathrm{d}x \ \mathrm{d}t &= \int_0^T \Big( \int_\Omega \tilde{\varrho}^2 |\mathbf{u}|^2 \frac{\mathrm{d}x}{\tilde{\varrho}} \Big) \ \mathrm{d}t. \end{aligned} \tag{6.24}$$

On the other hand, in accordance with (5.1),

$$\varrho_\varepsilon \to \tilde{\varrho} \text{ in } L^\infty(0,T; L^{\frac{5}{3}}(\Omega)),$$

and we deduce from (6.24) that

$$\mathbf{H}_{\tilde{\varrho}}[\tilde{\varrho} \mathbf{u}_\varepsilon] \to \tilde{\varrho} \mathbf{U} \text{ in } L^2(0,T; L^2(\Omega; R^3)),$$

which, by the same token, gives rise to (6.22).

Seeing that

$$\mathbf{H}^\perp_{\tilde{\varrho}}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \to 0 \text{ weakly-(*) in } L^\infty(0,T; L^{\frac{5}{4}}(\Omega; R^3)), \tag{6.25}$$

and keeping (6.22) in mind, we easily check that (6.20) reduces to showing

$$\int_0^T \int_\Omega \mathbf{H}^\perp_{\tilde{\varrho}}[\tilde{\varrho} \mathbf{Q}_\varepsilon] \otimes \mathbf{H}^\perp_{\tilde{\varrho}}[\tilde{\varrho} \mathbf{u}_\varepsilon] : \nabla_x \Big( \frac{\varphi}{\tilde{\varrho}} \Big) \frac{\mathrm{d}x}{\tilde{\varrho}} \ \mathrm{d}t \to 0 \text{ as } \varepsilon \to 0 \tag{6.26}$$

for any test function $\varphi$ satisfying (6.21), where $\mathbf{Q}_\varepsilon$ is the quantity introduced in (6.3).

Now we set

$$\left\{\mathbf{H}^\perp_{\tilde\varrho}[\tilde\varrho\mathbf{Z}]\right\}_M = \sum_{\{j;0<\Lambda_j\le M\}}\sum_{m=1}^{m_j}[\mathbf{Z}]_{j,m}\mathbf{V}_{j,m},$$

where, similarly to (6.18),

$$[\mathbf{Z}]_{j,m} = \int_\Omega \mathbf{Z}\cdot\mathbf{V}_{j,m}\mathrm{d}x$$

for any $Z\in L^1(\Omega;R^3)$.

A straightforward computation yields

$$\begin{aligned}\left[\mathbf{H}^\perp_{\tilde\varrho}[\tilde\varrho\mathbf{Q}_\varepsilon]\otimes\mathbf{H}^\perp_{\tilde\varrho}[\tilde\varrho\mathbf{u}_\varepsilon]\right] = {}&\Big[\left\{\mathbf{H}^\perp_{\tilde\varrho}[\tilde\varrho\mathbf{Q}_\varepsilon]\right\}_M + \Big[\mathbf{H}^\perp_{\tilde\varrho}[\tilde\varrho\mathbf{Q}_\varepsilon] - \left\{\mathbf{H}^\perp_{\tilde\varrho}[\tilde\varrho\mathbf{Q}_\varepsilon]\right\}_M\Big]\Big] \\ &\otimes\Big[\left\{\mathbf{H}^\perp_{\tilde\varrho}[\tilde\varrho\mathbf{u}_\varepsilon]\right\}_M + \Big[\mathbf{H}^\perp_{\tilde\varrho}[\tilde\varrho\mathbf{u}_\varepsilon] - \left\{\mathbf{H}^\perp_{\tilde\varrho}[\tilde\varrho\mathbf{u}_\varepsilon]\right\}_M\Big]\Big],\end{aligned}\tag{6.27}$$

where

$$\begin{aligned}&\mathbf{H}^\perp_{\tilde\varrho}[\tilde\varrho\mathbf{Q}_\varepsilon] - \left\{\mathbf{H}^\perp_{\tilde\varrho}[\tilde\varrho\mathbf{Q}_\varepsilon]\right\}_M \\ &\quad = \mathbf{H}^\perp_{\tilde\varrho}[(\varrho_\varepsilon-\tilde\varrho)\mathbf{u}_\varepsilon] - \left\{\mathbf{H}^\perp_{\tilde\varrho}[(\varrho_\varepsilon-\tilde\varrho)\mathbf{u}_\varepsilon]\right\}_M + \mathbf{H}^\perp_{\tilde\varrho}[\tilde\varrho\mathbf{u}_\varepsilon] - \left\{\mathbf{H}^\perp_{\tilde\varrho}[\tilde\varrho\mathbf{u}_\varepsilon]\right\}_M .\end{aligned}$$

Here, by virtue of the uniform estimates obtained in Section 5,

$$\mathbf{H}^\perp_{\tilde\varrho}[(\varrho_\varepsilon-\tilde\varrho)\mathbf{u}_\varepsilon] - \left\{\mathbf{H}^\perp_{\tilde\varrho}[(\varrho_\varepsilon-\tilde\varrho)\mathbf{u}_\varepsilon]\right\}_M \to 0 \text{ in } L^1(0,T;L^1(\Omega;R^3)).$$

On the other hand, using orthogonality of functions $\omega_{j,m}$, together with Parseval's identity with respect to the scalar product of $L^2_{1/\tilde\varrho}(\Omega)$, we get

$$\|\mathrm{div}_x(\tilde\varrho\mathbf{u}_\varepsilon)\|^2_{L^2_{1/\tilde\varrho}(\Omega)} = \textstyle\sum_{j=1}^\infty\sum_{m=1}^{m_j}\Lambda_j[\mathbf{u}_\varepsilon]^2_{j,m}\,;$$

whence

$$\|\mathbf{H}^\perp_{\tilde\varrho}[\tilde\varrho\mathbf{u}_\varepsilon] - \left\{\mathbf{H}^\perp_{\tilde\varrho}[\tilde\varrho\mathbf{u}_\varepsilon]\right\}_M\|^2_{L^2_{1/\tilde\varrho}(\Omega)} = \sum_{\{j;\Lambda_j>M\}}\sum_{m=1}^{m_j}[\mathbf{u}_\varepsilon]^2_{j,m} \le \frac{1}{M}\|\mathrm{div}_x(\tilde\varrho\mathbf{u}_\varepsilon)\|^2_{L^2_{1/\tilde\varrho}(\Omega)}$$

and we are allowed to conclude that

$$\mathbf{H}^\perp_{\tilde\varrho}[\tilde\varrho\mathbf{u}_\varepsilon] - \left\{\mathbf{H}^\perp_{\tilde\varrho}[\tilde\varrho\mathbf{u}_\varepsilon]\right\}_M \to 0 \quad \text{in } L^2(0,T;L^2_{\frac{1}{\tilde\varrho}}(\Omega;R^3)) \text{ as } M\to\infty \text{ uniformly in } \varepsilon.$$

In the light of the previous arguments, the proof of (6.26) reduces to showing that

$$\int_0^T\int_\Omega \left\{\mathbf{H}^\perp_{\tilde\varrho}[\tilde\varrho\mathbf{Q}_\varepsilon]\right\}_M\otimes\left\{\mathbf{H}^\perp_{\tilde\varrho}[\tilde\varrho\mathbf{u}_\varepsilon]\right\}_M : \nabla_x\Big(\frac{\varphi}{\tilde\varrho}\Big)\frac{\mathrm{d}x}{\tilde\varrho}\ \mathrm{d}t\to 0$$

or, equivalently,

$$\int_0^T\int_\Omega \left\{\mathbf{H}^\perp_{\tilde\varrho}[\tilde\varrho\mathbf{Q}_\varepsilon]\right\}_M\otimes\left\{\mathbf{H}^\perp_{\tilde\varrho}[\tilde\varrho\mathbf{Q}_\varepsilon]\right\}_M : \nabla_x\Big(\frac{\vec\varphi}{\tilde\varrho}\Big)\frac{\mathrm{d}x}{\tilde\varrho}\ \mathrm{d}t\to 0 \tag{6.28}$$

for all $\varphi$ satisfying (6.21) and for any fixed $M$.

In order to show (6.28), we first observe that, by means of (6.11),

$$\int_0^T \int_\Omega \left\{ \mathbf{H}^\perp_{\tilde\varrho}[\tilde\varrho \mathbf{Q}_\varepsilon] \right\}_M \otimes \left\{ \mathbf{H}^\perp_{\tilde\varrho}[\tilde\varrho \mathbf{Q}_\varepsilon] \right\}_M : \nabla_x \Big( \frac{\varphi}{\tilde\varrho} \Big) \frac{\mathrm{d}x}{\tilde\varrho} \ \mathrm{d}t$$

$$= \int_0^T \int_\Omega (\tilde\varrho \nabla_x \Psi_\varepsilon \otimes \nabla_x \Psi_\varepsilon) : \nabla_x \Big( \frac{\vec\varphi}{\tilde\varrho} \Big) \ \mathrm{d}x \ \mathrm{d}t,$$

where $\Psi_\varepsilon = \sum_{j\le M} \sum_{m=1}^{j_m} \frac{[\mathbf{Q}_\varepsilon]_{j,m}}{\sqrt{\Lambda_j}} \Big( \frac{\omega_{j,m}}{\tilde\varrho} \Big)$.

Integrating by parts and using the fact that $\vec\varphi$ is a solenoidal function, we get

$$\int_0^T \int_\Omega (\tilde\varrho \nabla_x \Psi_\varepsilon \otimes \nabla_x \Psi_\varepsilon) : \nabla_x \Big( \frac{\vec\varphi}{\tilde\varrho} \Big) \ \mathrm{d}x \ \mathrm{d}t = -\int_0^T \int_\Omega \mathrm{div}_x \Big( \tilde\varrho \nabla_x \Psi_\varepsilon \Big) \nabla_x \Psi_\varepsilon \cdot \Big( \frac{\vec\varphi}{\tilde\varrho} \Big) \ \mathrm{d}x \ \mathrm{d}t,$$

where, in accordance with (6.8), $-\mathrm{div}_x(\tilde\varrho \nabla_x \Psi_\varepsilon) = \sum_{j\le M} \sum_{m=1}^{j_m} \sqrt{\Lambda_j} [\mathbf{Q}_\varepsilon]_{j,m} \omega_{j,m}$.

It is only now when we use the fact that the quantities $[\mathbf{Q}_\varepsilon]_{j,m}$ satisfy the acoustic equation (6.4), (6.5) yielding

$$-\int_0^T \int_\Omega \mathrm{div}_x \Big( \tilde\varrho \nabla_x \Psi_\varepsilon \Big) \nabla_x \Psi_\varepsilon \cdot \Big( \frac{\vec\varphi}{\tilde\varrho} \Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$= \mathrm{i}\varepsilon \int_0^T \int_\Omega \sum_{j\le M} \sum_{m=1}^{j_m} \partial_t [r_\varepsilon]_{j,m} \frac{\omega_{j,m}}{\tilde\varrho} \nabla_x \Psi_\varepsilon \cdot \vec\varphi \ \mathrm{d}x \ \mathrm{d}t$$

$$= \mathrm{i}\varepsilon \int_0^T \int_\Omega \sum_{j\le M} \sum_{m=1}^{j_m} \frac{\omega_{j,m}}{\tilde\varrho} \Big( [r_\varepsilon]_{j,m} \nabla_x \Psi_\varepsilon \Big) \cdot \partial_t \vec\varphi \ \mathrm{d}x \ \mathrm{d}t$$

$$- \mathrm{i}\varepsilon \int_0^T \int_\Omega \sum_{j\le M} \sum_{m=1}^{j_m} \frac{\omega_{j,m}}{\tilde\varrho} [r_\varepsilon]_{j,m} \partial_t \nabla_x \Psi_\varepsilon \cdot \vec\varphi \ \mathrm{d}x \ \mathrm{d}t.$$

Thus in order to complete the proof of (6.20), it is enough to show that

$$\Big| \int_0^T \int_\Omega \sum_{j\le M} \sum_{m=1}^{j_m} \frac{\omega_{j,m}}{\tilde\varrho} [r_\varepsilon]_{j,m} \partial_t \nabla_x \Psi_\varepsilon \cdot \vec\varphi \ \mathrm{d}x \ \mathrm{d}t \Big| \le \frac{c}{\sqrt\varepsilon}. \tag{6.29}$$

To this end, we make use of equation (6.5) to obtain

$$\partial_t \nabla_x \Psi_\varepsilon = \frac{-\mathrm{i}}{\varepsilon} \sum_{j\le M} \sum_{m=1}^{j_m} [r_\varepsilon]_{j,m} \nabla_x \Big( \frac{\omega_{j,m}}{\tilde\varrho} \Big) + \frac{1}{\sqrt\varepsilon} \sum_{j\le M} \sum_{m=1}^{j_m} H^\varepsilon_{j,m} \nabla_x \Big( \frac{\omega_{j,m}}{\tilde\varrho} \Big),$$

where $H^\varepsilon_{j,m}$ satisfy (6.19).

Finally, as $\varphi$ is solenoidal, meaning $\mathrm{div}_x \varphi = 0$,

$$\int_0^T \int_\Omega \Big[ \sum_{j\le M} \sum_{m=1}^{j_m} [r_\varepsilon]_{j,m} \Big( \frac{\omega_{j,m}}{\tilde\varrho} \Big) \Big] \Big[ \sum_{j\le M} \sum_{m=1}^{j_m} [r_\varepsilon]_{j,m} \nabla_x \Big( \frac{\omega_{j,m}}{\tilde\varrho} \Big) \Big] \cdot \vec\varphi \ \mathrm{d}x \ \mathrm{d}t = 0;$$

whence (6.29), and, consequently (6.20), follow.

Thus we have proved that the limit quantities $\tilde{\varrho}$, $\overline{\vartheta}$, $\mathbf{U}$, and $\Theta$ satisfy the target system (1.29), (1.31), and (1.32) in the sense of distributions. Accordingly, in order to complete the proof of Theorem 3.1, we have to show (1.30). This will be done in the last section.

## 7. Asymptotic limit of the temperature

Similarly to (4.2), the rescaled entropy balance equation (1.21) can be written in the form

$$\int_0^T \int_\Omega \Big[ \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)\partial_t \varphi + \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)\mathbf{u}_\varepsilon \cdot \nabla_x \varphi + \frac{\mathbf{q}_\varepsilon}{\vartheta_\varepsilon} \cdot \nabla_x \varphi \Big] \mathrm{d}x \ \mathrm{d}t \tag{7.1}$$
$$+ \langle \sigma_\varepsilon, \varphi \rangle - \int_0^T \int_{\{x_3=1\}} \frac{\eta(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \ \mathrm{d}S_x \ \mathrm{d}t = 0$$

for any $\varphi \in \mathcal{D}((0,T) \times \overline{\Omega})$, where

$$\mathbf{q}_\varepsilon = -\Big( \kappa_0(\vartheta_\varepsilon) + \frac{1}{\varepsilon^2}\vartheta_\varepsilon^3 \Big) \nabla_x \vartheta_\varepsilon. \tag{7.2}$$

To begin with, in accordance with hypothesis (3.6) and the uniform estimates (4.19), (4.20),

$$\langle \sigma_\varepsilon, \varphi \rangle - \int_0^T \int_{\{x_3=1\}} \frac{\eta(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \ \mathrm{d}S_x \ \mathrm{d}t \to 0 \text{ as } \varepsilon \to 0 \tag{7.3}$$

for any fixed $\varphi$ as in (7.1).

Similarly, by virtue of (4.25), (4.26),

$$\int_0^T \int_\Omega \frac{\kappa_0(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \nabla_x \vartheta_\varepsilon \cdot \nabla_x \varphi \ \mathrm{d}x \to 0; \tag{7.4}$$

whence we have shown

$$\lim_{\varepsilon \to 0} \int_0^T \int_\Omega \Big[ \varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon)\partial_t \varphi + \varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon)\mathbf{u}_\varepsilon \cdot \nabla_x \varphi + \vartheta_\varepsilon^2 \nabla_x \Big( \frac{\vartheta_\varepsilon}{\varepsilon^2} \Big) \cdot \nabla_x \varphi \Big] \mathrm{d}x \ \mathrm{d}t = 0 \tag{7.5}$$

for any fixed $\varphi$ as in (7.1).

Now writing

$$\varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon) = [\varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\mathrm{ess}} + [\varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\mathrm{res}}$$

we check, by means of (4.27), that

$$[\varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\mathrm{res}} \to 0 \text{ in } L^1((0,T) \times \Omega), \tag{7.6}$$

while, by virtue of hypothesis (3.1),

$$[\varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\mathrm{ess}} \to \beta \tilde{\varrho} \Big( \frac{3}{2} \log(\overline{\vartheta}) - \log(\tilde{\varrho}) \Big) \text{ in } L^q((0,T) \times \Omega) \text{ for any } q \geq 1. \tag{7.7}$$

In particular, we get

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon,\vartheta_\varepsilon)\partial_t\varphi \mathrm{d}x\ \mathrm{d}t \to 0 \tag{7.8}$$

as soon as $\varphi$ is compactly supported in $(0,T)$, and, in accordance with (1.31),

$$\int_0^T\int_\Omega [\varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\mathrm{ess}}\mathbf{u}_\varepsilon\cdot\nabla_x\varphi \mathrm{d}x\ \mathrm{d}t \to -\beta\int_0^T\int_\Omega \tilde\varrho\log(\tilde\varrho)\mathbf{U}\cdot\nabla_x\varphi \mathrm{d}x\ \mathrm{d}t \tag{7.9}$$

for any fixed $\varphi$ as in (7.1).

On the other hand, we claim that

$$\int_0^T\int_\Omega [\varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\mathrm{res}}\mathbf{u}_\varepsilon\cdot\nabla_x\varphi \mathrm{d}x\ \mathrm{d}t \to 0. \tag{7.10}$$

In order to see this, it is enough to show, in view of (4.18), that

$$\{[\varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\mathrm{res}}\mathbf{u}_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^q(0,T;L^q(\Omega;R^3)) \text{ for a certain } q>1. \tag{7.11}$$

Writing

$$s_\varepsilon(\varrho,\vartheta) = \underbrace{s_\varepsilon^A(\varrho,\vartheta)}_{\text{atomic entropy}} + \underbrace{s_\varepsilon^R(\varrho,\vartheta)}_{\text{radiation entropy}},$$

$$s_\varepsilon^A(\varrho,\vartheta) = S\Big(\varepsilon\frac{\varrho}{\vartheta^{\frac32}}\Big) - S(\varepsilon),\ \ s_\varepsilon^R(\varrho,\vartheta) = \varepsilon\frac43\frac{\vartheta^3}{\varrho},$$

we obtain, by virtue of (4.17), (4.24),

$$\{\varrho_\varepsilon s_\varepsilon^R(\varrho_\varepsilon,\vartheta_\varepsilon)\mathbf{u}_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^2(0,T;L^{\frac{12}{11}}(\Omega;R^3)). \tag{7.12}$$

On the other hand, in accordance with hypothesis (3.3),

$$|S(\varepsilon Y) - S(\varepsilon)| \le c|\log(Y)| \text{ for all } Y>0;$$

whence, as a consequence of (1.25),

$$|\varrho_\varepsilon s_\varepsilon^A(\varrho_\varepsilon,\vartheta_\varepsilon)| \le c\varrho_\varepsilon(|\log(\varrho_\varepsilon)| + |\log(\vartheta_\varepsilon)|). \tag{7.13}$$

Now it follows from (4.13), (4.28) that

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\varrho_\varepsilon\log(\varrho_\varepsilon)\|_{L^q(\Omega)} \le c \text{ for any } 1\le q<\frac53. \tag{7.14}$$

Furthermore, one can deduce from (4.26), (5.1) that

$$\{\varrho_\varepsilon\log(\vartheta_\varepsilon)\mathbf{u}_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T;L^{\frac{30}{29}}(\Omega;R^3)). \tag{7.15}$$

Combining (7.12–7.15) we get (7.11), and, consequently (7.10).

Thus relation (7.5) reduces to

$$-\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \vartheta_\varepsilon^2\nabla_x\Big(\frac{\vartheta_\varepsilon}{\varepsilon^2}\Big)\cdot\nabla_x\varphi \mathrm{d}x\ \mathrm{d}t = \beta\int_0^T\int_\Omega \tilde\varrho\log(\tilde\varrho)\mathbf{U}\cdot\nabla_x\varphi \mathrm{d}x\ \mathrm{d}t \tag{7.16}$$

for any $\varphi\in\mathcal{D}((0,T)\times\overline{\Omega})$.

In order to identify the limit on the left-hand side of (7.16), we write

$$\int_0^T \int_\Omega \vartheta_\varepsilon^2 \nabla_x \left( \frac{\vartheta_\varepsilon}{\varepsilon^2} \right) \cdot \nabla_x \varphi \mathrm{d}x \ \mathrm{d}t$$
$$= \int_0^T \int_\Omega [\vartheta_\varepsilon]_{\mathrm{ess}}^2 \nabla_x \left( \frac{\vartheta_\varepsilon}{\varepsilon^2} \right) \cdot \nabla_x \varphi \mathrm{d}x \ \mathrm{d}t + \int_0^T \int_\Omega [\vartheta_\varepsilon]_{\mathrm{res}}^{3/2} \sqrt{\vartheta_\varepsilon} \nabla_x \left( \frac{\vartheta_\varepsilon}{\varepsilon^2} \right) \cdot \nabla_x \varphi \mathrm{d}x \ \mathrm{d}t.$$

Thus we can use (4.17), (4.22) in order to conclude that

$$\lim_{\varepsilon \to 0} \int_0^T \int_\Omega \vartheta_\varepsilon^2 \nabla_x \left( \frac{\vartheta_\varepsilon}{\varepsilon^2} \right) \cdot \nabla_x \varphi \mathrm{d}x \ \mathrm{d}t = \overline{\vartheta}^2 \int_0^T \int_\Omega \nabla_x \Theta \cdot \nabla_x \varphi \mathrm{d}x \ \mathrm{d}t, \tag{7.17}$$

where $\Theta$ is the quantity introduced in (5.18).

Relations (7.16), (7.17) give rise to (1.30). Theorem 3.1 has been proved.

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# New Perspectives in Fluid Dynamics: Mathematical Analysis of a Model Proposed by Howard Brenner

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**Abstract.** We study a model of a compressible, viscous and heat conducting fluid proposed in a series of papers by Howard Brenner. We show that the corresponding system of partial differential equations possesses global-in-time weak solutions for any finite energy initial data. In addition, the density of the fluid remains positive a.a. in the physical domain on any finite time interval.

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## 1. Introduction

### 1.1. Field equations

In a series of papers [3], [4], [5], Howard Brenner proposed a daring new approach to continuum fluid mechanics based on the concept of two different velocities: the mass-based (Eulerian) *mass velocity* $\mathbf{v}_m$ derived from the classical notion of mass transport, and the fluid-based (Lagrangian) *volume velocity* $\mathbf{v}$ associated to the motion of individual particles (molecules). According to the overwhelming majority of standard works and research studies on continuum fluid mechanics, these two velocities are implicitly assumed to be one and the same entity (see [3]). This point of view, remaining unchallenged from the time of Euler, led to the nowadays classical mathematical theory of fluid mechanics based on the *Navier-Stokes-Fourier* system of partial differential equations. Brenner argues that, in general, $\mathbf{v} \neq \mathbf{v}_m$,

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this inequality being significant for compressible fluids with high density gradients. He provides a number of purely theoretical as well as experimental arguments in support of his theory involving: (i) Öttinger's generic theory [20] of non-equilibrium irreversible processes; (ii) Klimontovich's molecularly based theory [15] of rarefied gases; (iii) existing thermophoretic and diffusiophoretic experimental data (see [3]).

At the level of mathematical modeling, Brenner's modification of the standard Navier-Stokes-Fourier system is significantly simpler than the alternatives provided by the theories of extended thermodynamics. In the absence of external body forces and heat sources, a mathematical description of a single-component fluid is based on the following trio of balance laws:

Brenner-Navier-Stokes-Fourier (BNSF) System:

Mass Conservation (continuity equation):

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{v}_m) = 0; \tag{1.1}$$

Balance of Linear Momentum:

$$\partial_t(\varrho \mathbf{v}) + \mathrm{div}_x(\varrho \mathbf{v} \otimes \mathbf{v}_m) + \nabla_x p = \mathrm{div}_x \mathbb{S}; \tag{1.2}$$

Total Energy Conservation:

$$\partial_t \left( \varrho(\frac{1}{2}|\mathbf{v}|^2 + e) \right) + \mathrm{div}_x \left( \varrho(\frac{1}{2}|\mathbf{v}|^2 + e)\mathbf{v}_m \right) \tag{1.3}$$

$$+\mathrm{div}_x(p\mathbf{v}) + \mathrm{div}_x \mathbf{q} = \mathrm{div}_x(\mathbb{S}\mathbf{v}),$$

where $\varrho$ is the *mass density*, $p$ is the *pressure*, $e$ the specific *internal energy*, $\mathbb{S}$ the *viscous stress* tensor, and $\mathbf{q}$ stands for the internal energy flux. In addition, we assume that the fluid occupies a bounded (regular) domain $\Omega \subset R^3$ so that all quantities depend on the time $t \in [0, T]$, and the spatial position $x \in \Omega$.

### 1.2. Constitutive relations

A constitutive equation relating $\mathbf{v}_m$ to $\mathbf{v}$ is a cornerstone of Brenner's approach. After a thorough discussion (see [3]–[5]), Brenner proposes a universal constitutive equation in the form:

$(\mathbf{v} - \mathbf{v}_m)$ – Constitutive Relation:

$$\mathbf{v} - \mathbf{v}_m = K \nabla_x \log(\varrho), \tag{1.4}$$

where $K \geq 0$ is a purely *phenomenological* coefficient. A specific relation of $K$ to other thermodynamic quantities is open to discussion. Note that Brenner's original hypothesis $K = \frac{\kappa}{c_p \varrho}$, with $\kappa$ the heat conductivity coefficient and $c_p$ the specific heat at constant volume, has been tested and subsequently modified by several authors (see Greenshields and Reese [12]). In the incompressible regime, namely when $\varrho = \text{const}$, the two velocities coincide converting (1.1–1.3) to the conventional

Navier-Stokes-Fourier system describing the motion of an "incompressible" fluid. It is interesting to note that a similar problem for the *incompressible* fluid with mass diffusion was studied by Kazhikhov and Smagulov [14].

The specific form of the remaining constitutive relations is determined, to a certain extent, by the Second Law of Thermodynamics. In accordance with the fundamental principles of statistical physics (see Gallavotti [11]), the pressure $p = p(\varrho, \vartheta)$ as well as the internal energy $e = e(\varrho, \vartheta)$ are numerical functions of the density $\varrho$ and the *absolute temperature* $\vartheta$ interrelated through

GIBBS' EQUATION:

$$\vartheta Ds(\varrho, \vartheta) = De(\varrho, \vartheta) + p(\varrho, \vartheta) D \left( \frac{1}{\varrho} \right), \tag{1.5}$$

where the symbol $s = s(\varrho, \vartheta)$ denotes the *specific entropy*.

Equation (1.2) multiplied on $\mathbf{v}$ provides, with the help of (1.1), the balance of kinetic energy

$$\partial_t \left( \frac{1}{2} \varrho |\mathbf{v}|^2 \right) + \mathrm{div}_x \left( \frac{1}{2} \varrho |\mathbf{v}|^2 \mathbf{v}_m \right) + \mathrm{div}_x (p\mathbf{v}) = \mathrm{div}_x (\mathbb{S}\mathbf{v}) + p \mathrm{div}_x \mathbf{v} - \mathbb{S} : \nabla_x \mathbf{v}, \tag{1.6}$$

which can be subtracted from (1.3) in order to obtain the balance of internal energy in the form

$$\partial_t (\varrho e) + \mathrm{div}_x (\varrho e \mathbf{v}_m) + \nabla_x \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{v} - p \mathrm{div}_x \mathbf{v}. \tag{1.7}$$

For the sake of simplicity, we shall assume that

$$p(\varrho, \vartheta) = \underbrace{p_e(\varrho)}_{\text{elastic pressure}} + \underbrace{\vartheta p_t(\varrho)}_{\text{thermal pressure}}. \tag{1.8}$$

Albeit rather restrictive, formula (1.8) still includes the physically relevant case of a *perfect gas*, where $p_e = 0$, $p_t = \mathrm{R}\varrho$. The reader may consult the book by Bridgeman [8] concerning general state equations in the form (1.8).

In accordance with Gibbs' equation (1.5), the internal energy splits into two parts:

$$e(\varrho, \vartheta) = \underbrace{e_e(\varrho)}_{\text{elastic energy}} + \underbrace{e_t(\vartheta)}_{\text{thermal energy}}, \text{ where } e_e(\varrho) = \int_1^\varrho \frac{p_e(z)}{z^2} \, \mathrm{d}z. \tag{1.9}$$

Simplifying again we take

$$e_t(\vartheta) = c_v \vartheta, \tag{1.10}$$

where $c_v > 0$ is the specific heat at constant volume. Accordingly, we have

$$s(\varrho, \vartheta) = c_v \log(\vartheta) - \int_1^\varrho \frac{p_t(z)}{z^2} \, \mathrm{d}z. \tag{1.11}$$

Thus, after a simple manipulation, we deduce from (1.7) the thermal energy balance

$$c_v\left(\partial_t(\varrho\vartheta)+\mathrm{div}_x(\varrho\vartheta\mathbf{v}_m)\right)+\mathrm{div}_x\left(\mathbf{q}+Kp_e(\varrho)\nabla_x\log(\varrho)\right) \tag{1.12}$$
$$=\mathbb{S}:\nabla_x\mathbf{v}+K\frac{p_e'(\varrho)}{\varrho}|\nabla_x\varrho|^2-\vartheta p_t(\varrho)\mathrm{div}_x\mathbf{v}.$$

Moreover, we suppose the *heat flux* obeys Fourier's law, specifically,

FOURIER'S LAW:

$$\mathbf{q}+Kp_e(\varrho)\nabla_x\log(\varrho)=-\kappa\nabla_x\vartheta, \tag{1.13}$$

where $\kappa$ is the heat conductivity coefficient.

Finally, dividing (1.12) by $\vartheta$ yields the *entropy balance*

$$\partial_t\left(\varrho s\right)+\mathrm{div}_x\left(\varrho s\mathbf{v}_m\right)-\mathrm{div}_x\left(\frac{\kappa}{\vartheta}\nabla_x\vartheta-Kp_t(\varrho)\nabla_x\log(\varrho)\right) \tag{1.14}$$
$$=\frac{1}{\vartheta}\left(\mathbb{S}:\nabla_x\mathbf{v}+K\frac{p_e'(\varrho)}{\varrho}|\nabla_x\varrho|^2+\frac{\kappa}{\vartheta}|\nabla_x\vartheta|^2+K\frac{p_t'(\varrho)\vartheta}{\varrho}|\nabla_x\varrho|^2\right).$$

By virtue of the Second Law of Thermodynamics, the quantity on the right-hand side of (1.14) representing the *entropy production rate* must be non-negative for any admissible physical process. Accordingly, and in sharp contrast to the standard theory, it is the velocity $\mathbf{v}$ rather than $\mathbf{v}_m$ that must appear in the rheological law for the viscous stress. For a linearly viscous (Newtonian) fluid such a stipulation yields:

NEWTON'S RHEOLOGICAL LAW:

$$\mathbb{S}=\mu\left(\nabla_x\mathbf{v}+\nabla_x^T\mathbf{v}-\frac{2}{3}\mathrm{div}_x\mathbf{v}\mathbb{I}\right)+\eta\mathrm{div}_x\mathbf{v}\ \mathbb{I}, \tag{1.15}$$

where $\mu\geq 0$ and $\eta\geq 0$ stand for the shear and bulk viscosity coefficients, respectively. By the same token, the quantities $K$, $\kappa$, $p_e'$, $p_t'$ must be non-negative.

### 1.3. Boundary conditions

Another innovative aspect of Brenner's theory is the claim that it is the volume velocity $\mathbf{v}$ rather than $\mathbf{v}_m$ that should be considered in the otherwise well-accepted *no-slip* boundary condition for viscous fluids

$$\mathbf{v}|_{\partial\Omega}=0 \tag{1.16}$$

(cf. Brenner [5]). On the other hand, the standard *impermeability* condition hypothesis keeps its usual mass-based form

$$\mathbf{v}_m\cdot\mathbf{n}|_{\partial\Omega}=0, \tag{1.17}$$

where $\mathbf{n}$ stands for the outer normal vector, or, equivalently,

$$\nabla_x \varrho \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{1.18}$$

As we will see below, such a stipulation is in perfect agreement with the variational formulation of the problem in the spirit of the modern theory of partial differential equations.

Finally, we focus in this study on energetically closed systems, in particular,

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0 \tag{1.19}$$

yielding, in view of (1.13), (1.18),

$$\nabla_x \vartheta \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{1.20}$$

### 1.4. Mathematics of Brenner's model

Besides the necessity of experimental evidence, an ultimate criterion of validity of any mathematical model is its solvability in the framework of physically relevant data. The main goal of the present study is to show that Brenner's model provides a very attractive alternative to both the classical Navier-Stokes-Fourier system and the mathematically almost untractable problems provided by extended thermodynamics. In particular, we establish *a priori* estimates for solutions of BNSF systems strong enough to ensure the property of *weak sequential stability*. That means any sequence of solutions bounded by *a priori* estimates possesses a subsequence converging weakly to a (weak) solution of the same problem. This is a remarkable property that allows us to develop a rigorous *existence theory* for the evolutionary problem (1.1–1.4), without any restriction imposed on the size of the data and the length of the time interval. Such a theory can be viewed as a counterpart of the seminal work of Leray [17] devoted to the classical incompressible Navier-Stokes system.

It is worth noting here that, despite the concerted effort of generations of mathematicians, the weak solutions identified by Leray [17], Hopf [13], and Ladyzhenskaya [16] provide the only available framework, where global-in-time existence for the (standard) incompressible Navier-Stokes model can be rigorously verified for any choice of (large) data. A comparable theory for the compressible isentropic fluids was developed by P.-L.Lions [18] and later extended to a specific class of solutions for the full Navier-Stokes-Fourier system (see [9], [10]), where the energy equation (1.3) is replaced by an entropy or thermal energy inequality supplemented with an integrated total energy balance.

Rigorous (large data) existence results for the classical Navier-Stokes-Fourier system, meaning system (1.1–1.3) with $\mathbf{v} = \mathbf{v}_m$ are in short supply. Quite recently, Bresch and Desjardins [6], [7] discovered an interesting integral identity yielding *a priori* bounds on the density gradient and the property of weak sequential stability for system (1.1–1.3) provided the viscosity coefficients $\mu$ and $\eta$ depend on the density $\varrho$ in a specific way and the pressure $p$ is given through formula (1.8), where the elastic component $p_e$ is assumed to be singular for the density $\varrho$ approaching zero.

The idea that a density dependent *bulk* viscosity coefficient $\eta$ may actually provide better *a priori* estimates goes back to the remarkable study by Vaigant and Kazhikhov [21]. To the best of our knowledge, this is the only result where the authors establish global existence of *smooth* solutions although conditioned by the 2-D periodic geometry of the physical space and rather unrealistic hypotheses concerning the viscosity coefficients. A suitable functional relation satisfied by the viscosity coefficients $\mu = \mu(\varrho)$, $\eta = \eta(\varrho)$ was also used in [19] to prove global existence for the isentropic Navier-Stokes system with a general pressure law.

The main stumbling blocks encountered in the mathematical theory of compressible fluids based on the classical Navier-Stokes-Fourier system that have been identified in the previously cited studies can be summarized as follows:

- absence of *uniform bounds* on the density $\varrho$, in particular, the hypothetical possibility of concentration phenomena in the pressure term;
- a possibility of appearance of *vacuum regions*, meaning sets of positive measure on which $\varrho = 0$, even in the situation when strict positivity is imposed on the initial density distribution;
- low regularity of the velocity field allowing for development of uncontrollable *oscillations* experienced by the transported quantities, in particular, the density.

Quite remarkably, Brenner's modifications of the Navier-Stokes-Fourier system offer a new insight and at least a partial remedy to each of the issues listed above. The principal features of this new approach based on the (BNSF) system read as follows:

- The new model provides a relatively simple and rather transparent modification of the classical system replacing the Eulerian mass velocity $\mathbf{v}_m$ by its volume counterpart $\mathbf{v}$ in the viscous stress tensor and the specific momentum, where $\mathbf{v}_m$ and $\mathbf{v}$ are interrelated through formula (1.4). The two velocities coincide in the "incompressible" regime when $\varrho = \text{const}$.
- The model conveniently unifies the principles of statistical mechanics with thermodynamics of continuum models of large multiparticle systems, in particular, it is consistent with the First and Second Laws of Thermodynamics.
- The associated mathematical theory developed below allows for a rather general class of state equations, in particular and unlike all comparable results for the classical Navier-Stokes-Fourier system, the perfect gas state equation expressed through Boyle-Marriot's law can be handled.
- Weak solutions of the (BNSF) system do not contain vacuum zones for positive times; the time-space Lebesgue measure of the set where $\varrho$ vanishes is zero.
- Possible oscillations of the density as well as other fields are effectively damped by diffusion; the weak stability property is preserved even under non-isotropic perturbations of the transport terms.

**Outline of the paper**

The mathematical theory of the (BNSF) system developed in this study is based on the standard framework of Sobolev spaces. Similarly to any *non-linear* problem, the class of function spaces is determined by means of the available *a priori* estimates established in Section 2. A remarkable new feature of the present setting is that the continuity equation (1.1) can be rewritten in terms of the volume velocity as

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{v}) = \operatorname{div}_x(K \nabla_x \varrho). \tag{1.21}$$

Equation (1.21) is *parabolic* yielding strong *a priori* estimates on $\varrho$ provided $\mathbf{v}$ enjoys certain smoothness. On the other hand, the standard energy and gradient estimates for the velocity $\mathbf{v}$ and the temperature $\vartheta$ are obtained on the basis of entropy equation (1.14).

Section 3 develops refined estimates on $\mathbf{v}$ and $\vartheta$ necessary in order to establish equi-integrability of the fluxes appearing in the total energy balance (1.3). We point out that the bounds obtained for the volume velocity $\mathbf{v}$ are actually *better* than those for the velocity field considered in the standard incompressible Navier-Stokes system because we control the pressure in a rather strong $L^2-$norm.

Section 4 discusses the issue of *weak sequential stability.* It is shown that any sequence of (regular) solutions bounded by *a priori* bounds established in the previous part converges weakly to a distributional solution of the same problem. The most delicate task here is to control possible *concentrations* rather than *oscillations* of the weakly converging fields.

Finally, Section 5 proposes an approximation scheme analogous to that developed in [9] in order to establish a rigorous existence result for the corresponding initial-boundary value problem without any essential restrictions imposed on the size of the initial data and the length of the time interval.

## 2. A priori estimates

*A priori* estimates are natural bounds imposed on a family of solutions to a system of partial differential equations by the data, boundary conditions, and other parameters as the case may be. When deriving *a priori* estimates it is customary to assume that all quantities appearing in the equations are as smooth as necessary unless such a stipulation violated some obvious physical principles. In order to fix ideas, and in addition to the hypotheses discussed in Section 1, we suppose throughout the whole text the technical assumptions shown on top of the next page.

While hypotheses (A2.2–A2.4) have obvious physical interpretations, hypotheses (A2.1), (A2.5) are of a technical nature facilitating analysis of the problem, in particular at the level of *a priori* estimates. Note however that an active discussion is going on concerning the appropriate value of the phenomenological coefficient $K$ (see [12]), while (A 2.5) is physically relevant as radiation heat conductivity at least for large values of $\vartheta$ (see [22]).

HYPOTHESES:

**(A 2.1)** The phenomenological coefficient $K$ introduced in (1.4) is a positive constant, say, $K \equiv 1$.

**(A 2.2)** The pressure $p$ obeys the classical Boyle-Marriot law:
$$p(\varrho, \vartheta) = \mathrm{R}\varrho\vartheta,$$
with a positive constant R.

**(A 2.3)** The specific internal energy $e$ satisfies
$$e = c_v\vartheta,$$
with a positive constant $c_v$.

**(A 2.4)** The viscosity coefficients are constant satisfying
$$\mu > 0, \ \eta = 0.$$

**(A 2.5)** The heat conductivity coefficient $\kappa$ depends on the temperature, specifically,
$$\kappa(\vartheta) = \kappa_0(1 + \vartheta^3),$$
with a positive constant $\kappa_0$.

### 2.1. A priori estimates based on mass and energy conservation

Equation (1.1), or, equivalently, (1.21), expresses the physical principle of mass conservation. To begin, the standard maximum principle applies to the parabolic equation (1.21) yielding
$$\varrho(t, x) \geq 0 \text{ for all } t \in (0, T), \ x \in \Omega \tag{2.1}$$
provided
$$\varrho(0, \cdot) \equiv \varrho_0 \geq 0 \text{ in } \Omega. \tag{2.2}$$
Moreover, integrating (1.21) over $\Omega$ gives rise to
$$\int_\Omega \varrho(t, \cdot) \, \mathrm{d}x = \int_\Omega \varrho_0 \, \mathrm{d}x \text{ for all } t \in [0, T]$$
provided $\varrho$, $\mathbf{v}$ satisfy the boundary conditions (1.16), (1.18). As both $\varrho_0$ and $\varrho$ are non-negative, we conclude that
$$\sup_{t \in (0,T)} \|\varrho(t, \cdot)\|_{L^1(\Omega)} = \int_\Omega \varrho_0 \, \mathrm{d}x \equiv M_0. \tag{2.3}$$

Similarly, the total energy balance equation (1.3) integrated over $\Omega$ yields
$$\int_\Omega \varrho \left( \frac{1}{2}|\mathbf{v}|^2 + c_v\vartheta \right)(t, \cdot) \, \mathrm{d}x = \int_\Omega \varrho_0 \left( \frac{1}{2}|\mathbf{v}_0|^2 + c_v\vartheta_0 \right) \, \mathrm{d}x \equiv E_0, \tag{2.4}$$
where $\mathbf{v}_0$ and $\vartheta_0$ denote the initial distribution of the volume velocity and the temperature, respectively. Thus
$$\sup_{t \in (0,T)} \|\sqrt{\varrho}\mathbf{v}\|_{L^2(\Omega; R^3)} + \sup_{t \in (0,T)} \|\varrho\vartheta\|_{L^1(\Omega)} \leq c(E_0). \tag{2.5}$$

Note that we have tacitly anticipated that the absolute temperature $\vartheta$ is a non-negative quantity. This stipulation is justified in the next section.

**2.2. A priori estimates stemming from the Second Law of Thermodynamics**

The Second Law of Thermodynamics is expressed through the entropy balance (1.14). In accordance with the boundary conditions (1.17), (1.18), the normal component of the entropy flux vanishes on the boundary; whence

$$\begin{aligned}\int_\Omega \varrho s(\tau,\cdot)\ \mathrm{d}x - \int_\Omega \varrho_0 s(\varrho_0,\vartheta_0)\ \mathrm{d}x \\ = \int_0^T \int_\Omega \frac{1}{\vartheta}\left(\mathbb{S}:\nabla_x\mathbf{v} + \kappa_0 \frac{1+\vartheta^3}{\vartheta}|\nabla_x\vartheta|^2 + K\frac{\vartheta}{\varrho}|\nabla_x\varrho|^2\right)\ \mathrm{d}x\ \mathrm{d}t.\end{aligned} \tag{2.6}$$

Gibbs' equation (1.5) combined with hypotheses (A 2.2), (A 2.3) yields

$$s(\varrho,\vartheta) = c_v \log(\vartheta) - \mathrm{R}\log(\varrho),$$

in particular, since $\Omega$ is assumed to be bounded, the uniform bounds already established in (2.3), (2.5) imply that

$$\int_\Omega \varrho[\log(\vartheta)]^+\ \mathrm{d}x - \int_\Omega \varrho[\log(\varrho)]^-\ \mathrm{d}x \leq c(M_0,E_0) \text{ uniformly in } (0,T),$$

where we have written $[z]^+ = \max\{z,0\}$, $[z]^- = \min\{z,0\}$.

Consequently, we easily deduce from (2.6) that

$$\sup_{t\in(0,T)} \|\varrho\log(\varrho)\|_{L^1(\Omega)} + \sup_{t\in(0,T)} \|\varrho\log(\vartheta)\|_{L^1(\Omega)} \leq c(M_0,E_0,S_0), \tag{2.7}$$

and

$$\int_0^T \int_\Omega \frac{1}{\vartheta}\mathbb{S}:\nabla_x\mathbf{v}\ \mathrm{d}x\ \mathrm{d}t \leq c(M_0,E_0,S_0), \tag{2.8}$$

$$\|\nabla_x \log(\vartheta)\|_{L^2(0,T;L^2(\Omega;R^3)} + \|\nabla_x\vartheta^{3/2}\|_{L^2(0,T;L^2(\Omega;R^3))} \leq c(M_0,E_0,S_0), \tag{2.9}$$

$$\|\nabla_x\sqrt{\varrho}\|_{L^2(0,T;L^2(\Omega;R^3))} \leq c(M_0,E_0,S_0), \tag{2.10}$$

where we have written

$$S_0 = \int_\Omega \varrho_0 s(\varrho_0,\vartheta_0)\ \mathrm{d}x.$$

At this stage we need the following version of *Poincaré's inequality*:

**Lemma 2.1.** *Let $\Omega\subset R^N$ be a bounded Lipschitz domain. Let $B\subset\Omega$ be a measurable set such that $|B|\geq m>0$.*

*Then we have*

$$\|v\|_{W^{1,2}(\Omega;R^N)} \leq c(m,\beta)\left(\|\nabla_x v\|_{L^2(\Omega;R^N)} + \left(\int_B |v|^\beta\ \mathrm{d}x\right)^{1/\beta}\right)$$

*for any $v\in W^{1,2}(\Omega)$, where the constant $c=c(m,\beta)$ depends solely on $m$ and the parameter $\beta>0$.*

Our goal is to apply Lemma 2.1 first to $v = \vartheta^{3/2}$ and then $v = \log(\vartheta)$. To this end, we first show that there exist $\delta > 0$, $m > 0$ independent of $t \in (0, T)$ such that

$$|\{x \in \Omega \mid \varrho(t, x) > \delta\}| > m. \tag{2.11}$$

In order to see this, use (2.7) to deduce that there exists $\alpha > 0$ such that

$$\int_{\{\varrho(t,\cdot) \geq \alpha\}} \varrho(t, \cdot)\ \mathrm{d}x \leq \frac{M_0}{3}$$

for any $t \in (0, T)$, where $M_0$ is the total mass of the fluid defined in (2.3). We fix $\delta = M_0/(3|\Omega|)$.

On the other hand, in accordance with (2.3),

$$M_0 = \int_\Omega \varrho(t, \cdot)\ \mathrm{d}x = \int_{\{\varrho(t,\cdot) \leq \delta\}} \varrho(t, \cdot)\,\mathrm{d}x + \int_{\{\delta < \varrho(t,\cdot) < \alpha\}} \varrho(t, \cdot)\,\mathrm{d}x + \int_{\{\varrho(t,\cdot) \geq \alpha\}} \varrho(t, \cdot)\,\mathrm{d}x$$
$$\leq \delta|\Omega| + \alpha|\{x \in \Omega \mid \varrho(t, x) > \delta\}| + \frac{M_0}{3};$$

whence we can take $m = M_0/3\alpha$. Note that the value of $m$, $\delta$, derived on the basis of (2.3), (2.7), depend only on $M_0$, $E_0$, $S_0$.

Consequently, combining the uniform bounds established in (2.5), (2.9) with the conclusion of Lemma 2.1 we obtain

$$\|\vartheta^{3/2}\|_{L^2(0,T;W^{1,2}(\Omega;R^3))} \leq c(M_0, E_0, S_0). \tag{2.12}$$

Similarly, estimates (2.7), (2.9) yield

$$\|\log(\vartheta)\|_{L^2(0,T;W^{1,2}(\Omega;R^3))} \leq c(M_0, E_0, S_0), \tag{2.13}$$

and, finally, (2.3), (2.10) give rise to

$$\|\sqrt{\varrho}\|_{L^2(0,T;W^{1,2}(\Omega;R^3))} \leq c(M_0, E_0, S_0). \tag{2.14}$$

### 2.3. A priori bounds based on maximal regularity

*A priori* bounds established so far depend solely on the integral means $M_0$, $E_0$, $S_0$ representing the total amount of mass, energy, and entropy at the initial instant $t = 0$. In order to get more information, better summability of the initial data is necessary.

Equation (1.21) can be written in the form

$$\partial_t \varrho - \Delta\varrho = -\mathrm{div}_x(\varrho\mathbf{v}),\ \nabla_x\varrho \cdot \mathbf{n}|_{\partial\Omega} = 0,\ \varrho(0, \cdot) = \varrho_0 \tag{2.15}$$

that can be viewed as a non-homogeneous linear parabolic equation, where, by virtue of (2.5), (2.14) combined with the standard imbedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$,

$$\|\varrho\mathbf{v}\|_{L^2(0,T;L^{3/2}(\Omega;R^3))} \leq c(M_0, E_0, S_0).$$

Now, we evoke the *maximal regularity estimates* applicable to the parabolic problem (2.15) (see Amann [1], [2]):

MAXIMAL REGULARITY ESTIMATES:

**Proposition 2.1.** *Let $\Omega \subset R^3$ be a regular bounded domain. Assume that*

$$f \in L^p(0,T;[W^{1,q'}(\Omega)]^*),\ 1 < p,q < \infty$$

*is a given function.*
*Then problem*

$$\partial_t r - \Delta r = f,\ \nabla_x r \cdot \mathbf{n}|_{\partial\Omega} = 0,\ r(0,\cdot) = 0$$

*admits a unique weak solution $r$ in the class*

$$r \in L^p(0,T;W^{1,q}(\Omega)),\ \partial_t r \in L^p(0,T;[W^{1,q'}(\Omega)]^*),$$
$$r \in C\left([0,T];\{[W^{1,q'}(\Omega)]^*;W^{1,q}(\Omega)\}_{1/p',p}\right),$$

*where the symbol $\{;\}_{1/p',p}$ stands for the real interpolation space. Moreover,*

$$\begin{aligned}&\sup_{t\in[0,T]} \|r(t,\cdot)\|_{\{[W^{1,q'}(\Omega)]^*;W^{1,q}(\Omega)\}_{1/p',p}}\\ &\quad + \|\partial_t r\|_{L^p(0,T;[W^{1,q'}(\Omega)]^*)} + \|r\|_{L^p(0,T;W^{1,q}(\Omega))}\\ &\quad \le \|f\|_{L^p(0,T;[W^{1,q'}(\Omega)]^*)}.\end{aligned}$$

Thus, writing $\varrho = \varrho_1 + \varrho_2$, where $\varrho_1$ solves the homogeneous problem

$$\partial_t \varrho_1 - \Delta \varrho_1 = 0,\ \nabla_x \varrho_1 \cdot \mathbf{n}|_{\partial\Omega} = 0,\ \varrho_1(0,\cdot) = \varrho_0,$$

while

$$\partial_t \varrho_2 - \Delta \varrho_2 = -\mathrm{div}_x(\varrho \mathbf{u}),\ \nabla_x \varrho_2 \cdot \mathbf{n}|_{\partial\Omega} = 0,\ \varrho_2(0,\cdot) = 0,$$

we obtain

$$\|\varrho_2\|_{L^2(0,T;W^{1,3/2}(\Omega))} \le c\|\varrho \mathbf{v}\|_{L^2(0,T;L^{3/2}(\Omega))} \le c(M_0,E_0,S_0), \tag{2.16}$$

and

$$\sup_{t\in(0,T)} \|\varrho_1(t,\cdot)\|_{L^3(\Omega)} \le c\|\varrho_0\|_{L^3(\Omega)}.$$

On the other hand, as $W^{1,3/2}(\Omega) \hookrightarrow L^3(\Omega)$, we deduce, exactly as above,

$$\|\varrho \mathbf{v}\|_{L^4(0,T;L^{3/2}(\Omega;R^3))} \le c(M_0,E_0,S_0,\|\varrho_0\|_{L^3(\Omega)}).$$

Consequently, a simple iteration of the previous argument (bootstrap) yields, finally,

$$\|\varrho\|_{L^p(0,T;L^3(\Omega))} \le c(M_0,E_0,S_0,\|\varrho_0\|_{L^3(\Omega)}) \text{ for any } 1 \le p < \infty. \tag{2.17}$$

At this stage, we are ready to exploit the thermal energy balance (1.12) in order to obtain uniform bounds on the volume velocity gradient. Indeed integrating (1.12) over $\Omega$ we easily deduce a uniform bound

$$\int_0^T \int_\Omega \mathbb{S} : \nabla_x \mathbf{v}\ \mathrm{d}x\ \mathrm{d}t \le c$$

provided we are able to control the term

$$\int_0^T \int_\Omega \varrho\vartheta \mathrm{div}_x \mathbf{v} \ \mathrm{d}x \ \mathrm{d}t.$$

To this end, we evoke (2.12), (2.17), which, together with the imbedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, yield

$$\|\varrho\vartheta\|_{L^q(0,T;L^{9/4}(\Omega))} \leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, q) \text{ for any } 1 \leq q < 3. \tag{2.18}$$

On the other hand, the linear form associated to the viscous stress $\mathbb{S}$ satisfies a variant of Korn's inequality

$$\int_\Omega \mathbb{S} : \nabla_x \mathbf{v} \ \mathrm{d}x \geq c\|\nabla_x \mathbf{v}\|^2_{L^2(\Omega;R^{3\times 3})} \tag{2.19}$$

that can be easily verified by means of by parts integration since the volume velocity $\mathbf{v}$ vanishes on $\partial\Omega$.

Consequently, using (2.18), (2.19), and the standard Poincaré inequality, we infer that

$$\|\mathbf{v}\|_{L^2(0,T;W^{1,2}(\Omega;R^3))} \leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}). \tag{2.20}$$

### 2.4. Positivity of the density

A remarkable feature of Brenner's model is the possibility to eliminate the regions with vanishing density. To this end, we multiply equation (2.15) on $1/\varrho$ to obtain

$$\partial_t \log(\varrho) - \Delta \log(\varrho) = |\nabla_x \log(\varrho)|^2 - \mathrm{div}_x \mathbf{v} - \mathbf{v} \cdot \nabla_x \log(\varrho). \tag{2.21}$$

It follows from (2.3) that

$$\sup_{t\in(0,T)} \|[\log(\varrho)(t,\cdot)]^+\|_{L^p(\Omega)} \leq c(M_0) \text{ for all } 1 \leq p < \infty; \tag{2.22}$$

whence integrating (2.21) over $\Omega$ yields

$$\sup_{t\in(0,T)} \|\log(\varrho)(t,\cdot)\|_{L^1(\Omega)} \leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\log(\varrho_0)\|_{L^1(\Omega)}), \tag{2.23}$$

$$\|\log(\varrho)\|_{L^2(0,T;W^{1,2}(\Omega))} \leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\log(\varrho_0)\|_{L^1(\Omega)}), \tag{2.24}$$

where we have used the uniform bound on the velocity field established in (2.20). In particular, we have

$$|\{x \in \Omega \mid \varrho(t,x) = 0\}| = 0 \text{ for any } t \in (0,T), \tag{2.25}$$

meaning, the vacuum zones, if any, have zero Lebesgue measure.

The lower bound on $\log(\varrho)$ can be improved by means of the classical comparison argument. Specifically, we deduce from (2.21) that

$$\log(\varrho) \geq V,$$

where $V$ is a solution to the problem

$$\partial_t V - \Delta V = -\mathrm{div}_x \mathbf{v} - \frac{1}{2}|\mathbf{v}|^2, \ \nabla_x V \cdot \mathbf{n}|_{\partial\Omega} = 0, \ V(0,\cdot) = \log(\varrho_0), \tag{2.26}$$

with $\mathbf{v}$ fixed. Seeing that, by virtue of (2.20), the right-hand side $-\mathrm{div}_x\mathbf{v}-(1/2)|\mathbf{v}|^2$ is bounded in the space

$$L^2((0,T)\times\Omega)\oplus L^1(0,T;L^3(\Omega)),$$

we conclude, using the standard parabolic theory, that

$$\sup_{t\in(0,T)}\|V(t,\cdot)\|_{L^3(\Omega)}\le c(M_0,E_0,S_0,\|\varrho_0\|_{L^3(\Omega)},\|\log(\varrho_0)\|_{L^3(\Omega)}),$$

which, together with (2.22), yields

$$\sup_{t\in(0,T)}\|\log(\varrho)(t,\cdot)\|_{L^3(\Omega)}\le c(M_0,E_0,S_0,\|\varrho_0\|_{L^3(\Omega)},\|\log(\varrho_0)\|_{L^3(\Omega)}). \tag{2.27}$$

## 3. Refined velocity and temperature estimates

The *a priori* bounds derived in this section are quite non-standard and to a certain extent even better than those that can be obtained for the weak solutions of the classical incompressible Navier-Stokes system. This is due to the fact that we are able to control the pressure by means of estimate (2.18).

### 3.1. Refined estimates of the volume velocity

We start rewriting the momentum equation (1.2) in the form

$$\varrho\left(\partial_t\mathbf{v}+\mathbf{v}_m\cdot\nabla_x\mathbf{v}\right)+\mathrm{R}\nabla_x(\varrho\vartheta)=\mu\Delta\mathbf{v}+\frac{1}{3}\mu\nabla_x\mathrm{div}_x\mathbf{v}. \tag{3.1}$$

Following [19], the main idea is to multiply (3.1) on $|\mathbf{v}|^{2\alpha}\mathbf{v}$, where $\alpha>0$ is a positive parameter to be fixed below. Integrating the resulting expression, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2(\alpha+1)}\int_\Omega\varrho|\mathbf{v}|^{2(\alpha+1)}\ \mathrm{d}x+\mu\int_\Omega\left(|\mathbf{v}|^{2\alpha}|\nabla_x\mathbf{v}|^2+\frac{1}{3}|\mathbf{v}|^{2\alpha}|\mathrm{div}_x\mathbf{v}|^2\right)\ \mathrm{d}x\\&=\mathrm{R}\int_\Omega\left(|\mathbf{v}|^{2\alpha}\varrho\vartheta\mathrm{div}_x\mathbf{v}+\varrho\vartheta\nabla_x|\mathbf{v}|^{2\alpha}\cdot\mathbf{v}\right)\ \mathrm{d}x\\&\quad-\mu\int_\Omega\left([(\nabla_x\mathbf{v})\mathbf{v}]\cdot\nabla_x|\mathbf{v}|^{2\alpha}+\frac{1}{3}\mathrm{div}_x\mathbf{v}\nabla_x|\mathbf{v}|^{2\alpha}\cdot\mathbf{v}\right)\ \mathrm{d}x.\end{aligned} \tag{3.2}$$

It is easy to check that the second integral on the right-hand side of (3.2) is controlled by its counterpart on the left-hand side as soon as $\alpha>0$ is small enough. By the same token, using estimates (2.18), (2.20) we get

$$\left|\int_0^T\int_\Omega|\mathbf{v}|^{2\alpha}\varrho\vartheta\mathrm{div}_x\mathbf{v}\ \mathrm{d}x\ \mathrm{d}t\right|\le c(M_0,E_0,S_0,\|\varrho_0\|_{L^3(\Omega)})$$

provided $\alpha>0$ is sufficiently small.

Finally,

$$\left| \int_0^\tau \int_\Omega \varrho\vartheta \nabla_x |\mathbf{v}|^{2\alpha} \cdot \mathbf{v} \ \mathrm{d}x \ \mathrm{d}t \right|$$
$$\leq \alpha \int_0^\tau \int_\Omega \varrho^2\vartheta^2 |\mathbf{v}|^{2\alpha} \ \mathrm{d}x \ \mathrm{d}t + \alpha \int_0^T \int_\Omega |\mathbf{v}|^{2\alpha} |\nabla_x \mathbf{v}|^2 \ \mathrm{d}x \ \mathrm{d}t \text{ for any } \tau \in (0,T).$$

Consequently, relation (3.2) gives rise to the following bounds:

$$\sup_{t\in(0,T)} \int_\Omega \varrho |\mathbf{v}|^{2(1+\alpha)}(t,\cdot) \ \mathrm{d}x \leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0|\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)}), \tag{3.3}$$

and

$$\| |\mathbf{v}|^\alpha \mathbf{v}\|_{L^2(0,T;W^{1,2}(\Omega;R^3))} \leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0|\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)}) \tag{3.4}$$

for a certain $\alpha > 0$.

Since $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ and $W^{1,3/2}(\Omega) \hookrightarrow L^3(\Omega)$, estimate (3.4) combined with (2.17) imply

$$\|\varrho \mathbf{v}\|_{L^2(0,T;L^2(\Omega;R^3))} \leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0|\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)}), \tag{3.5}$$

which can be used in (2.15) in order to obtain

$$\sup_{t\in(0,T)} \|\varrho(t,\cdot)\|_{L^2(\Omega)} \leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0|\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)}), \tag{3.6}$$

together with

$$\|\varrho\|_{L^2(0,T;W^{1,2}(\Omega))} \leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0|\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)}). \tag{3.7}$$

As a matter of fact, a slightly better estimate may be obtained, namely

$$\|\varrho\|_{L^{(2+\alpha)}(0,T;W^{1,(2+\alpha)}(\Omega))} \leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0|\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)}) \tag{3.8}$$

for a certain $\alpha > 0$. Here, similarly to (3.4), the symbol $\alpha > 0$ denotes a generic positive parameter that may be different in different formulas.

Combining the previous estimates, in particular (3.4), (3.6), (3.7), we observe that

$$\mathrm{div}_x(\varrho\mathbf{v}) \text{ belongs to the space } L^{(1+\alpha)}((0,T)\times\Omega;R^3) \text{ for a certain } \alpha > 0;$$

whence equation (2.15), together with the standard $L^p-$theory for linear parabolic problems, yield

$$\begin{aligned} &\|\partial_t \varrho\|_{L^{(1+\alpha)}((0,T)\times\Omega)} + \|\varrho\|_{L^{(1+\alpha)}(0,T;W^{2,(1+\alpha)}(\Omega))} \\ &\quad \leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0|\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)}) \end{aligned} \tag{3.9}$$

for a certain $\alpha > 0$.

On the point of conclusion, we note that the *a priori* bounds established in this section, notably (3.4), imply equi-integrability of the "viscous flux" of the total energy that is necessary in order to handle (1.3).

### 3.2. Temperature estimates

A short inspection of the total energy balance (1.3) reveals immediately one of the main technical problems involved in Brenner's model, namely a possibility of concentrations in the heat flux

$$\mathbf{q} = -\kappa_0(1+\vartheta^3)\nabla_x\vartheta.$$

Note that, for the time being, we have shown only (2.12), which is clearly insufficient not even to control the $L^1-$norm of $\mathbf{q}$.

To begin, we exploit again the thermal energy balance. Multiplying (1.12) on $H'(\vartheta)$, where $H$ is a suitable function specified below, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho H(\vartheta)\ \mathrm{d}x + \kappa_0\int_\Omega H''(\vartheta)(1+\vartheta^3)|\nabla_x\vartheta|^2\ \mathrm{d}x$$
$$= \int_\Omega H'(\vartheta)\mathbb{S}:\nabla_x\mathbf{v}\ \mathrm{d}x - \mathrm{R}\int_\Omega H'(\vartheta)\varrho\vartheta\mathrm{div}_x\mathbf{v}\ \mathrm{d}x.$$

By virtue of estimates (2.18), (2.20), the right-hand side is integrable in $t$ as soon as the derivative $H'$ is bounded. Thus the choice $H(\vartheta) = (1+\vartheta)^{1-\omega}$, with $\omega > 0$, leads to a uniform bound

$$\int_0^T\int_\Omega \frac{\vartheta^3}{(1+\vartheta)^{1+\omega}}|\nabla_x\vartheta|^2\ \mathrm{d}x\ \mathrm{d}t \tag{3.10}$$
$$\leq c(\omega, M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0|\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)})$$

for any $\omega > 0$. Here, the best estimates would be obtained in the limit case $\omega \to 0$ unfortunately not attainable.

Writing $\vartheta^3\nabla_x\vartheta \approx \nabla_x\vartheta^4$ we need uniform bounds on $\vartheta^4$ that would be "slightly better" than in $L^1$, more precisely, we need *equi-integrability* of $\vartheta^4$ in the Lebesgue space $L^1((0,T)\times\Omega)$. To this end, we claim first that such a bound follows immediately from (2.5), (3.10) at least on the region where $\varrho$ is bounded below away from zero. Indeed by virtue of the standard imbedding relation $W^{1,2}(\Omega)\hookrightarrow L^6(\Omega)$ estimate (3.10) yields

$$\|\vartheta\|_{L^{(4-\omega)}(0,T;L^{(12-\omega)}(\Omega))} \tag{3.11}$$
$$\leq c(\omega, M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0|\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)})$$

for any $\omega > 0$. Consequently, by means of (2.5) and a simple interpolation argument, we deduce that for any $\varepsilon > 0$ there exists $\alpha = \alpha(\varepsilon) > 0$ such that

$$\int_{\{\varrho>\varepsilon\}} |\vartheta|^{4+\alpha}\ \mathrm{d}x\ \mathrm{d}t \leq c(\varepsilon, M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0|\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)}). \tag{3.12}$$

Boundedness of $\vartheta$ on the vacuum set, though the latter is of zero measure, is a more delicate task. The first step is to obtain $L^4-$ integrability of $\vartheta$ on the whole set $(0,T)\times\Omega$. To this end, we multiply the thermal energy equation (1.12) by

$$\varphi = \Delta_N^{-1}\left[\varrho - \frac{1}{|\Omega|}\int_\Omega \varrho\ \mathrm{d}x\right],$$

where $\Delta_N$ stands for the Laplacian defined on the space of functions of zero mean and supplemented with the homogeneous Neumann boundary conditions. Observe that, by virtue of estimates (2.17), (3.6) combined with the standard elliptic regularity for $\Delta_N$,

$$\left\| \Delta_N^{-1} \left[ \varrho - \frac{1}{|\Omega|} \int_\Omega \varrho \, \mathrm{d}x \right] \right\|_{L^\infty((0,T)\times\Omega)} \tag{3.13}$$
$$\leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0 |\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)}),$$

and

$$\left\| \nabla_x \Delta_N^{-1} \left[ \varrho - \frac{1}{|\Omega|} \int_\Omega \varrho \, \mathrm{d}x \right] \right\|_{L^p(0,T;L^p(\Omega))} \tag{3.14}$$
$$\leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0 |\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)})$$

for any $1 < p < \infty$.

Thus multiplying the thermal energy equation on $\varphi$ yields

$$-\kappa_0 \int_0^T \int_\Omega \vartheta^4 \left( \varrho - \frac{1}{|\Omega|} \int_\Omega \varrho \, \mathrm{d}x \right) \, \mathrm{d}x \, \mathrm{d}t = \sum_{i=1}^3 I_i, \tag{3.15}$$

where we have set

$$I_1 = \int_0^T \int_\Omega (\mathbb{S} : \nabla_x \mathbf{v} - \mathrm{R} \varrho \vartheta \mathrm{div}_x \mathbf{v}) \, \Delta_N^{-1} \left[ \varrho - \frac{1}{|\Omega|} \int_\Omega \varrho \, \mathrm{d}x \right] \, \mathrm{d}x \, \mathrm{d}t,$$

$$I_2 = c_v \int_0^T \int_\Omega \varrho \vartheta \mathbf{v}_m \cdot \nabla_x \Delta_N^{-1} \left[ \varrho - \frac{1}{|\Omega|} \int_\Omega \varrho \, \mathrm{d}x \right] \, \mathrm{d}x \, \mathrm{d}t,$$

and

$$I_3 = -c_v \int_0^T \int_\Omega \partial_t(\varrho \vartheta) \Delta_N^{-1} \left[ \varrho - \frac{1}{|\Omega|} \int_\Omega \varrho \, \mathrm{d}x \right] \, \mathrm{d}x \, \mathrm{d}t.$$

In accordance with the uniform bounds (2.18), (2.20), and (3.13), the integral $I_1$ is bounded in terms of the norm of the initial data.

Similarly, writing

$$\varrho \vartheta \mathbf{v}_m = \varrho \vartheta \mathbf{v} - \vartheta \nabla_x \varrho,$$

we can use estimates (2.12), (2.20), (3.6), and (3.9) in order to conclude that

$$\|\varrho \vartheta \mathbf{v}_m\|_{L^q(0,T;L^q(\Omega;R^3))} \tag{3.16}$$
$$\leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0 |\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)}) \text{ for a certain } q > 1.$$

Thus, in view of (3.14), the integral $I_2$ is controlled by the data.

Next, we use equation (2.15) in order to write $I_3$ in the form

$$I_3 = c_v \left[ \int_\Omega \varrho \vartheta \Delta_N^{-1} \left[ \varrho - \frac{1}{|\Omega|} \int_\Omega \varrho \, \mathrm{d}x \right] \, \mathrm{d}x \right]_{t=0}^{t=T}$$
$$+ \int_0^T \int_\Omega \varrho \vartheta \Delta_N^{-1} \left[ \mathrm{div}_x(\varrho \mathbf{v}) + \Delta \varrho \right] \, \mathrm{d}x \, \mathrm{d}t,$$

where, by virtue of (2.5), (3.13), the first term on the right-hand side is bounded. Moreover, by virtue of the standard elliptic regularity estimates combined with (2.20), (3.6), we have

$$\|\Delta_N^{-1}[\mathrm{div}_x(\varrho\mathbf{v})]\|_{L^2(0,T;L^3(\Omega))} \le c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0|\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)});$$

whence, evoking (2.18), we control

$$\int_0^T \int_\Omega \varrho\vartheta\Delta_N^{-1}[\mathrm{div}_x(\varrho\mathbf{v})]\ \mathrm{d}x\ \mathrm{d}t.$$

Finally, seeing that

$$\Delta_N^{-1}[\Delta\varrho] = \varrho - \frac{1}{|\Omega|}\int_\Omega \varrho\ \mathrm{d}x,$$

we can use (3.6) in order to conclude that $I_3$ is bounded.

Now, returning to (3.15) we write

$$\begin{aligned} &-\int_0^T \int_\Omega \vartheta^4 \left(\varrho - \frac{1}{|\Omega|}\int_\Omega \varrho\ \mathrm{d}x\right)\ \mathrm{d}x\ \mathrm{d}t \\ &\ge \frac{M_0}{2|\Omega|}\int_{\{\varrho<M_0/(2|\Omega|)\}} \vartheta^4\ \mathrm{d}x\ \mathrm{d}t - \int_{\{\varrho\ge M_0/(2|\Omega|)\}} \vartheta^4 \left(\varrho - \frac{1}{|\Omega|}\int_\Omega \varrho\ \mathrm{d}x\right)\mathrm{d}x\ \mathrm{d}t, \end{aligned}$$

where the last integral is bounded. Indeed interpolating (2.5), (3.11) on the set $\{\varrho \ge M_0/(2|\Omega|)\}$ yields

$$\begin{aligned} &\|1_{\{\varrho>M_0/(2|\Omega|)}\vartheta\|_{L^{(4+\alpha)}(0,T;L^{(8+\alpha)}(\Omega))} \\ &\quad\le c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0|\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)}) \end{aligned} \tag{3.17}$$

for a certain $\alpha > 0$, which, combined with (3.6) yields the desired conclusion.

Thus we infer from (3.15) that

$$\int_{\{\varrho<M_0/(2|\Omega|)\}} \vartheta^4\ \mathrm{d}x\ \mathrm{d}t \le c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0|\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)}),$$

which yields, together with (3.12),

$$\|\vartheta^4\|_{L^1((0,T)\times\Omega)} \le c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0|\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)}). \tag{3.18}$$

Our ultimate goal is to improve the estimates on $\vartheta^4$ in the area where the density $\varrho$ is small. To this end, we use "test" functions in the form

$$\varphi = \chi\Delta^{-1}[\chi\log(\varrho)],$$

where $\chi \in C_c^\infty(\Omega)$, $\chi \ge 0$, and the symbol $\Delta$ denotes the standard Laplace operator defined via its Fourier symbol $-|\xi|^2$ on the whole space $R^3$.

To begin, we claim that, on the basis of the refined velocity estimates obtained in (3.4), we are allowed to use the comparison argument exactly as in Section 2.4, in order to strengthen (2.27) to

$$\sup_{t\in(0,T)} \|\log(\varrho)(t,\cdot)\|_{L^{(3+\alpha)}(\Omega)} \le c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\log(\varrho_0)\|_{L^3(\Omega)}) \tag{3.19}$$

for a certain $\alpha > 0$. In particular, by virtue of the standard elliptic theory,

$$\|\Delta^{-1}[\chi \log(\varrho)]\|_{L^\infty((0,T)\times\Omega)} + \|\nabla_x \Delta^{-1}[\chi \log(\varrho)]\|_{L^\infty((0,T)\times\Omega;R^3)} \le c(\chi, M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\log(\varrho_0)\|_{L^3(\Omega)}). \tag{3.20}$$

Similarly to the preceding step, we use $\varphi$ as "test" functions in the thermal energy equation (2.12). Using the same arguments as above combined with the bounds established in (3.20), we deduce

$$\frac{\kappa_0}{4}\int_0^T \int_\Omega \Delta\vartheta^4 \chi \Delta^{-1}[\chi \log(\varrho)] \, \mathrm{d}x \, \mathrm{d}t = c_v \int_0^T \int_\Omega \partial_t(\varrho\vartheta)\chi\Delta^{-1}[\chi\log(\varrho)] \, \mathrm{d}x \, \mathrm{d}t, \tag{3.21}$$

where, furthermore,

$$\int_0^T \int_\Omega \Delta\vartheta^4 \chi \Delta^{-1}[\chi \log(\varrho)] \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_\Omega \chi^2 \vartheta^4 \log(\varrho) \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_\Omega \Big( 2\vartheta^4 \nabla_x \chi \cdot \nabla_x \Delta^{-1}[\chi \log(\varrho)] + \vartheta^4 \Delta\chi \Delta^{-1}[\chi\log(\varrho)] \Big) \, \mathrm{d}x \, \mathrm{d}t. \tag{3.22}$$

In order to handle the right-hand side of (3.21), we use equation (2.21) to obtain

$$\begin{aligned}
\int_0^T \int_\Omega \partial_t(\varrho\vartheta)\chi\Delta^{-1}[\chi\log(\varrho)] \, \mathrm{d}x \, \mathrm{d}t &= \left[ \int_\Omega \varrho\vartheta\chi\Delta^{-1}[\chi\log(\varrho)] \, \mathrm{d}x \right]_{t=0}^{t=T} \\
&- \int_0^T \int_\Omega \varrho\vartheta\chi\Delta^{-1} \left[ \chi\Delta\log(\varrho) + \chi|\nabla_x \log(\varrho)|^2 - \chi \mathrm{div}_x \mathbf{v} - \chi \mathbf{v}\cdot\nabla_x \log(\varrho) \right] \, \mathrm{d}x \, \mathrm{d}t \\
&\ge - \int_0^T \int_\Omega \varrho\vartheta\chi\Delta^{-1} \left[ \chi\Delta\log(\varrho) - \chi\mathrm{div}_x\mathbf{v} - \frac{1}{2}|\mathbf{v}|^2 \right] \, \mathrm{d}x \, \mathrm{d}t,
\end{aligned} \tag{3.23}$$

where, similarly to Section 2.4, we have used positivity of the operator $-\Delta$.

At this stage, we can use the uniform estimates (2.5), (2.18), (2.20) and (3.13) in order to observe that the last integral on the right-hand side of (3.23) is bounded. Consequently, relations (3.22), (3.23) allow us to conclude that

$$\int_0^T \int_{\mathcal{K}} \vartheta^4 |\log(\varrho)| \mathrm{d}x \, \mathrm{d}t \le c(\mathcal{K}, M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\log(\varrho_0)\|_{L^3(\Omega)}) \tag{3.24}$$

for any *compact* $\mathcal{K} \subset \Omega$ (keep in mind that $\log(\varrho)$ is negative on the "vacuum" set where $\varrho$ approaches zero).

Estimate (3.24), together with (3.12), (3.18), imply *equi-integrability* of $\vartheta^4$ at least on any compact subset of $\Omega$. In order to extend this property up to the boundary, we simply use

$$\varphi = \Delta_N^{-1}[\omega]$$

as a "test" function in (1.12), where

$$\omega = \omega(x), \ \omega(x) \ge -\underline{\omega} \text{ for all } x \in \Omega, \ \omega \in L^4(\Omega), \ \lim_{x\to\partial\Omega} \omega(x) = \infty,$$

and

$$\int_\Omega \omega \, \mathrm{d}x = 0.$$

Thus we have shown the following result:

EQUI-INTEGRABILITY OF THE TEMPERATURE FLUX:

For any $\varepsilon > 0$, there exists $\delta > 0$,

$$\delta = \delta(\varepsilon, M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\log(\varrho_0)\|_{L^3(\Omega)}),$$

such that

$$\int_B \vartheta^4 \, \mathrm{d}x \, \mathrm{d}t < \varepsilon \tag{3.25}$$

for any measurable set $B \subset (0,T) \times \Omega$ such that $|B| < \delta$.

## 4. Weak sequential stability

The problem of weak sequential stability can be formulated in the following way:

Assume that $\{\varrho_n, \mathbf{v}_n, \vartheta_n\}_{n=1}^\infty$ is a sequence of, say, regular solutions to problem (1.1–1.4), supplemented with the constitutive equations (1.13), (1.15), and the boundary conditions (1.16), (1.20). In addition, suppose that

$$\varrho_n(0,\cdot) = \varrho_{0,n}, \ \mathbf{v}_n(0,\cdot) = \mathbf{v}_{0,n}, \ \vartheta_n(0,\cdot) = \vartheta_{0,n},$$

where the initial data $\varrho_{0,n}$, $\mathbf{v}_{0,n}$, $\vartheta_{0,n}$ satisfy:

$$\left.\begin{array}{c} 0 < \underline{\varrho} \le \varrho_{0,n}(x) \le \overline{\varrho} \text{ for all } x \in \Omega, \\ 0 < \underline{\vartheta} \le \vartheta_{0,n}(x) \le \overline{\vartheta} \text{ for all } x \in \Omega, \\ \\ \|\mathbf{v}_{0,n}\|_{L^\infty(\Omega;R^3)} \le U_0 \end{array}\right\} \tag{4.1}$$

uniformly for $n = 1, 2, \ldots$

Thus, if hypotheses (A 2.1–A 2.5) are satisfied, the sequence $\{\varrho_n, \mathbf{v}_n, \vartheta_n\}_{n=1}^\infty$ admits the uniform bounds established in the preceding section. In particular, passing to subsequences if necessary, we may assume that

$$\left\{\begin{array}{c} \varrho_n \to \varrho \text{ weakly in } L^1((0,T) \times \Omega), \\ \mathbf{v}_n \to \mathbf{v} \text{ weakly in } L^1((0,T) \times \Omega; R^3), \\ \vartheta_n \to \vartheta \text{ weakly in } L^1((0,T) \times \Omega). \end{array}\right\}$$

The problem of *weak sequential stability* consists in showing that the limit quantities $\varrho$, $\mathbf{v}$, and $\vartheta$ represent a weak solution to the same system. To this end, two fundamental properties have to be verified: **(i)** pointwise a.a. convergence of all field variables, **(ii)** equi-integrability of all fluxes and production terms in the field equations (1.1–1.4).

### 4.1. Pointwise convergence

With the relatively strong *a priori* estimates at hand, the pointwise a.a. convergence of the field variables can be resolved easily. To begin, the uniform bound established in (3.9) is sufficient to conclude that

$$\left.\begin{aligned} \varrho_n &\to \varrho, \\ \nabla_x \varrho_n &\to \nabla_x \varrho \end{aligned}\right\} \text{ a.a. in } (0,T)\times\Omega. \tag{4.2}$$

Next, using (2.5), (3.6), we deduce from the momentum equation (1.2) that

$$\varrho_n \mathbf{v}_n \to \varrho \mathbf{v} \text{ in } C_{\text{weak}}([0,T]; L^{4/3}(\Omega; R^3)).$$

On the other hand, by virtue of (2.20),

$$\mathbf{v}_n \to \mathbf{v} \text{ weakly in } L^2(0,T; W^{1,2}(\Omega; R^3));$$

whence

$$\int_0^T \int_\Omega \varrho_n |\mathbf{v}_n|^2 \, \mathrm{d}x \, \mathrm{d}t \to \int_0^T \int_\Omega \varrho |\mathbf{v}|^2 \, \mathrm{d}x \, \mathrm{d}t. \tag{4.3}$$

Now it is easy to check that (4.3), together with the uniform bounds derived in the previous section, imply pointwise (a.a.) convergence of $\mathbf{v}_n$ on the set $\{x \mid \varrho(x) > 0\}$. But since the latter is of full measure, we conclude that

$$\mathbf{v}_n \to \mathbf{v} \text{ a.a. in } (0,T)\times\Omega. \tag{4.4}$$

Exactly the same argument can be used to show

$$\vartheta_n \to \vartheta \text{ a.a. in } (0,T)\times\Omega. \tag{4.5}$$

To conclude, let us remark that, by virtue of (4.2), (4.4), and (2.24),

$$\mathbf{v}_{mn} \to \mathbf{v}_m \text{ a.a. in } (0,T)\times\Omega. \tag{4.6}$$

### 4.2. Equi-integrability of the fluxes and production rates

We concentrate only on the most difficult terms appearing in (1.2), (1.3), namely,

$$\{\varrho_n \mathbf{v}_n \otimes \mathbf{v}_{mn}\}_{n=1}^\infty,\ \{\varrho_n |\mathbf{v}_n|^2 \mathbf{v}_{mn}\}_{n=1}^\infty,\ \{\varrho_n \vartheta_n \mathbf{v}_{mn}\}_{n=1}^\infty,\ \{\mathbf{q}\}_{n=1}^\infty,\ \{\mathbb{S}_n \mathbf{v}_n\}_{n=1}^\infty.$$

We recall that the term $\varrho\vartheta\mathbf{v}_m$ has already been handled in (3.16), while equi-integrability of $\{\mathbb{S}_n \mathbf{v}_n\}_{n=1}^\infty$ follows directly from the refined velocity estimates (3.4). Moreover, writing

$$\int_0^T \int_\Omega \mathbf{q}_n \cdot \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t = \frac{\kappa_0}{4} \int_0^T \int_\Omega \vartheta_n^4 \Delta\varphi \, \mathrm{d}x \, \mathrm{d}t$$

for any test function $\varphi$ satisfying $\nabla_x \varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$, we observe that convergence of the integral on the left-hand side follows from (3.25).

Furthermore,

$$\varrho_n \mathbf{v}_n \otimes \mathbf{v}_{mn} = \varrho_n \mathbf{v}_n \otimes \mathbf{v}_n - \mathbf{v}_n \otimes \nabla_x \varrho_n;$$

whence equi-integrability of the sequence $\{\varrho_n \mathbf{v}_n \otimes \mathbf{v}_{mn}\}_{n=1}^\infty$ follows directly from (3.6), (3.7), and the refined velocity estimates (3.4).

Finally,

$$\varrho_n |\mathbf{v}_n|^2 \mathbf{v}_{mn} = \varrho_n |\mathbf{v}_n|^2 \mathbf{v}_n - |\mathbf{v}_n|^2 \nabla_x \varrho_n.$$

Seeing that

$$\varrho_n |\mathbf{v}_n|^2 \mathbf{v}_n = \sqrt{\varrho_n} \sqrt{\varrho_n} \mathbf{v}_n |\mathbf{v}_n|^2,$$

we can deduce equi-integrability of this expression from (2.5), (2.17), and (3.4). Furthermore,

$$\int_0^T \int_\Omega |\mathbf{v}_n|^2 \nabla_x \varrho_n \cdot \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t$$
$$= - \int_0^T \int_\Omega \varrho_n |\mathbf{v}_n|^2 \Delta \varphi \ \mathrm{d}x \ \mathrm{d}t - 2 \int_0^T \int_\Omega \varrho_n [\nabla_x \mathbf{v}_n \mathbf{v}_n] \cdot \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t$$

for any $\varphi \in C_c^\infty((0,T) \times \Omega)$, where the last integral can be controlled by means of (2.17) and the refined estimates on the volume velocity established in (3.4).

## 5. Global existence

The question of *existence* of solutions is an ultimate criterion of validity of any mathematical model. Fortunately, Brenner's model exhibits strong similarity to the *approximate system* of equations introduced in [9] in order to show existence of weak solutions to the standard Navier-Stokes-Fourier system. Taking advantage of this remarkable coincidence, we propose the following family of approximate problems:

APPROXIMATE SYSTEM:

$$\partial_t \varrho - \Delta \varrho = -\mathrm{div}_x(\varrho \mathbf{v}), \tag{5.1}$$
$$\nabla_x \varrho \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{5.2}$$
$$\varrho(0, \cdot) = \varrho_0; \tag{5.3}$$

$$\partial_t(\varrho \mathbf{v}) + \mathrm{div}_x(\varrho \mathbf{v} \otimes \mathbf{v}) + \mathrm{R} \nabla_x(\varrho \vartheta)$$
$$-\mathrm{div}_x(\mathbf{v} \otimes \nabla_x \varrho) = \mathrm{div}_x \mathbb{S} - \varepsilon |\mathbf{v}|^{\Gamma - 2} \mathbf{v}, \tag{5.4}$$
$$\mathbf{v}|_{\partial\Omega} = 0, \tag{5.5}$$
$$(\varrho \mathbf{v})(0, \cdot) = \varrho_0 \mathbf{v}_0; \tag{5.6}$$

$$c_v \left( \partial_t(\varrho\vartheta) + \mathrm{div}_x(\varrho \vartheta \mathbf{v}) \right) - \kappa_0 \mathrm{div}_x \left( (1 + \vartheta^3) \nabla_x \vartheta \right)$$
$$-c_v \mathrm{div}_x(\vartheta \nabla_x \varrho) = \mathbb{S} : \nabla_x \mathbf{v} - \mathrm{R} \varrho \vartheta \mathrm{div}_x \mathbf{v} + \varepsilon |\mathbf{v}|^\Gamma, \tag{5.7}$$
$$\nabla_x \vartheta \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{5.8}$$
$$(\varrho\vartheta)(0, \cdot) = \varrho_0 \vartheta_0. \tag{5.9}$$

Here $\varepsilon > 0$ is a small parameter and $\Gamma > 0$ a fixed (large) number, the value of which is to be chosen below. A general strategy developed in [9, Chapter 7] applies almost literally to the above system. Specifically, the approximate momentum equation (5.4) can be solved via the Faedo-Galerkin approximation scheme while equations (5.1), (5.7) are solved directly by means of the standard theory of parabolic problems. Even more specifically, replacing (5.4) by a system of ordinary differential equations resulting from the Galerkin projections on a finite number of modes, we fix $\mathbf{v}$ satisfying the initial condition (5.6), solve (5.1–5.3) with this $\mathbf{v}$ obtaining $\varrho$, then solve (5.7–5.9) with given $\varrho$, $\vartheta$, and go back to solve (5.4–5.6) closing the circle via the Schauder fixed point argument.

This procedure, carried out and discussed in detail in [9, Chapter 7], yields the existence of solutions to the approximate system (5.1–5.9) provided we are able to show that our scheme is compatible with the *a priori* estimates obtained in Section 2. In order to see this, we note first that the total energy balance

$$\partial_t \left( \varrho \left( \frac{1}{2}|\mathbf{v}|^2 + c_v\vartheta \right) \right) + \mathrm{div}_x \left( \varrho \left( \frac{1}{2}|\mathbf{v}|^2 + c_v\vartheta \right) \mathbf{v} \right)$$
$$- \mathrm{div}_x \left( \left( \frac{1}{2}|\mathbf{v}|^2 + c_v\vartheta \right) \nabla_x \varrho \right) - \kappa_0 \mathrm{div}_x \left( (1+\vartheta^3)\nabla_x\vartheta \right) + \mathrm{div}_x(p\mathbf{v}) = \mathrm{div}_x(\mathbb{S}\mathbf{v})$$

deduced on the basis of (5.4), (5.7) does not contain any $\varepsilon$-dependent terms.

### 5.1. Regularity of the approximate velocities

Unfortunately, the refined velocity estimates obtained in Section 3.1 are not compatible with the Faedo-Galerkin approximations as they are based on multiplying the momentum equation on a *nonlinear* function of $\mathbf{v}$. In order to substitute for these estimates at the first level of the approximation procedure, the $\varepsilon$-dependent quantities have been added in (5.4), (5.7). In particular, integrating the thermal energy balance equation (5.7), we deduce that the approximate volume velocities are bounded in the Lebesgue space $L^\Gamma((0,T)\times\Omega;R^3)$. As $\varrho$ solves the parabolic equation (5.1), better summability of $\mathbf{v}$ gives rise to higher regularity of $\varrho$. Specifically, we have the result shown on top of the next page.

*Proof.* **(i)** Assume first that $\varrho(0,\cdot)=0$. Following the line of arguments used in Section 2.3, specifically the maximal regularity estimates established in Proposition 2.1, we deduce from (5.11), (5.12) that

$$\mathrm{ess}\sup_{t\in(0,T)} \|\varrho\|_{L^p(\Omega)} + \|\varrho\|_{L^q(0,T;L^3(\Omega))} \le c(p,q,r,\overline{m}) \text{ for any } 1\le p<3,\ q<\infty. \tag{5.14}$$

**(ii)** Estimate (5.14) combined with hypothesis (5.10) can be used iteratively to improve integrability of $\varrho$. To begin, employing again Proposition 2.1 we get

$$\varrho \in L^{p(\Gamma)}(0,T;W^{1,q(\Gamma)}(\Omega)),$$

where $p(\Gamma)\nearrow\infty$, $q(\Gamma)\nearrow 3$ provided $\Gamma\to\infty$. Since $W^{1,3}(\Omega)\hookrightarrow L^q(\Omega)$ for any finite $q$, we infer that

$$\varrho \in L^{p(\Gamma)}((0,T)\times\Omega), \text{ with } p(\Gamma)\nearrow\infty \text{ for } \Gamma\to\infty.$$

**Lemma 5.1.** *Let $\Omega \subset R^3$ be a bounded regular domain. Let $\mathbf{v}$ be a given velocity field satisfying*

$$\|\mathbf{v}\|_{L^\Gamma((0,T)\times\Omega;R^3)} + \|\mathbf{v}\|_{L^2(0,T;W^{1,2}(\Omega;R^3))} \leq \overline{v}. \tag{5.10}$$

*Assume that $\varrho \geq 0$ is a weak solution of problem (5.1–5.3) belonging to the class $L^1(0,T;L^3(\Omega))$, and such that $\varrho_0 \in C(\overline{\Omega})$,*

$$\frac{1}{r} \leq \inf_{x\in\Omega} \varrho_0(x) \leq \sup_{x\in\Omega} \varrho_0(x) + \|\varrho\|_{L^1(0,T;L^3(\Omega))} \leq r. \tag{5.11}$$

*Finally, suppose that*

$$\operatorname{ess}\sup_{t\in(0,T)} \|(\sqrt{\varrho}\mathbf{v})(t,\cdot)\|_{L^2(\Omega;R^3)} \leq \overline{m}. \tag{5.12}$$

*Then, for any $\Gamma > 0$ large enough, the density $\varrho$ belongs to the spaces $C([0,T]\times\overline{\Omega})$ and is strictly positive in $[0,T]\times\overline{\Omega}$. In addition,*

$$\partial_t\varrho,\ \Delta\varrho \in L^2((0,T)\times\Omega) \tag{5.13}$$

*provided $\varrho_0 \in W^{1,2}(\Omega)$. The norm of $\varrho$ in the aforementioned spaces depends only on $\overline{v}$, $r$, and $\overline{m}$.*

Thus, another application of Proposition 2.1 yields the desired conclusion

$$\varrho \in C([0,T]\times\overline{\Omega}).$$

The same can be shown, of course, in the case $\varrho_0 \neq 0 \in C(\overline{\Omega})$.

**(iii)** Strict positivity of $\varrho$ can be shown via a comparison argument exactly as in Section 2.4. Specifically, we have

$$\log(\varrho) \geq V,$$

where $V$ solves problem (2.26). Accordingly, for $\Gamma > 0$ large enough, $V$ is bounded from below as required.

**(iv)** It remains to show that $\varrho$ belongs to the "optimal" regularity class (5.13). Here again, it is enough to handle the case $\varrho_0 = 0$. Since we already know that $\varrho$ is bounded, we deduce from Proposition 2.1 that

$$\varrho \in L^{p(\Gamma)}(0,T;W^{1,p(\Gamma)}(\Omega)),\ p(\Gamma) \nearrow \infty \text{ as } \Gamma \to \infty.$$

Thus the desired conclusion follows from the standard parabolic case as

$$\partial_t\varrho - \Delta\varrho = -\nabla_x\varrho\cdot\mathbf{v} - \varrho\operatorname{div}_x\mathbf{v} \in L^2((0,T)\times\Omega). \qquad \square$$

### 5.2. Refined velocity estimates revisited

As already pointed out, the refined velocity estimates based on *non-linear* multipliers $|\mathbf{v}|^{2\alpha}\mathbf{v}$ are not compatible with the Faedo-Galerkin approximations applied to problem (5.4–5.6). Fortunately, as we have observed in Lemma 5.1, even better regularity of $\varrho$ is obtained as a consequence of the presence of the extra $\varepsilon$-terms in

(5.4), (5.7). In view of the general arguments discussed in Sections 2, 4, we may therefore expect the approximate system (5.1–5.9) to be solvable in the regularity class induced by the *a priori* estimates obtained in Section 2, where, in addition, $\varrho$ enjoys the same regularity as in the conclusion of Lemma 5.1. Thus, our ultimate goal is to carry out the limit $\varepsilon \to 0$.

At this stage, the $\varepsilon$-dependent bounds on the volume velocity field $\mathbf{v}$ have to be replaced by those obtained in Section 3.1. In other words, we have to show that the quantities $|\mathbf{v}|^{2\alpha}\mathbf{v}$ can be used as test functions in the weak formulation of (5.4) which reads

$$\left|\int_\Omega \varrho\mathbf{v}\cdot\varphi \,\mathrm{d}x\right|_{t=0}^{t=\tau} - \int_0^\tau\int_\Omega \varrho\mathbf{v}\cdot\partial_t\varphi \,\mathrm{d}x\,\mathrm{d}t - \int_0^\tau\int_\Omega \varrho\mathbf{v}\otimes(\mathbf{v}-\nabla_x\varrho):\nabla_x\varphi\,\mathrm{d}x\,\mathrm{d}t$$
$$-\mathrm{R}\int_0^\tau\int_\Omega \varrho\vartheta\mathrm{div}_x\varphi\,\mathrm{d}x\,\mathrm{d}t = -\int_0^\tau\int_\Omega \mathbb{S}:\nabla_x\varphi\,\mathrm{d}x\,\mathrm{d}t - \varepsilon\int_0^\tau\int_\Omega |\mathbf{v}|^{\Gamma-2}\mathbf{v}\cdot\varphi\,\mathrm{d}x\,\mathrm{d}t \tag{5.15}$$

for any $\tau \in [0,T]$ and any test function $\varphi \in C_c^\infty(R\times\Omega;R^3)$. Obviously, the principal difficulty stems from the lack of information on the *time derivative* of $\varphi \approx |\mathbf{v}|^\alpha\mathbf{v}$.

Extending $\varrho$, $\mathbf{v}$ to be $\varrho_0$, $\mathbf{v}_0$ for $t<0$ and $\varrho(\tau,\cdot)$, $\mathbf{v}(\tau,\cdot)$ for $\tau\in[T,\infty)$ we can use the quantities

$$\varphi(t,x) = \eta_\delta(\tau-t)\phi(x)$$

as test functions in (5.15), where $\{\eta\}_\delta$ is a suitable family of regularizing kernels with respect to the time variable. Writing $[v]_\delta = \eta_\delta * v$ we deduce

$$\partial_t[\varrho\mathbf{v}]_\delta = [\mathrm{div}_x\mathbb{S}]_\delta - [\mathrm{div}_x(\varrho\mathbf{v}\otimes\mathbf{v}_m)]_\delta - \mathrm{R}[\nabla_x(\varrho\vartheta)]_\delta - \varepsilon[|\mathbf{v}|^{\Gamma-2}\mathbf{v}]_\delta \tag{5.16}$$

for $t\in R$ provided all quantities in the brackets on the right-hand side have been extended to be zero outside the interval $[0,T]$. Since $\mathbf{v}$ and $\varrho$ belong to the regularity class specified in Lemma 5.1, we can identify the mapping $t\mapsto[\mathrm{div}_x\mathbb{S}]_\delta$ with a smooth function of time ranging in the dual $W^{-1,2}(\Omega)$, while the remaining terms on the right-hand side of (5.16) belong to $C^\infty(R;L^q(\Omega;R^3))$ for a certain $q>1$.

Now, consider the commutator

$$\omega_\delta = \partial_t[\varrho\mathbf{v}]_\delta - \partial_t(\varrho[\mathbf{v}]_\delta) \text{ on the time interval } [0,T].$$

Since $\partial_t\varrho$ belongs to the Lebesgue space $L^2((0,T)\times\Omega)$ and $\mathbf{v}\in L^\Gamma((0,T)\times\Omega;R^3)$, the classical regularity estimates of Friedrichs (see [9, Lemma 4.3]) yield

$$\omega_\delta \to 0 \text{ as } \delta\to 0 \text{ in } L^p((0,T)\times\Omega),\ \text{where } \frac{1}{p} = \frac{1}{2}+\frac{1}{\Gamma}.$$

Thus we are allowed to replace $\partial_t[\varrho\mathbf{v}]_\delta$ by $\partial_t(\varrho[\mathbf{v}]_\delta)$ in (5.16) and multiply the resulting expression by $T_k(|[\mathbf{v}]_\delta|^{2\alpha})[\mathbf{v}]_\delta$, where $T_k$ are the cut-off functions

$$T_k(z) = \min\{k,z\},$$

to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho H_k(|[\mathbf{v}]_\delta|)\ \mathrm{d}x + \int_\Omega T_k(|[\mathbf{v}]_\delta|^{2\alpha})\mathrm{div}_x[(\varrho\mathbf{v}_m\otimes\mathbf{v})]_\delta\cdot[\mathbf{v}]_\delta\ \mathrm{d}x \tag{5.17}$$
$$-\mathrm{R}\int_\Omega [\varrho\vartheta]_\delta T_k(|[\mathbf{v}]_\delta|^{2\alpha})\mathrm{div}_x([\mathbf{v}]_\delta)\ \mathrm{d}x - \mathrm{R}\int_\Omega [\varrho\vartheta]_\delta\nabla_x T_k(|[\mathbf{v}]_\delta|^{2\alpha})\cdot[\mathbf{v}]_\delta\ \mathrm{d}x$$
$$+\mu\int_\Omega T_k(|[\mathbf{v}]_\delta|^{2\alpha})|\nabla_x[\mathbf{v}]_\delta|^2\ \mathrm{d}x + \frac{\mu}{3}\int_\Omega T_k(|[\mathbf{v}]_\delta|^{2\alpha})|\mathrm{div}_x[\mathbf{v}]_\delta|^2\ \mathrm{d}x$$
$$+\varepsilon\int_\Omega T_k(|[\mathbf{v}]_\delta|^{2\alpha})[|\mathbf{v}|^{\Gamma-2}\mathbf{v}]_\delta\cdot[\mathbf{v}]_\delta\ \mathrm{d}x$$
$$=\int_\Omega T_k(|[\mathbf{v}]_\delta|^{2\alpha})\omega_\delta\cdot[\mathbf{v}]_\delta\ \mathrm{d}x$$
$$-\mu\int_\Omega\left([(\nabla_x[\mathbf{v}]_\delta)[\mathbf{v}]_\delta]\cdot\nabla_x T_k(|[\mathbf{v}]_\delta|^{2\alpha}) + \frac{1}{3}\mathrm{div}_x[\mathbf{v}]_\delta\nabla_x T_k(|[\mathbf{v}]_\delta|^{2\alpha})\cdot[\mathbf{v}]_\delta\right)\ \mathrm{d}x,$$

where

$$H_k'(z) = \begin{cases} z^{2\alpha+1} \text{ for } 0\le z\le k^{1/2\alpha}, \\ kz \text{ if } z\ge k^{1/2\alpha}. \end{cases}$$

Thus letting first $\delta\to 0$ and then $k\to\infty$ we deduce the same estimates as in Section 3.1 that are independent of $\varepsilon$.

### 5.3. Global existence – conclusion

In accordance with the previous discussion, the existence of global-in-time solutions for the initial-boundary value problem associated to system (1.1–1.4) can be established in two steps:

- Solutions of the approximate system (5.1–5.9) are obtained by the method described in detail in [9].
- The approximate solutions enjoy the regularity properties established in Lemma 5.1. In particular, the approximate velocity field $\mathbf{v}$ belongs to the regularity class identified in Section 5.2, where the bounds are independent of $\varepsilon$.
- We let $\varepsilon\to 0$ to recover a weak solution of the original system. The limit is carried over by the same arguments as in Section 4.

On the point of conclusion, let us state our main existence result (top of the next page).

**Remark:** *In accordance with our considerations in Section* 4*, we write*

$$\mathbf{q} = -\kappa_0\nabla_x\vartheta - \frac{\kappa_0}{4}\nabla_x\vartheta^4,$$

*and similarly.*

$$\varrho|\mathbf{v}|^2\mathbf{v}_m = \varrho|\mathbf{v}|^2\mathbf{v} - \mathrm{div}_x(\varrho|\mathbf{v}|^2) + 2\varrho(\nabla_x\mathbf{v}\ \mathbf{v})$$

*in the weak formulation of the total energy balance* (1.3).

GLOBAL-IN-TIME EXISTENCE FOR LARGE DATA:

**Theorem 5.1.** *Let* $\Omega \subset R^3$ *be a bounded domain of class* $C^{2+\nu}$, $\nu > 0$. *Assume that the thermodynamic functions satisfy hypotheses* (A 2.1–A 2.5). *Finally, let the initial data be chosen so that*

$$\varrho_0,\ \vartheta_0 \in L^\infty(\Omega),\ \mathbf{v}_0 \in L^\infty(\Omega; R^3),$$
$$\operatorname{ess}\inf_\Omega \varrho_0 > 0,\ \operatorname{ess}\inf_\Omega \vartheta_0 > 0.$$

*Then the initial-boundary value problem associated to* (1.1–1.5), (1.13), (1.15–1.20) *possesses at least one weak solution* $\varrho$, $\mathbf{v}$, $\vartheta$ *on* $(0,T) \times \Omega$. *Moreover,*

$$\varrho(t,x) > 0,\ \vartheta(t,x) > 0 \text{ for a.a. } (t,x) \in (0,T) \times \Omega.$$

Under the hypotheses (A 2.1–A 2.5), Brenner's model or (BNSF) system represents an interesting alternative to the classical approach. The velocity field $\mathbf{v}$ enjoys more regularity than the weak solutions to the incompressible Navier-Stokes system constructed by Leray. In addition, both $\varrho$ and $\vartheta$ are *positive* although with a possible exception of a set of zero Lebesgue measure. To the best of our knowledge, such a result for the *standard* Navier-Stokes-Fourier system lies beyond the scope of the available existence theory. Of course, the model is open to discussion regarding, in particular, the relevant value of the phenomenological coefficient $K$ set constant in the present study.

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# Existence of a Regular Periodic Solution to the Rothe Approximation of the Navier–Stokes Equation in Arbitrary Dimension

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**Abstract.** In this paper we show the existence of regular solutions of the Rothe approximation of the unsteady Navier–Stokes equations with periodic boundary condition in arbitrary dimension. The result relies on techniques developed by the authors in the study of the higher-dimensional steady Navier–Stokes equations.

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## 1. Introduction

In previous papers [FR94a], [FR95a], [FR95b] (cf. [Str95], [BF02]) we proved the existence of regular solutions to the steady Navier–Stokes equations

$$\begin{aligned} -\mu\Delta u + u\cdot\nabla u + \nabla p = f \qquad &\text{in } \Omega,\\ \operatorname{div} u = 0 \qquad &\text{in } \Omega, \end{aligned} \tag{1}$$

with periodic boundary conditions, where $\Omega = (0,L)^N$, $0 < L$, $5 \le N < 15$. The function $f \in L^\infty$ and the constant viscosity $\mu > 0$ are given. In the case of Dirichlet boundary conditions (steady problem) the authors succeeded to prove the existence of locally regular solutions up to dimension 6, cf. [FR96a]. The motivation to consider higher-dimensional steady problems ($n \ge 5$) comes from the similarity in the regularity theory and in the scaling behaviour of solutions to the steady $n$-dimensional and the unsteady $(n-2)$-dimensional problem. This similarity allows us to develop new analytical techniques in the steady context, which might be also useful in the unsteady case. (cf. [NRŠ96], [Tsa98]).

In the periodic case it is likely that restriction to the dimension $N \leq 15$ is not needed. From this background it is interesting that there is *no* dimensional restriction concerning the existence of a regular solution if one considers the *Rothe* approximation of the unsteady Navier–Stokes equations with periodic boundary condition. It reads $\big(u(t) = u(t,.),\ u : [0,T] \times \Omega \to \mathbb{R}^N\big)$

$$\begin{aligned} h^{-1}\big(u(t) - u(t-h)\big) - \mu\Delta u(t) + u(t)\cdot\nabla u(t) + \nabla p(t) &= f\,, \\ \operatorname{div} u &= 0\,, \end{aligned} \tag{2}$$

where $u(0) = u_0$ is given.

Obviously (2) is solved successively for $t = kh$, $k = 1, 2, \ldots, N$, $hN = T$. Given $u(t-h)$ being smooth one has to prove the existence of a periodic regular solution, say $u(t) \in W^{2,q}$ of (2) for all $q < \infty$, if the data are smooth enough. Proving this theorem is the purpose of the present paper.

By a scaling argument in the time and space variables, the unknown functions and a redefinition of $h$, we may assume without loss of generality that $\mu = 1$.

## 2. Notation and formulation of the theorem

Let us first introduce some notation. By $\big(L^q(\Omega), \|\cdot\|_q\big)$, $\big(W^{k,q}(\Omega), \|\cdot\|_{k,p}\big)$ we denote the usual Lebesgue, resp. Sobolev, spaces of periodic functions. We denote the mean value of any function $g$ by $\bar{g}$. By $K$ we denote a generic positive constant. Here we consider the periodic case. Let $\Omega = (0,L)^N$ be a cube of length $L > 0$ and let us write $\Gamma_j = \partial\Omega \cap \{x_j = 0\}$, $\Gamma_{j+N} = \partial\Omega \cap \{x_j = L\}$. We consider the problem (2) with $\mu = 1$ and periodic boundary conditions

$$\begin{aligned} u|_{\Gamma_j} = u|_{\Gamma_{j+N}}\,, \qquad p|_{\Gamma_j} &= p|_{\Gamma_{j+N}} \qquad \forall j = 1, \ldots, N \\ \frac{\partial u}{\partial x_k}|_{\Gamma_j} &= \frac{\partial u}{\partial x_k}|_{\Gamma_{j+N}} \qquad \forall j,k = 1, \ldots, N\,. \end{aligned}$$

We shall prove the following theorem:

**Theorem 2.1.** *Let $\Omega = (0,L)^N$, $N \geq 2$, be a periodicity cube in $\mathbb{R}^N$ and for $t = kh$, $k \in \mathbb{N}$ let*

$$f(t) \in L^\infty(\Omega)\,, \tag{3}$$

$$\operatorname{div} f(t) \in L^\infty(\Omega)\,. \tag{4}$$

*Furthermore, let*

$$u_0 \in W^{2,q}(\Omega) \qquad \text{for all } q \in [1,\infty)\,, \qquad \operatorname{div} u_0 = 0\,. \tag{5}$$

*Then there exists a solution*

$$u(t) \in W^{2,q}(\Omega)\,, \qquad t = kh\,, \quad k \in \mathbb{N}\,,$$

*to the Rothe approximation* (2).

For the proof of this theorem regularity is successively established for $u(kh)$, $k \in \mathbb{N}$. The proof of this relies on our result from [FR94b] which states that if the quantity (called head pressure)

$$w = p(t) + \frac{1}{2}|u(t)|^2 ,$$

is bounded from above, then a solution of (2) with this property is regular. To achieve this criterium, we use a Moser type technique to derive the desired $L^\infty$-bound. The head pressure also played a crucial role in the resolution of a long-lasting problem concerning the existence of self-similar solutions of the unsteady Navier–Stokes equations (cf. [NRŠ96], [Tsa98], [Tsa99]).

## 3. Regularized Rothe approximation

We assume that for $t = h, 2h, \ldots, (k-1)$ regular solutions $u(t)$, $p(t)$ have been constructed and we have to construct a regular pair $u(t)$, $p(t)$, $t = kh$. For this we consider – as in the first paper [FR94a] in this subject – the approximation

$$\begin{aligned} h^{-1}u(t) - \Delta u(t) + u(t)\cdot\nabla u(t) + \delta_0|u(t)|^2\, u(t) + \nabla p(t) \\ = f + h^{-1}u(t-h), \\ \operatorname{div} u(t) = 0 \end{aligned} \tag{6}$$

in the weak sense, for $\delta_0 > 0$, $\delta_0 \to 0$.

By routine methods (say, the usual Ritz–Galerkin method) we obtain a weak solution $v = u(t)$ with $\operatorname{div} v = 0$ such that

$$\|\nabla v\|_2^2 + \frac{1}{2}h^{-1}\|v\|_2^2 + \delta_0\|v\|_4^4 \le \frac{1}{2}h^{-1}\|v(t-h)\|_2^2 + \|f\|_2\|v\|_2 . \tag{7}$$

The norms are taken over $\Omega$. Then, obviously, the right-hand side of (7) remains bounded as $\delta_0 \to 0$. With the usual procedure from the theory of Navier–Stokes equations we obtain a pressure $p$ with mean value $\bar{p} = 0$ satisfying the equation

$$\Delta p = -\sum_{i,k=1}^{N} D_i(v_k D_k v_i) + \operatorname{div} f - \delta_0 \operatorname{div}(|v|^2 v) \tag{8}$$

in the weak sense. Since $\|\nabla v\|$ and $\delta_0\|v\|_4^4$ are bounded uniformly and $\operatorname{div} f \in L^\infty$ we obtain via linear elliptic regularity theory that

$$\|p\|_{N/(N-2)} \le K , \qquad \|\nabla p\|_{N/(N-1)} \le K \tag{9}$$

uniformly as $\delta_0 \to 0$. In fact, due to Sobolev embedding, $v \in W^{1,2}$ and Hölder's inequality we have $v\cdot\nabla v \in L^{N/(N-1)}$ and due to the uniform bound for $\delta_0\|v\|_4^4$ we further have

$$\|\delta_0|v|^2 v\|_{4/3} \to 0 \qquad \text{as } \delta_0 \to 0 \tag{10}$$

and

$$\|\delta_0|v|^2 v\|_{N/(N-1)} \le K .$$

Thus the right-hand side of (8) is the divergence of a $L^{N/(N-1)}$-function and (9) follows via $W^{1,N/(N-1)}$-duality.

We confine ourselves to the case $N \geq 4$ since $N = 2, 3$ correspond to the classical steady Navier–Stokes equation (up to the term $h^{-1}u$ which does not change the situation).

Due to the estimates (7) and (9) the following lemma is clear.

**Lemma 3.1.** *The solutions $v = v_{\delta_0}$ of (6) converge weakly, for a subsequence, to a weak solution of (2) as $\delta_0 \to 0$, and are uniformly bounded in $W^{1,2}$. Moreover, there hold the inclusions*

$$|v||\nabla v| \in L^{4/3}\,, \qquad \nabla^2 v \in L^{4/3}\,, \qquad \nabla p \in L^{4/3} \tag{11}$$

*for fixed $\delta_0 > 0$.*

*Proof.* The first statement is routine analysis. Inequality (11) follows via duality in $W^{1,q}$ using $v \cdot \nabla v \in L^{4/3}$ since $v \in L^4$; and $|v|^2 v \in L^{4/3}$. So we derive $\nabla p \in L^{4/3}$ from the pressure equation and finally $\Delta v$ and $\nabla^2 v \in L^{4/3}$ due to elliptic $W^{2,p}$-theory. ☐

## 4. The head pressure equation

Let $v = u(t)$ and $p(t)$ be the solutions of (2) constructed in the previous section. It is easy to see that the function

$$w = \frac{|v|^2}{2} + p(t)\,,$$

called the "head pressure" is a weak solution of the equation

$$\begin{aligned} h^{-1}|v|^2 - \Delta w + |\nabla v|^2 + \sum_{k,j=1}^{N} D_k v_j D_j v_k + v \cdot \nabla w + \delta_0 |v|^4 \\ = -\operatorname{div} f + f \cdot v + h^{-1} v \cdot u(t-h) + \operatorname{div}(\delta_0 v |v|^2)\,, \end{aligned} \tag{12}$$

where we used $\sum\limits_{i,k=1}^{n} D_i v_k D_k v_i = \sum\limits_{i,k=1}^{n} D_k (v_k D_k v_i)$. Clearly $w$ depends on $\delta_0$. Equation (12) holds strongly in $L^1$ and in the weak sense for test functions $\varphi \in L^\infty \cap W^{1,2}$. This is due to the fact that $\nabla v \in L^2$, $v \cdot \nabla w \in L^1$ for all $\delta_0$ and $\operatorname{div}(\delta_0 v |v^2|)$, $\delta_0 v^4 \in L^1$ for fixed $\delta_0$, cf. (11) and (7).

**Proposition 4.1.** *For every $r > 2$, the head pressure $w$ satisfies the inequality*[1]

$$\int |\nabla w|^2 w_+^{r-2}\, dx \leq K_h \int w_+^{r-1}\, dx$$

*uniformly as $\delta_0 \to 0$ and uniformly in $r \to \infty$.*

[1] We use the convention that the integral is taken over $\Omega = (0, L)^N$ if not otherwise stated.

*Proof.* Let $M, r \geq 2$ be real numbers (which shall tend to $\infty$ later) and let $\omega*$ be the usual mollification operation converging to the Dirac functional. We test equation (12) with the test function

$$\omega * [\omega * w_+]_M^{r-1}$$

where $w_+ = \max\{0, w\}$, $[\xi]_M = \min(\xi, M)$ and obtain an equation

$$T_1 + T_2 = T_3\,, \tag{13}$$

where $T_1, T_2, T_3$ are defined below. We analyze the summands arising as $\omega*$ converges to the Dirac functional. We have

$$\begin{aligned} T_1 := \int \Big\{ & h^{-1}|v|^2 + |\nabla v|^2 + \sum_{k,j=1}^{N} D_k v_j D_j v_k + u \cdot \nabla w + \delta_0 |v|^4 \Big\} \\ & \times \omega * \big[(\omega * w)_+\big]_M^{r-1}\, dx \\ \to \int & \Big\{ h^{-1}|v|^2 + |\nabla v|^2 + \sum_{k,j=1}^{N} D_k v_j D_j v_k + v \cdot \nabla w + \delta_0 |v|^4 \Big\} [w_+]_M^{r-1}\, dx \end{aligned}$$

since the terms in $\{\ldots\}$ are in $L^1$ for fixed $\delta_0$. Approximating $w$ with periodic $C^\infty$-functions $w_m$ with $w_m \to w$ in $W^{1,4/3}$ we see that

$$T_{14} := \int v \cdot \nabla w [w_+]_M^{r-1}\, dx = o(1) + \int v \cdot \nabla(w_m)\big[(w_m)_+\big]_M^{r-1}\, dx = o(1)$$

hence $T_{14} = 0$. We have used the usual *Navier–Stokes Identity*

$$\int v \cdot \nabla\varphi\, b(\varphi)\, dx = 0\,.$$

So, we proved

$$T_1 \to \int \{h^{-1}|v|^2 + |\nabla v|^2 + \sum_{j,k=1}^{N} D_j v_k D_k v_j + \delta_0 |v|^4\} [w_+]_M^{r-1}\, dx =: T_{10} \tag{14}$$

as $\omega*$ converges to the Dirac functional. The term

$$T_2 := -\int \Delta w\, \omega * \big[(\omega * w)_+\big]_M^{r-1}\, dx$$

is re-written via partial integration as

$$\begin{aligned} T_2 &= \int \nabla(\omega * w) \nabla[(\omega * w)_+]_M^{r-1}\, dx \\ &= (r-1) \int_M |\nabla(\omega * w)|^2 [(\omega * w)_+]_M^{r-2}\, dx\,, \end{aligned}$$

where the symbol $M$ under the integral sign indicate, that the integration runs over the set $\{\omega * w \leq M\}$ resp. over $\{w \leq M\}$ thereafter. By Fatou's lemma we obtain for $\omega*$ converging to the Dirac functional

$$\liminf T_2 \geq (r-1) \int_M |\nabla w|^2 [w_+]_M^{r-2}\, dx\,. \tag{15}$$

The terms arising due to the right-hand side of (12) do not give trouble since $v \in L^4$, $\operatorname{div} f$, $f$, $u(t-h) \in L^\infty$, $|\nabla v||v|^2 \in L^1$. Thus we obtain that

$$\begin{aligned} T_3 := \int & \big\{ -\operatorname{div} f + f \cdot v + h^{-1} v \cdot u(t-h) + \operatorname{div}(\delta_0 v |v|^2) \big\} \\ & \times \omega * [(\omega * w)_+]_M^{r-1}\, dx \end{aligned}$$

converges to

$$T_{30} := \int \big\{ -\operatorname{div} f + f \cdot v + h^{-1} v \cdot u(t-h) + \operatorname{div}(\delta_0 v |v|^2) \big\} [w_+]_M^{r-1}\, dx\,.$$

All together we proved that

$$T_{10} + (r-1) \int_M |\nabla w|^2 [w_+]_M^{r-2}\, dx \leq T_{30}\,. \tag{16}$$

We estimate the term

$$\begin{aligned} & \int \big\{ f \cdot v + h^{-1} v \cdot u(t-h) \big\} [w_+]_M^{r-1}\, dx \\ & \leq \frac{1}{2} h^{-1} \int |v|^2 [w_+]_M^{r-1}\, dx + K_h \int [w_+]_M^{r-1}\, dx \end{aligned} \tag{17}$$

where we used $\operatorname{div} f, f \in L^\infty$, $u(t-h) \in L^\infty$. The term $-\operatorname{div} f$ in $T_{30}$ is estimated by a constant and the term $|\nabla v|^2 + \sum_{j,k=1}^{N} D_j v_k D_k v_j$ in $T_{10}$ is estimated from below by zero. Now, the term $h^{-1} \int |v|^2 [w_+]_M^{r-1}\, dx$ in $T_{10}$ dominates the corresponding term in (17) which is the main difference in the treatment of the Rothe approximation compared to the steady Navier–Stokes equations. The term $\delta_0 \operatorname{div}(v|v|^2)$ is estimated by

$$\frac{1}{2} \delta_0 |v|^4 + c(N) \delta_0 |\nabla v|^2\,,$$

where the first one is absorbed on the left-hand side. Using all this we conclude from (14), (16)

$$\begin{aligned} & (r-1) \int_M |\nabla w|^2 [w_+]_M^{r-2}\, dx \\ & \quad \leq K_h \int [w_+]_M^{r-1}\, dx + c(N)\, \delta_0 \int |\nabla v|^2 [w_+]_M^{r-1}\, dx\,. \end{aligned}$$

The constant $K_h$ does not depend on $r, M$, $\delta_0$, $\delta_0 \leq \delta_0'$. Using Fatou's lemma we pass to the limit $\delta_0 \to 0$ and we see that

$$[w_+]_M^{r/2} \in W^{1,2}$$

and

$$(r-1)\int\limits_M |\nabla w|^2 [w_+]_M^{r-2}\,dx \le K_h \int [w_+]_M^{r-1}\,dx\,, \tag{18}$$

since the term with integrand $|\nabla v|^2 [w_+]_M^{r-1}$ does not make trouble for fixed $M$, because of the $L^2$-bound for $\nabla v$ and the truncation $[w_+]_M^{r-1}$.

It is well known from the theory of non-linear elliptic equations, [LU68], [GT01], that (18) implies

$$w_+ \in L^\infty\,.$$

Let us remind the reader: Start an iteration procedure in (18) with $r_0 = 2$. Thus $[w_+]^{r_0-1} \in L^1$ (at least) and we may pass to the limit $M \to \infty$. This implies

$$c\Big(\int w_+^{\frac{2N}{N-2}}\,dx\Big)^{\frac{N-2}{N}} \le \int |\nabla w_+|^2\,dx \le K_h \int w_+\,dx$$

and we have the improved inclusion

$$w_+ \in L^{\frac{2N}{N-2}}.$$

We choose successively $r_j = \frac{N}{N-2} r_{j-1}$ and obtain from (18)

$$w_+ \in L^q \qquad \text{with } q = \left(\frac{N}{N-2}\right)^s r_0\,, \qquad s \to \infty$$

and we conclude with (18)

$$(r-1)\int |\nabla w|^2 |w_+|^{r-2}\,dx \le K_h \int |w_+|^{r-1}\,dx\,.$$

This is the basic inequality for Moser's iteration method, which yields:

**Lemma 4.1.** $w_+ \in L^\infty$.

## 5. Proof of the main theorem

In [FR94b] we showed that solutions to the steady Navier–Stokes equation are regular, provided that the head pressure is bounded from above. This worked in arbitrary dimensions. In principle, we want to apply this theorem in our situation, however our equation has the additional term $h^{-1}v$ coming from the Rothe approximation. Recall that $v = u(t)$ is the solution of (2). So we have to go through the old proofs and explain that this additional term does not give problems.

*First step* (cf. [FR94a, Theorem 2.1], [FR96b]): One uses in the pressure equation (8) the test function $(|x-x_0|^2+h^2)^{-(N-2)/2}\tau^2$ and $(|x-x_0|^2+h^2)^{-q+2}\tau^2$, $q < N-2$, where $\tau$ is a localization function and passes to the limit $h \to 0$. Then using the boundedness of the positive part of the head pressure, one obtains the inequalities

$$\int \Big|\frac{|v|^2}{2} + p\Big| \frac{\tau^2}{|x-x_0|^{N-2}}\,dx \le c, \tag{19}$$

$$\int \frac{(v\cdot(x-x_0))^2}{|x-x_0|^N}\tau^2\,dx \le c, \tag{20}$$

$$\int \Big(\frac{|v|^2}{2} + |p|\Big)\frac{\tau^2}{|x-x_0|^q}\,dx \le c_q\,, \qquad q < N-2\,, \tag{21}$$

$q < N-2$. This procedure is the same in the case considered here since the pressure equations do not differ, due to the fact that $\operatorname{div} v = \operatorname{div} u(t-h) = 0$.

*Second step*: Let $1 \le r < 2$. We use the function

$$\varphi_0(x_0) = \int |p|^{r-1}\operatorname{sign} p|x-x_0|^{2-N}\tau^2\,dx$$

as a test function in the pressure equation. Due to (21) one obtains that $\varphi_0 \in L^\infty$, so the arising term of the right-hand side in the pressure equation is bounded, and the integral

$$\int \nabla p\cdot\nabla\varphi_0\,dx$$

will lead to

$$\int |p|^r\,\tau^2\,dx \le c_r\,. \tag{22}$$

From Lemma 4.1 thus also follows

$$\int |v|^{2r}\,\tau^2\,dx \le c_r\,. \tag{23}$$

Indeed we have used Lemma 4.1 and (22):

$$\begin{aligned}\int |v|^{2r}\,\tau^2\,dx &\le c\int (\frac{|v|^2}{2} + p - p)^r\,\tau^2\,dx\\ &\le c\int \Big(\Big(\frac{|v|^2}{2}+p\Big)_+ + |p|\Big)^r\,\tau^2\,dx\\ &\le c\,.\end{aligned}$$

Note, that $\varphi_0 \in W^{1,2}$ since $\Delta\varphi_0 \in L^1$ and $\varphi_0 \in L^\infty$. These arguments are worked out for $N = 5$ in [FR94a, Theorem 2.11] and work completely analogously for arbitrary dimension $N$. Since the pressure equations in the present and in the cited paper are the same we have (22) and (23).

With similar arguments, only using the pressure equation and the fact that $\big(\frac{|v|^2}{2}+p\big)_+ \le c$, we obtain (cf. [FR94a, Corollary 2.15])

$$\int |p|\,\Big|\frac{v^2}{2}+p\Big|\,\tau^2\,dx \le c\,, \tag{24}$$

$$\int \Big|\frac{|v|^2}{2}+p\Big|^2 \tau^2\,dx \le c\,. \tag{25}$$

*Third step*: We have to establish the Navier–Stokes inequality:

$$\begin{aligned} &h^{-1}\int |v|^2\varphi\,dx + \int \nabla v\cdot\nabla(v\varphi)\,dx \\ &\quad\le \int \Big(\frac{|v|^2}{2}+p\Big)v\cdot\nabla\varphi\,dx + \int \big(f\cdot v - \operatorname{div} f + h^{-1}u(t-h)\cdot v\big)\varphi\,dx \end{aligned} \tag{26}$$

for smooth non-negative test function $\varphi$. The corresponding inequality without the terms coming from the Rothe approximation has been proved in [FR94b]. Formally, it follows from the Navier–Stokes equation or the Rothe approximation using the function $v\varphi$ as a test function. However, this cannot be done directly since we do not know in the present state of the proof that $v\cdot\nabla(\frac{|v|^2}{2}+p)$ is integrable. We can justify (26) using that $v$ is the limit of the approximate Rothe problem (6). Indeed, using in the weak formulation of (6) $v\varphi$ as a test function we obtain

$$\begin{aligned} &h^{-1}\int |v|^2\varphi\,dx + \int \nabla v\cdot\nabla(v\varphi)\,dx + \delta_0\int |v|^4\varphi\,dx \\ &\quad= \int \Big(\frac{|v|^2}{2}+p\Big)v\cdot\nabla\varphi\,dx + \int \big(f\cdot v - \operatorname{div} f + h^{-1}u(t-h)\cdot v\big)\varphi\,dx\,, \end{aligned}$$

where we used $\int (v\cdot\nabla v+\nabla p)v\varphi\,dx = -\int \big(\frac{|v|^2}{2}+p\big)v\cdot\nabla\varphi\,dx$. Note that all integrals are defined since $v\in W^{1,2}\cap L^4$. Thus the limit $\delta_0\to 0$ yields (26).

*Step four*: We introduce the function $g$ defined by

$$\begin{aligned} \Delta g &= \Big(\frac{|v|^2}{2}+p\Big) \qquad &&\text{in } B_R, \\ g &= 0 &&\text{on } \partial B_R\,. \end{aligned} \tag{27}$$

We may assume $[-L,L]^N \subset\subset B_R$. In [FR94b, Lemma 2.4] it was shown that $g\in W^{2,\frac{N}{N-2}}\cap W_0^{1,\frac{N}{N-3}}$ satisfies the inequalities

$$\|\nabla^2 g\|_{2,\mathrm{loc}} \le c, \tag{28}$$

$$\|g\|_{\infty,\mathrm{loc}} \le c, \tag{29}$$

$$\int_{B_r} |\nabla g|^2|x-x_0|^{2-N}\,dx \le c\,, \qquad B_r\subset\subset B_R\,. \tag{30}$$

In fact, we work with $\tau^2|x-x_0|^{2-N}$ and $g|x-x_0|^{2-N}\tau^2$ as test functions, combined with some mollification argument to justify the integrations.

For the proof, only (25) and (19) are used. So we have these inequalities also in our setting. The introduction of this function $g$ is an important trick to deal with the term $(\frac{|v|^2}{2}+p)v\cdot\nabla\varphi$, for special $\varphi$, in the next steps.

*Step five*: In the Navier–Stokes inequality (26) we choose

$$\varphi = (|x-x_0|^2+h^2)^{(N-4)/2}\tau^2\,,$$

where $\tau$ is a localization function. We replace the term $\frac{|v|^2}{2}+p$ by $\Delta g$, where $g$ is defined in (27). Using similar calculations as in [FR94b, Proposition 2.8], we arrive at the estimate

$$\begin{aligned}\int |\nabla v|^2(|x-x_0|^2+h^2)^{-(N-4)/2}\tau^2\,dx + \int |v|^2(|x-x_0|^2+h^2)^{-(N-2)/2}\tau^2\,dx \\ \le \sum_{i=1}^N \int \nabla g \nabla\big(v_i D_i(|x-x_0|^2+h^2\big)^{-(N-4)/2}\tau^2\big)\,dx + K\,,\end{aligned} \tag{31}$$

where several absorbing steps have been applied. The estimates for $g$ are such that the convective term at the right-hand side of (31) can be estimated such that the terms with factor $v$ and $\nabla v$ in the integrand can be absorbed by the left-hand side of (31). Thereafter, we pass to the limit $h\to 0$ and end up with

$$\int |\nabla v|^2|x-x_0|^{-N+4}\tau^2\,dx + \int |v|^2|x-x_0|^{-N+2}\,dx \le K\,. \tag{32}$$

This is a Morrey condition for $\nabla v$. From (32) we obtain via a refinement of embedding theorem due to Chiarenza, Frasca [CF87] that

$$\int |v|^4|x-x_0|^{-N+4}\,dx \le K\,.$$

*Step six – the hole-filling step*: The Navier–Stokes inequality is used once more with

$$\varphi = |x-x_0|^{-N+4}\tau_R^2\,,$$

where $\tau_R$ is a Lipschitz continuous localization function with $\tau_R = 1$ on $B_R(x_0)$, support $\tau_R = B_{2R}(x_0)$, $|\nabla\tau_R|\le R^{-1}$. Repeating the arguments of the 5th step, with a more precise analysis of the $R$ dependence of the constants, and using also the equation for $g$ and $p$ we obtain similarly as in ([FR94b, Proof of Theorem 1.5] the hole-filling condition for the quantity

$$\begin{aligned}\phi = |\nabla v|^2|x-x_0|^{4-N} + |v|^2|x-x_0|^{2-N} + (v\cdot(x-x_0))^2|x-x_0|^{-N} \\ + \Big|\frac{|v|^2}{2}+p\Big|\,|x-x_0|^{2-N} + |\nabla g|^2|x-x_0|^{2-N}\,,\end{aligned}$$

namely

$$\int_{B_R}\phi\,dx \le K\int_{T_R}\phi\,dx + KR^\alpha\,, \qquad T_R = B_{2R}\setminus B_R\,. \tag{33}$$

Note that the term $h^{-1}v$ coming from the Rothe method does not disturb the Navier–Stokes inequality; it has the correct sign and could be dropped or used to dominate terms to have better constants. From (33) one concludes a Morrey condition (cf. [BF02, Section 1.2.3], [FR94b])

$$\int_{B_R} \phi \, dx \leq K R^{\gamma} ,$$

which gives an improved Morrey condition compared to (32):

$$\int_{B_R} |\nabla v|^2 \, dx \leq K R^{N-4+\gamma} , \qquad \int_{B_R} |\nabla v|^2 \, dx \leq K R^{N-2+\gamma} . \tag{34}$$

*Seventh step*: Already in the first paper [FR94a, Theorem 3.28] the authors showed via a bootstrap argument, using weighted $W^{2,p}$-estimates for the Laplacian, that (34) and $w_+ \in L^\infty$ imply full regularity (say $W^{2,q}$ for all $q < \infty$) for the solution $u$ of the Navier–Stokes equation. For the Rothe approximation, this is just the same argument since $h^{-1}v$ is only a lower order term not disturbing the bootstrap procedure. (Anyhow, it is likely that the operator $h^{-1}u - \Delta u$ is stable with respect to Morrey norms.) This proves our theorem. $\square$

## References


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# Optimal Neumann Control for the Two-dimensional Steady-state Navier-Stokes equations

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*To the memory of Alexander Vasil'evich Kazhikhov*

**Abstract.** An optimal control problem, the minimization of drag, is considered for the 2D stationary Navier-Stokes equations. The control is of Neumann kind and acts at a part of the boundary which is contiguous to the rigid boundary where the no-slip condition holds. Further, certain constraints are imposed on the control and the phase variable. We derive an existence theorem as well as the corresponding optimality system

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## 1. Introduction

This paper is devoted to the study of an optimal control problem for the Navier-Stokes equations defined in a bounded domain $\Omega$. We are interested in the existence of optimal solutions as well as in the derivation of the corresponding "optimality system", i.e., the first-order optimality conditions. These problems have been studied already for the stationary Navier-Stokes equations (see [GHS1], [GHS2], [CH], [A], [ALT]) and the nonstationary Navier-Stokes equations (see [F1], [F2], [AT], [S] [F], [FGH]) for small as well as large Reynolds numbers. However, not all aspects of these optimization problems have been completely investigated, yet.

In this paper, we concentrate on the following questions arisen in optimal control problems. First of all the extremal problem we study contains restrictions not only on the control but on the phase variable as well. The restriction is imposed

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that the component $v_1(x)$ of the fluid velocity should be nonnegative on a certain subdomain $\omega$ of $\Omega$.

The derivation of the optimality system in such a situation needs a specific Lagrange principle. A general Lagrange principle of such kind was worked out by I.V. Girsanov [G] and A.A. Milutin, A.V. Dmitruk, N.P. Osmolovskiy [MDO]. In this paper, we have to adapt the approach from [G], [MDO] to the optimal control problem for the Navier-Stokes equations.

Usually in applications the boundary control is acting not on the whole boundary $\partial\Omega$ but only on a certain part $\Gamma$. Besides, often it is more reasonable to use Neumann control on $\Gamma$ instead of Dirichlet control. Moreover $\Gamma$ is contiguous with the part of the boundary where the adhesion condition is posed. In such a situation Neumann control causes a local singularity of the state at $\partial\Gamma$. This effect was studied in many papers beginning with V.A. Kondrat'ev's work [Kon1]. This effect is not essential in the proof of the existence theorem for the optimal control problem, but it becomes important in the derivation of the optimality system.

In this paper, we derive the optimality system for an optimal control problem in which all the aforementioned complications take place. In order to focus on the essential aspects, we minimize all other possible difficulties by only considering an optimal control problem for the 2D steady-state Navier-Stokes equations. However we are sure that the results of this paper can be extended to the 3D case as well as to the nonstationary Navier-Stokes equations.

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The authors thank Dominik Meidner for providing the numerical results (see Section 8) by the software package GASCOIGNE [GA].

## 2. Setting of the optimal control problem

Let $\Omega$ be the two-dimensional domain shown in Figure 1, i.e., a rectangle without the set bounded by the curve $S$. We introduce the following notation for parts of the boundary $\partial\Omega$: $AH = \Gamma_{\text{in}}$, $DE = \Gamma_{\text{out}}$, $AB \cup CD \cup FE \cup HG = S'$, $BC = \Gamma_1$, $GF = \Gamma_2$, $\Gamma_1 \cup \Gamma_2 = \Gamma$ and $\partial\Omega = \Gamma_{\text{in}} \cup \Gamma_{\text{out}} \cup \Gamma \cup S \cup S'$. We shall use the abbreviated notation $(\cdot,\cdot) = (\cdot,\cdot)_{L^2(\Omega)}$ for the $L^2$ scalar product over $\Omega$ and $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$ for the associated norm. For subdomains $D \subset \Omega$ and $\Gamma \subset \partial\Omega$, we write $\|\cdot\|_D = \|\cdot\|_{L^2(D)}$ and $\|\cdot\|_\Gamma = \|\cdot\|_{L^2(\Gamma)}$, respectively, and similarly for the corresponding scalar products. We will not distinguish notations of norms and scalar products for scalar functions and corresponding vector fields: this should not lead to misunderstandings.

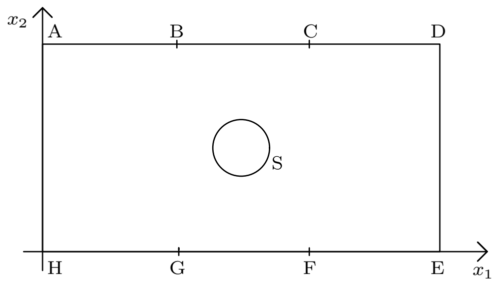


FIGURE 1. Domain

On $\Omega$, we consider the Navier-Stokes equations

$$-\Delta v + v \cdot \nabla v + \nabla p = 0 \quad \text{in } \Omega, \tag{2.1}$$

$$\nabla \cdot v = 0 \quad \text{in } \Omega, \tag{2.2}$$

where $v = (v_1, v_2)$ is the velocity vector field, $\nabla p = (\partial_1 p, \partial_2 p)$ the pressure gradient, $v \cdot \nabla v = \sum_{j=1}^{2} v_j \partial_j v$, and $\nabla \cdot v = \sum_{j=1}^{2} \partial_j v_j$. The system (2.1), (2.2) is supplemented by the boundary conditions

$$v_{|\Gamma_{\text{in}}} = v^{\text{in}}, \quad (\partial_n v - pn)_{|\Gamma_{\text{out}}} = 0, \quad v_{|S \cup S'} = 0, \tag{2.3}$$

where $v^{\text{in}}$ is a given inflow vector field, and $n = n(x)$, $x \in \partial\Omega$, is the outside normal unit vector field to $\partial\Omega$. The goal is to minimize the drag functional of $S$,

$$J = \int_S n \cdot \sigma \cdot e_1 \, dx \quad \rightarrow \quad \inf, \tag{2.4}$$

under the action of a control $u(x_1) = (u^1(x_1), u^2(x_1))$ at the horizontal boundary component $\Gamma = \Gamma_1 \cup \Gamma_2$,

$$(\partial_n v - pn)_{|\Gamma_1} = u^1, \quad (\partial_n v - pn)_{|\Gamma_2} = u^2. \tag{2.5}$$

Here $e_1$ is the unit vector in the $x_1$ direction, and

$$n \cdot \sigma = -pn + 2\mathcal{D}(v)n, \quad 2\mathcal{D}(v) = (\partial_j v_i + \partial_i v_j)_{i,j=1,2} \, . \tag{2.6}$$

This control problem is supplemented by the following additional constraint on the phase variable $v_1(x)$:

$$v_1(x) \geq 0, \quad x \in \omega, \tag{2.7}$$

where $\omega \subset \Omega$ is a prescribed closed subset. Further, we impose the following restriction on the controls $u = (u^1, u^2)$:

$$\|u^1\|_{\Gamma_1}^2 + \|u^2\|_{\Gamma_2}^2 \leq \gamma^2, \tag{2.8}$$

where $\gamma > 0$ is a given constant.

Our goal is to prove an existence theorem for the optimal control problem (2.1)–(2.8) and to derive the corresponding optimality system.

## 3. Boundary value problems

In this section, we prove an existence theorem for several boundary value problems that will be used to prove the existence theorem for the optimal control problem (2.1)–(2.8).

### 3.1. The Stokes boundary value problem

On the domain $\Omega$, we consider the Stokes system

$$-\Delta v + \nabla p = f, \quad \nabla \cdot v = 0, \quad \text{in } \Omega, \tag{3.1}$$

supplemented by the boundary condition (2.3), (2.5). For simplicity let the coordinates of the points $A, B, \ldots, H$ in Figure 1 be as follows:

$$\begin{aligned} A &= (0,\pi), \quad B = (b,\pi), \quad C = (c,\pi), \quad D = (d,\pi), \\ H &= (0,0), \quad G = (b,0), \quad F = (c,0), \quad E = (d,0). \end{aligned} \tag{3.2}$$

We suppose that

$$u^1(x_1) \in L^2(\Gamma_1)^2, \quad u^2(x_1) \in L^2(\Gamma_2)^2, \quad v^{\text{in}} \in H_0^1(\Gamma_{\text{in}})^2 \tag{3.3}$$

where

$$H_0^1(\Gamma_{\text{in}}) = \big\{ w \in L^2(\Gamma_{\text{in}}) \,\big|\, \|\partial_2 w\|_{\Gamma_{\text{in}}} < \infty,\, w(0) = w(\pi) = 0 \big\}. \tag{3.4}$$

It is convenient for us to suppose that in (3.1)

$$f(x) \in L^{3/2}(\Omega)^2. \tag{3.5}$$

In this subsection, we prove an existence and uniqueness theorem for the generalized solution of the boundary value problem (3.1), (2.3), (2.5). To define the notion of "generalized solution", we introduce the space

$$\Phi = \big\{ v \in H^1(\Omega)^2 \,\big|\, \nabla \cdot v = 0,\, v_{|S\cup S'} = 0 \big\}, \tag{3.6}$$

where as above $H^1(\Omega)^2 = H^1(\Omega) \times H^1(\Omega)$ and $H^1(\Omega)$ is the usual Sobolev space over $\Omega$. We recall that for natural $k$ the Sobolev space $H^k(\Omega)$ is defined as follows: $H^k(\Omega) = W_2^k(\Omega)$, and for each integer $k \ge 1$ and $1 \le p < \infty$:

$$W_p^k(\Omega) = \Big\{ \varphi \in L^p(\Omega) \,\Big|\, \|\varphi\|_{W_p^k(\Omega)}^p = \sum_{|\alpha| \le k} \|D^\alpha \varphi\|_{L^p(\Omega)}^p < \infty \Big\},$$

with $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| = \alpha_1 + \alpha_2$, $\alpha_i$ nonnegative integers. For arbitrary $s > 0$ the Sobolev space $H^s(\Omega)$ can be defined by interpolation (see [LM]). Further, we introduce the space

$$\Phi_0 := \big\{ v \in H^1(\Omega)^2 \mid \nabla \cdot v = 0,\, v_{|\Gamma_{\text{in}}} = 0,\, v_{|S\cup S'} = 0 \big\}, \tag{3.7}$$

and supply the spaces $\Phi$ and $\Phi_0$ with the norms

$$\|\varphi\|_\Phi := \|\varphi\|_{H^1(\Omega)}, \quad \|\varphi\|_{\Phi_0} := \|\varphi\|_{H^1(\Omega)}.$$

**Definition 3.1.** Let $u^1 \in L^2(\Gamma_1)^2$, $u^2 \in L^2(\Gamma_2)^2$, $f \in L^{3/2}(\Omega)^2$, and $v^{\text{in}} \in H_0^1(\Gamma_{\text{in}})^2$. The vector function $v \in \Phi$ satisfying $v|_{\Gamma_{\text{in}}} = v^{\text{in}}$ and

$$(\nabla v, \nabla\varphi) - (u^2, \varphi)_{\Gamma_2} - (u^1, \varphi)_{\Gamma_1} = (f, \varphi) \quad \forall\varphi \in \Phi_0, \tag{3.8}$$

is called a "generalized solution" of problem (3.1), (2.3), (2.5).[1]

The following result clarifies the connection between the generalized solution satisfying (3.8) and the solution of problem (3.1), (2.3), (2.5).

**Proposition 3.1.** *Let $v \in \Phi$ be the generalized solution of problem* (3.1), (2.3), (2.5)*. Then, there exists a $p \in L^{3/2}(\Omega)$ such that the pair $(v, p)$ satisfies* (3.1)*. Moreover, if $(v, p) \in W^2_{3/2}(\Omega)^2 \times W^1_{3/2}(\Omega)$, then $p$ is unique, and this pair satisfies* (2.3), (2.5)[2]*.*

*Proof.* Integration by parts in (3.8) with $\varphi \in \Phi_0 \cap C_0^\infty(\Omega)^2$ implies

$$(\Delta v + f, \varphi) = 0 \quad \forall\varphi \in \Phi_0 \cap C_0^\infty(\Omega)^2. \tag{3.9}$$

Then, by the De Rham theorem (see [T]) there exists $p \in L^{3/2}(\Omega)$ such that $(v, p)$ satisfies (3.1) in the distributional sense. Notice that though $\nabla p$ is defined uniquely in (3.1), $p$ is determined only up to a constant. To define it uniquely, we substitute $f = -\Delta v + \nabla p$ in the right-hand side of (3.8) and integrate by parts in this term. As a result, we get

$$\sum_{i=1}^{2} (\partial_n v - pn - u^i, \varphi)_{\Gamma_i} + (\partial_n v - pn, \varphi)_{\Gamma_{\text{out}}} = 0 \quad \forall\varphi \in \Phi_0. \tag{3.10}$$

Equality (3.10) implies that

$$(\partial_n v - pn - u^i + cn)_{|\Gamma_i} = 0, \ i = 1, 2, \quad (\partial_n v - pn + cn)_{|\Gamma_{\text{out}}} = 0, \tag{3.11}$$

where the constant $c$ in all the equalities is the same. We choose the constant component of the pressure $p$ such that $c$ in equations (3.11) becomes zero.[3] □

**Theorem 3.2.** *Let $u^i \in L^2(\Gamma_i)^2$, $i = 1, 2$, $f \in L^{3/2}(\Omega)^2$. Then, there exists a unique generalized solution of problem* (3.1), (2.3), (2.5)*.*

*Proof.* Let us consider the extremal problem

$$J_0(v) := \frac{1}{2}\|\nabla v\|^2 - (f, v) - \sum_{j=1}^{2} (u^j, v)_{\Gamma_j} \ \to \ \inf, \tag{3.12}$$

$$v \in \Phi, \qquad v|_{\Gamma_{\text{in}}} = v^{\text{in}}, \tag{3.13}$$

for $v \in \Phi$ with $v_{|\Gamma_{\text{in}}} = v^{\text{in}}$, where $\Phi$ is defined in (3.6). The functional $J_0(v)$ is convex and continuous on $H^1(\Omega)^2$. Therefore it is semi-continuous on $H^1(\Omega)^2$

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[1]As we will show, a generalized solution exists even under weaker assumptions on $u^1, u^2, v^{\text{in}}$.
[2]Notice that by virtue of the ellipticity of the system (3.1) in the Douglas-Nirenberg sense the inclusion $(v, p) \in W^2_{3/2}(\Omega')^2 \times W^1_{3/2}(\Omega')$ holds for an arbitrary subdomain $\Omega' \Subset \Omega$.
[3]The uniqueness of $p$ without the additional assumption $(v, p) \in W^2_{3/2}(\Omega)^2 \times W^1_{3/2}(\Omega)$ will be proved below in Theorem 4.1.

with respect to the weak convergence in $H^1(\Omega)^2$. Besides, being a closed convex subset of $H^1(\Omega)^2$, the set of restrictions (3.13) is sequentially weakly closed in $H^1(\Omega)^2$. At last, $J_0(v_k) \to \infty$, as $v_k \in \Phi$, $\|v_k\|_{H^1(\Omega)} \to \infty$. Therefore (see [F]) there exists a unique solution $\hat{v} \in \Phi$ of problem (3.12), (3.13). The conditions $\hat{v} \in \Phi$, $\hat{v}+\varphi \in \Phi$, and $\widehat{v}|_{\Gamma_{\rm in}} = (\widehat{v}+\varphi)|_{\Gamma_{\rm in}} = v^{\rm in}$ imply the inclusion $\varphi \in \Phi_0$. Since $\hat{v}$ is a solution of (3.12), (3.13),

$$0 = \lim_{\lambda \to 0} \frac{J_0(\hat{v}+\lambda\varphi) - J_0(\hat{v})}{\lambda} = (\nabla\hat{v}, \nabla\varphi) - (f, \varphi) - \sum_{j=1}^{2} (u^j, \varphi)_{\Gamma_j},$$

for all $\varphi \in \Phi_0$. □

### 3.2. An extension result

We recall a well-known extension result using the notation

$$V^1(\Omega) = \{v \in H^1(\Omega)^2 \,|\, \nabla \cdot v = 0\}, \quad V_0^1(\Omega) = \{v \in V^1(\Omega) \,|\, v_{|\partial\Omega} = 0\}.$$

**Lemma 3.3.** *For each function $g \in H^{1/2}(\partial\Omega)^2$ satisfying*

$$(g, n)_S = 0 \quad \textit{and} \quad (g, n)_{\partial\Omega\setminus S} = 0,$$

*where $n$ is the outer normal to $\partial\Omega$, there exists $u \in V^1(\Omega)$ such that $u_{|\partial\Omega} = g$. Moreover*

$$\inf_{v \in V_0^1} \|u+v\|_{H^1(\Omega)} \le c\|g\|_{H^{1/2}(\partial\Omega)}, \tag{3.14}$$

*where the constant $c$ does not depend on $g$.*

*Proof.* For the proof of this lemma we refer to [GR], [ALT]. □

We introduce the space

$$\Psi^1 := \{v \in H^1(G)^2 \big| \nabla \cdot v = 0,\, v_{|S \cup S' \cup \Gamma_1 \cup \Gamma_2} = 0\}, \tag{3.15}$$

For each $v \in \Psi^1$ only the components $v_{|\Gamma_{\rm in}} = v^{\rm in}$ and $v_{|\Gamma_{\rm out}} = v^{\rm out}$ of the restriction $v_{|\partial\Omega}$ can differ from zero and

$$(v^{\rm in}, n)_{\Gamma_{\rm in}} + (v^{\rm out}, n)_{\Gamma_{\rm out}} = 0. \tag{3.16}$$

We set

$$\widehat{H}^{1/2}(\Gamma_{\rm in} \cup \Gamma_{\rm out}) = \{v^{\rm in} \in H_{00}^{1/2}(\Gamma_{\rm in})^2,\, v^{\rm out} \in H_{00}^{1/2}(\Gamma_{\rm out})^2 \,\big|\, (3.16) \text{ holds}\},$$

where $H_{00}^{1/2}$ is the space defined in [LM], Chapter 1, Theorem 11.7 [4]

**Lemma 3.4.** *There exists a bounded extension operator*

$$E\, : \widehat{H}^{1/2}(\Gamma_{\rm in} \cup \Gamma_{\rm out}) \to \Psi^1,$$

*i.e., the operator satisfying $E(v^{\rm in}, v^{\rm out})_{|\Gamma_{\rm in}} = v^{\rm in}$, $E(v^{\rm in}, v^{\rm out})_{|\Gamma_{\rm out}} = v^{\rm out}$.*

*Proof.* This lemma follows directly from Lemma 3.3. □

[4] Actually, $H_{00}^{1/2}(a,b)$ consists of restrictions on $[a,b]$ of functions from the space $\{f \in H^{1/2}(\mathbb{R}) :$ supp $f \subseteq [a,b]\}$

**Corollary 3.5.** *There exists a bounded extension operator*

$$E : H^1_0(\mathit{\Gamma}_{\rm in}) \to \mathit{\Psi}^1.$$

*Proof.* Since the embedding $H^1_0(\mathit{\Gamma}_{\rm in}) \subset H^{1/2}_{00}(\mathit{\Gamma}_{\rm in})$ is continuous, for each $v^{\rm in} \in H^1_0(\mathit{\Gamma}_{\rm in})$, we have to choose $v^{\rm out} \in H^1_0(\mathit{\Gamma}_{\rm out})$ satisfying (3.16) and to apply Lemma 3.4. $\square$

### 3.3. Estimates for the solution of the Stokes problem

We introduce the solution operator

$$R : L^{3/2}(\mathit{\Omega})^2 \times H^{1/2}_{00}(\mathit{\Gamma}_{\rm in})^2 \times H^{-1/2}_{00}(\mathit{\Gamma}_1)^2 \times H^{-1/2}_{00}(\mathit{\Gamma}_2)^2 \;\to\; \mathit{\Phi} \subset H^1(\mathit{\Omega})^2$$

where $H^{-1/2}_{00} = (H^{1/2}_{00})'$, that maps the data $(f, v^{\rm in}, u^1, u^2)$ to the generalized solution $\hat{v}$ of problem (3.1), (2.3), (2.5), i.e., $R\left(f, v^{\rm in}, u^1, u^2\right)(x) = \hat{v}\,(x)$. (Proof of Theorem 3.2 does not change if data belong to aforementioned spaces.)

**Lemma 3.6.** *The solution operator $R$ is bounded,*

$$\|R(f, v^{\rm in}, u^1, u^2)\|^2_{H^1(\mathit{\Omega})^2} \le c\Big( \|f\|^2_{L^{3/2}(\mathit{\Omega})^2} + \|v^{\rm in}\|^2_{H^{1/2}_{00}(\mathit{\Gamma}_{\rm in})^2} + \sum_{i=1}^{2} \|u^i\|^2_{H^{-1/2}_{00}(\mathit{\Gamma}_i)^2}\Big), \tag{3.17}$$

*where $c > 0$ is independent of the data $(f, v^{\rm in}, u^1, u^2)$.*

*Proof.* By virtue of Lemma 3.4, the following decomposition is true for the solution $\hat{v}(x) = R(f, v^{\rm in}, u^1, u^2)(x):$

$$\hat{v} = Ev^{\rm in} + \hat{\varphi}, \quad \hat{\varphi} = \hat{v} - Ev^{\rm in} \in \mathit{\Phi}_0. \tag{3.18}$$

The equalities (3.18) and (3.8) imply

$$\begin{aligned} \|\nabla\hat{v}\|^2 &= (\nabla\hat{v}, \nabla Ev^{\rm in}) + (\nabla\hat{v}, \nabla\hat{\varphi}) \\ &= (\nabla\hat{v}, \nabla Ev^{\rm in}) + \sum_{i=1}^{2}(u^i, \hat{\varphi})_{\mathit{\Gamma}_i} + (f, \hat{\varphi}). \end{aligned} \tag{3.19}$$

By virtue of Lemma 3.4, we get

$$|(\nabla\hat{v}, \nabla Ev^{\rm in})| \le c\|\nabla\hat{v}\|\|v^{\rm in}\|_{H^{1/2}_{00}(\mathit{\Gamma}_{\rm in})} \le \epsilon\|\nabla\hat{v}\|^2 + \frac{c}{\epsilon}\|v^{\rm in}\|^2_{H^{1/2}_{00}(\mathit{\Gamma}_{\rm in})}. \tag{3.20}$$

By means of the trace theorem and the Poincaré inequality,

$$\begin{aligned} \Big|\sum_{i=1}^{2}(u^i, \hat{\varphi})_{\mathit{\Gamma}_i}\Big| &\le c\Big(\sum_{i=1}^{2} \|u^i\|_{H^{-1/2}_{00}(\mathit{\Gamma}_i)}\Big)\|\nabla\hat{\varphi}\| \\ &\le \frac{c}{\epsilon}\Big(\sum_{i=1}^{2} \|u^i\|^2_{H^{-1/2}_{00}(\mathit{\Gamma}_i)}\Big) + \epsilon\|\nabla\hat{\varphi}\|^2. \end{aligned} \tag{3.21}$$

Using the Sobolev embedding theorem $H^1(\mathit{\Omega}) \subset L^3(\mathit{\Omega})$ and the Poincaré inequality, we get

$$|(f, \hat{\varphi})| \le c\|f\|_{L^{3/2}(\mathit{\Omega})}\|\nabla\hat{\varphi}\| \le \frac{c}{\epsilon}\|f\|^2_{L^{3/2}(\mathit{\Omega})} + \epsilon\|\nabla\hat{\varphi}\|^2. \tag{3.22}$$

At last, (3.18) and Lemma 3.4 imply

$$\|\nabla\hat{\varphi}\|^2 \le c\Big(\|\nabla\hat{v}\|^2 + \|v^{\mathrm{in}}\|^2_{H^{1/2}_{00}(\Gamma_{\mathrm{in}})}\Big). \tag{3.23}$$

After substituting inequalities (3.20)–(3.23) into (3.19), we obtain that

$$\|\nabla\hat{v}\|^2 \le c\Big(\|f\|^2_{L^{3/2}(\Omega)} + \|v^{\mathrm{in}}\|^2_{H^{1/2}_{00}(\Gamma_{\mathrm{in}})} + \sum_{i=1}^{2}\|u^i\|^2_{H^{-1/2}_{00}(\Gamma_i)}\Big).$$

This bound and again the Poincaré inequality imply the asserted estimate (3.17). □

### 3.4. The Navier-Stokes boundary value problem

Now, we consider the Navier-Stokes equations

$$-\Delta v + v\cdot\nabla v + \nabla p = 0, \quad \nabla\cdot v = 0, \quad \text{in } \Omega, \tag{3.24}$$

with boundary conditions (2.3), (2.5).

**Definition 3.2.** Let $u^i \in L^2(\Gamma_i)^2$, $i = 1, 2$, and $v^{\mathrm{in}} \in H^1_0(\Gamma_{\mathrm{in}})^2$. The vector field $v \in \Phi$ is called a "generalized solution" of problem (3.24), (2.3), (2.5) if $v|_{\Gamma_{\mathrm{in}}} = v^{\mathrm{in}}$ and the following equality holds:

$$(\nabla v, \nabla\varphi) + (v\cdot\nabla v, \varphi) - \sum_{i=1}^{2}(u^i, \varphi)_{\Gamma_i} = 0 \quad \forall\varphi \in \Phi_0, \tag{3.25}$$

where $\Phi$ and $\Phi_0$ are defined in (3.6), (3.7).

Our goal is now to prove the following theorem.

**Theorem 3.7.** *Suppose that*

$$\|v^{\mathrm{in}}\|^2_{H^1_0(\Gamma_{\mathrm{in}})} + \sum_{i=1}^{2}\|u^i\|^2_{L^2(\Gamma_i)} \le \epsilon, \tag{3.26}$$

*where* $\epsilon > 0$ *is sufficiently small. Then, there exists a unique generalized solution* $v \in \Phi$ *of problem* (3.24), (2.3), (2.5). *This solution satisfies the inequality*

$$\|v\|^2_{H^1(\Omega)} \le \alpha\Big(\|v^{\mathrm{in}}\|^2_{H^1_0(\Gamma_{\mathrm{in}})} + \sum_{i=1}^{2}\|u^i\|^2_{L^2(\Gamma_i)}\Big), \tag{3.27}$$

*with function* $\alpha(\lambda) = c\left(\lambda^2 + \lambda\right)$.

*Proof.* We look for a generalized solution $v$ of (3.24), (2.3), (2.5) in the form $v = R(f, v^{\mathrm{in}}, u^1, u^2)$ where $R$ is the solution operator of the Stokes boundary value problem, and $f \in L^{3/2}(\Omega)^2$ is an unknown vector field. We substitute $v = R(f, v^{\mathrm{in}}, u^1, u^2)$ into (3.25) and take into account that $v = R(\cdot)$ satisfies (3.8). As a result we get the equation

$$(\varphi, f + R\cdot\nabla R) = 0 \quad \forall\varphi \in \Phi_0. \tag{3.28}$$

Since $H^1(\Omega) \subset L^6(\Omega)$, we get using the Lipschitz inequality:

$$\int_\Omega |(R,\nabla)\, R|^{3/2}\, dx \le \|\, R\,\|_{L^6}^{3/2} \|\, \nabla R\,\|_{L^2}^{3/2}, \le c\|\, \nabla R\,\|_{L^2}^3 \tag{3.29}$$

and therefore $R\cdot\nabla R \in L^{3/2}(\Omega)^2$. We set, for $p>1$,

$$\hat{L}^p\,(\Omega) = \{\, \varphi \in L^p\,(\Omega)^2 \mid \operatorname{div}\varphi = 0,\ \varphi\cdot n\big|_{\Gamma_{\text{in}}\cup S\cup S'} = 0\,\} \tag{3.30}$$

and define the projection operator $P : L^{3/2}\,(\Omega)^2 \to \hat{L}^{3/2}\,(\Omega)$ as follows: for each $f\in L^{3/2}\,(\Omega)^2$ the function $Pf\in\hat{L}^{3/2}(\Omega)$ is defined as the unique solution of the equation

$$(\varphi, Pf) = (\varphi, f) \quad \forall\varphi\in\hat{L}^3(\Omega). \tag{3.31}$$

Since the space $\Phi_0$ defined in (3.7) is dense in $\hat{L}^3(\Omega)$, for each $f\in\hat{L}^{3/2}(\Omega)$ equation (3.28) is equivalent to the equality

$$f + P\big(R(f)\cdot\nabla R(f)\big) = 0. \tag{3.32}$$

We use the notation $R(f) = R(f, v^{\text{in}}, u^1, u^2)$ since $v^{\text{in}}, u^1, u^2$ are given and fixed. To prove the theorem, we have to check that the operator

$$S(f) = -P\big(R(f)\cdot\nabla R(f)\big)\ :\ \hat{L}^{3/2}(\Omega)\to\hat{L}^{3/2}(\Omega) \tag{3.33}$$

is a contraction operator. Using (3.29) and (3.17), we have

$$\begin{aligned}\|S(f_1)-S(f_2)\|_{L^{3/2}} &\le \|(R(f_1-f_2,0,0,0),\nabla)R(f_1)\|_{L^{3/2}}\\ &\qquad + \|(R(f_2),\nabla)R(f_1-f_2,0,0,0)\|_{L^{3/2}}\\ &\le c\|\nabla R(f_1-f_2,0,0,0)\|_{L^2}\big(\|\nabla R(f_1)\|_{L^2}\\ &\qquad + \|\nabla R(f_2)\|_{L^2}\big) \le \hat{c}\|f_1-f_2\|_{L^{3/2}},\end{aligned} \tag{3.34}$$

where, by virtue of (3.17),

$$\begin{aligned}\hat{c} &= c\big(\|\nabla R(f_1)\|_{L^2} + \|\nabla R(f_2)\|_{L^2}\big)\\ &\le 2c\Big(\tfrac{1}{2}\big(\|f_1\|_{L^{3/2}} + \|f_2\|_{L^{3/2}}\big) + \sum_{i=1}^{2}\|u^i\|_{L^2(\Gamma_i)} + \|v^{\text{in}}\|_{H_0^1(\Gamma_{\text{in}})}\Big).\end{aligned} \tag{3.35}$$

By the assumption of the theorem the right-hand side of (3.35) is small enough if $\|f_j\|_{L^{3/2}}$, $j=1,2$ are sufficiently small. Therefore $\hat{c}<1$ and the operator in(3.33) is a contraction. Hence equation (3.32) has a unique solution $f\in\hat{L}^{3/2}(\Omega)$.

As is well known, the solution of (3.32), i.e. of the equation $f = S(f)$, has the form $f = \lim_{k\to\infty} f_k$ where $f_1 = S(0),\ldots,f_k = S(f_{k-1})$. Since $f_k = \sum_{j=1}^{k}(f_j - f_{j-1})$,

$$\begin{aligned}\|f\|_{L^{3/2}} &\le \lim_{k\to\infty}\sum_{j=1}^{k}\|f_j - f_{j-1}\|_{L^{3/2}} \le \sum_{j=1}^{\infty}\hat{c}^j\|S\,(0)\,\|_{L^{3/2}}\\ &\le \frac{\hat{c}}{1-\hat{c}}\|R(0,v^{\text{in}},u^1,u^2)\|_{H^1}^2.\end{aligned} \tag{3.36}$$

This completes the proof. □

## 4. Existence theorem for the optimal control problem

In this section, we prove the existence of the solution for the extremal problem (2.1)–(2.4), (2.7), (2.8). For this, we need a smoothness result for the solution of the Navier-Stokes equations, which we recall in Subsection 4.1.

### 4.1. The smoothness theorem

For small enough $\delta > 0$ denote by $\partial\Omega_\delta$ the curve belonging to $\Omega$ which is the rectangle with sides parallel to the sides $AD$, $EH$, $HA$ of $\partial\Omega$ placed with distance $\delta$ from them and extending up to the side $DE$. Denote by $\Omega_\delta$ the open subset of $\Omega$ with boundary $\partial\Omega_\delta \cup S$. Let $\chi(x) \in C^\infty(\overline{\Omega})$ be a corresponding cut-off function satisfying

$$\chi(x) = \begin{cases} 1, & x \in \Omega_\delta, \\ 0, & x \in \Omega \setminus \Omega_{\delta/2}. \end{cases} \tag{4.1}$$

and near $DE$ $\chi(x_1, x_2) \equiv \chi(x_2)$. The following theorem holds.

**Theorem 4.1.** *Let $v$ be the generalized solution constructed in Theorem* 3.7*. Then, $v \in W^2_{3/2}(\Omega_\delta)^2$ and there exists unique $p \in L^2(\Omega)$ satisfying* (3.24) *and $p \in W^1_{3/2}(\Omega_\delta)$. Moreover*

$$\|v\|_{W^2_{3/2}(\Omega_\delta)} + \|p\|_{W^1_{3/2}(\Omega_\delta)} \le \rho\Big(\|v^{\text{in}}\|_{H^1_0(\Gamma_{\text{in}})} + \sum_{j=1}^{2} \|u^j\|_{L^2(\Gamma_j)}\Big), \tag{4.2}$$

*where $\rho(\lambda)$ is a continuous function for $\lambda > 0$ and $\rho(0) = 0$.*

*Proof.* Since $v$ is a generalized solution of the Navier-Stokes equations, (3.25) implies

$$(-\Delta v + v \cdot \nabla v, \varphi) = 0 \quad \forall \varphi \in \Phi_0 \cap C_0^\infty(\Omega)^2.$$

This equality, identity $v \cdot \nabla v = \sum_{j=1}^2 \partial_j(v_j v)$, inclusions $v_j v \in L^2(\Omega)$, $j = 1, 2$, and the De Rham theorem (see [T]) yield that there exists $p \in L^2(\Omega)$ such that

$$-\Delta v + \nabla p = -v \cdot \nabla v, \quad \nabla \cdot v = 0 \quad \text{in } \Omega. \tag{4.3}$$

Since $v \cdot \nabla v \in L^{3/2}(\Omega)$, and the Stokes system with right-hand side $-v \cdot \nabla v$ and boundary condition $\partial_n v - np = 0$ on $\Gamma_{\text{out}} \cap \Omega_{\delta/4}$ is a Douglas-Nirenberg elliptic system, we get from (4.3) that $v \in W^2_{3/2}(\Omega_{\delta/4})$ and $p \in W^1_{3/2}(\Omega_{\delta/4})$. To prove (4.2), we note that equations (4.3) imply

$$-\Delta(\chi v) + \nabla(\chi p) = g, \quad \nabla \cdot (\chi v) = g_1, \tag{4.4}$$

where

$$g = -\chi v \cdot \nabla v - 2(\nabla\chi \cdot \nabla)v - v\Delta\chi + p\nabla\chi, \quad g_1 = \nabla\chi \cdot v. \tag{4.5}$$

Using (3.27), we obtain that

$$\|g_1\|_{H^1(\Omega)} \le c\|v\|_{H^1(\Omega)} \le c\alpha^{1/2}\Big\{\|v^{\text{in}}\|^2_{H^1_0(\Gamma_{\text{in}})} + \sum_{j=1}^{2} \|u^i\|^2_{L^2(\Gamma_i)}\Big\}, \tag{4.6}$$

and

$$\|g\|_{L^{3/2}(\Omega)} \le c_1\alpha\Big\{\|v^{\rm in}\|^2_{H^1_0(\Gamma_{\rm in})} + \sum_{j=1}^{2}\|u^i\|^2_{L^2(\Gamma_i)}\Big\} + c\|p\nabla\chi\|_{L^2(\Omega)}. \tag{4.7}$$

Below, we will prove that

$$\|p\nabla\chi\|_{L^2(\Omega)} \le c_2\beta\Big\{\|v^{\rm in}\|^2_{H^1_0(\Gamma_{\rm in})} + \sum_{j=1}^{2}\|u^i\|^2_{L^2(\Gamma_i)}\Big\}, \tag{4.8}$$

where the function $\beta(\lambda) > 0$ is continuous, and $\beta(0) = 0$. Let us identify the sides $AD$ and $HE$ of the rectangle $ADEH$ (see FIGURE 1). Then this rectangle turns into a lateral area $LC$ of a cylinder with boundary $\partial LC = \hat\Gamma_{\rm in} \cup \hat\Gamma_{\rm out} \cup S$ where $\hat\Gamma_{\rm in} = \Gamma_{\rm in}$ with points $A$ and $H$ being identified, and $\hat\Gamma_{\rm out} = \Gamma_{\rm out}$ with points $D$ and $E$ being identified. By virtue of the properties of the cut-off function in (4.1), we can consider (4.4) as a system defined on $LC$. Evidently, the pair $(\chi v, \chi p)$ from (4.4) satisfies the following boundary conditions:

$$\chi v|_{\hat\Gamma_{\rm in}} = 0, \quad \chi v|_S = 0, \quad (\partial_1 v(x_1,x_2) - p(x_1,x_2)n)\,\chi(x_2)|_{\hat\Gamma_{\rm out}} = 0. \tag{4.9}$$

Since this boundary value problem is elliptic in the Douglas-Nirenberg sense, inequalities (4.6), (4.7), (4.8), and the evident bound

$$\|v\|_{W^2_{3/2}(\Omega_\delta)} + \|p\|_{W^1_{3/2}(\Omega_\delta)} \le \|\chi v\|_{W^2_{3/2}(LC)} + \|\chi p\|_{W^1_{3/2}(LC)}$$

imply the asserted estimate (4.2).

Let us prove estimate (4.8). The following bound holds (see inequality (6.12) of Chapter 1 in [T]):

$$\|p\partial_j\chi\|_{L^2(\Omega)} \le c\Big\{\Big|\int_\Omega p\partial_j\chi\,dx\Big| + \|\nabla(p\partial_j\chi)\|_{H^{-1}(\Omega)}\Big\}, \quad j = 1,2. \tag{4.10}$$

We estimate the first term in the right side of (4.10). Let $\psi \in C^\infty(\overline{\Omega})$ be a function satisfying $\psi(x)\partial_2\chi = \partial_2\chi$, and $\psi(x) \equiv 0$ outside a small neighborhood of $\operatorname{supp}(\partial_2\chi)$. Then integrating by parts and using (4.3), we get

$$\begin{aligned}\int_\Omega p\partial_2\chi\,dx &= -\int_\Omega \partial_2 p\chi\psi\,dx = \int_\Omega(\Delta v_2 - v\cdot\nabla v_2)\chi\psi\;dx\\ &= \int_{\Gamma_{\rm out}}\partial_1 v_2\chi\psi\,dx - \int_\Omega \nabla v_2\cdot\nabla(\chi\psi)\;dx - \int_\Omega(v\cdot\nabla v_2)\chi\psi\,dx.\end{aligned} \tag{4.11}$$

The boundary condition $(\partial_n v - pn)|_{\Gamma_{\rm out}} = 0$ implies $\partial_1 v_2|_{\Gamma_{\rm out}} = 0$. Therefore estimation of other terms in the right side of (4.11) yields

$$\Big|\int_\Omega p\partial_2\chi\,dx\Big| \le c\big(\|v\|_{H^1(\Omega)} + \|v\|^2_{H^1(\Omega)}\big). \tag{4.12}$$

Since $\operatorname{supp}(\partial_1\chi) \Subset \Omega$, we can choose $\psi \in C_0^\infty(\Omega)$ such that $\psi\partial_1\chi \equiv \partial_1\chi$. Therefore the inequality

$$\Big|\int_\Omega p\partial_1\chi\,dx\Big| \le c\big(\|v\|_{H^1(\Omega)} + \|v\|^2_{H^1(\Omega)}\big) \tag{4.13}$$

can be obtained similarly to (4.11), (4.12), but now without any boundary term. By virtue of (4.3) there holds

$$\begin{aligned}\|\nabla(p\partial_j\chi)\|_{H^{-1}(\Omega)} &\le \|\Delta v\partial_j\chi\|_{H^{-1}(\Omega)} + \|v\cdot\nabla v\partial_j\chi\|_{H^{-1}(\Omega)}\\ &\quad + \|p\nabla\partial_j\chi\|_{H^{-1}(\Omega)}\\ &\le c(\|v\|_{H^1(\Omega)} + \|v\|^2_{H^1(\Omega)}) + \|p\nabla\partial_j\chi\|_{H^{-1}(\Omega)}.\end{aligned} \tag{4.14}$$

Now we estimate the last term on the right side of (4.14). We choose a function $\psi \in C^\infty(\overline{\Omega})$ satisfying $\psi(x)\partial_i\partial_j\chi \equiv \partial_i\partial_j\chi$, $\psi(x) \equiv 0$ outside a small neighborhood of $\mathrm{supp}(\partial_i\partial_j\chi)$. Besides, we take an arbitrary function $\varphi \in W_3^1(\Omega)$ satisfying $\varphi|_{\partial\Omega} = 0$, and set $w := \varphi\psi\partial_i\partial_j\chi$. Then, for any fixed point $x^0 = (x_1^0, x_2^0) \in \Omega\setminus\mathrm{supp}(\psi)$, there holds

$$\begin{aligned}\int_\Omega p\partial_i\partial_j\chi\varphi\,dx &= \int_\Omega\int_{x_1^0}^{x_1}\partial_y p(y,x_2)\,dy w(x)\,dx\\ &= \int_\Omega\int_{x_1^0}^{x_1}\Delta v_1\ dyw\,dx - \int_\Omega\int_{x_1^0}^{x_1} v\cdot\nabla v_1\,dy\,w\,dx\\ &= \int_\Omega\Big(\partial_1 v_1 - \int_{x_1^0}^{x_1} v\cdot\nabla v_1\,dy\Big)w\,dx - \int_\Omega\Big(\int_{x_1^0}^{x_1}\partial_2 v_1\,dy\Big)\partial_2 w\,dx.\end{aligned}$$

Estimating the right side of this equality, we get

$$\Big|\int_\Omega p\partial_i\partial_j\chi\varphi\,dx\Big| \le c\big(\|v\|_{H^1(\Omega)} + \|v\|^2_{H^1(\Omega)}\big)\|\varphi\|_{H^1(\Omega)}. \tag{4.15}$$

Estimates (4.10), (4.12)–(4.15) imply

$$\|p\nabla\chi\|_{L^2(\Omega)} \le c(\|v\|_{H^1(\Omega)} + \|v\|^2_{H^1(\Omega)}). \tag{4.16}$$

Finally, the bound (4.8) follows from (4.16) and (3.27). □

*Remark* 4.1. The generalized solution $v \in H^1(\Omega)$ of problem (3.24), (2.3), (2.5) constructed in Theorem 3.7 together with the function $p \in L^2(\Omega)$ constructed in Theorem 4.1 possess enough smoothness in order to define traces $(\partial_n v - pn)|_{\Gamma_1}$ and $(\partial_n v - pn)|_{\Gamma_2}$. Moreover, the relations (2.5) hold. To prove this assertion, one has to use methods of [LM], [F] Chapter 2.5, and of Theorem 4.1 proved above.

### 4.2. Existence theorem for the extremal problem

In order to prove the existence theorem for problem (2.1)–(2.5), (2.7), (2.8), we have to describe the set of admissible elements for this problem. First of all, for given boundary condition $v^{\mathrm{in}} \in H_0^1(\Gamma_{\mathrm{in}})$ and controls $u^i \in L^2(\Gamma_i)$, $i = 1, 2$, satisfying

$$\sum_{i=1}^2 \|u^i\|^2_{L^2(\Gamma_i)} \le \gamma^2; \quad \gamma^2 + \|v^{\mathrm{in}}\|^2_{H_0^1(\Gamma_{\mathrm{in}})} \le \epsilon, \tag{4.17}$$

where $\epsilon$ is small enough, we have to define uniquely the pair $(v, p)$ that is the solution of the boundary value problem (2.1)–(2.3), (2.5). By virtue of the second condition in (4.17), by Theorem 3.7 there exists a unique generalized solution

$v \in \Phi$ of the Navier-Stokes equation. By virtue of the De Rham Theorem and the argument in the proof of Theorem 4.1 there exists a unique $p \in L^2(\Omega)$ that together with $v$ satisfies equation (4.3). Assuming that $v^{\text{in}} \in H_0^1(\Gamma_{\text{in}})$ is fixed and small enough, we define the map $NR_\delta(u^1, u^2)$ that maps the pair $(u^1, u^2)$ to the corresponding generalized solution $(v, p)$ of problem (2.1)–(2.5),

$$NR_\delta(u^1, u^2) = (v(u^1, u^2), p(u^1, u^2)) \in \Phi \times L^2(\Omega). \tag{4.18}$$

We introduce the following notation:

$$B := \big\{(u^1, u^2) \,\big|\, \|u^1\|^2_{L^2(\Gamma_1)} + \|u^2\|^2_{L^2(\Gamma_2)} \le \gamma^2\big\}, \tag{4.19}$$

$$VP(\Omega) := \big\{(v, p) \in \Phi \times L^2(\Omega) : (v, p)|_{\Omega_\delta} \in W^2_{3/2}(\Omega_\delta)^2 \times W^1_{3/2}(\Omega_\delta)\big\}, \tag{4.20}$$

$$\|(v, p)\|_{VP(\Omega)} := \|v\|_{H^1(\Omega)} + \|\nabla p\|_{H^{-1}(\Omega)} + \|v\|_{W^2_{3/2}(\Omega_\delta)} + \|p\|_{W^1_{3/2}(\Omega_\delta)}. \tag{4.21}$$

**Lemma 4.2.** *Let $\|v^{\text{in}}\|_{H_0^1(\Gamma_{\text{in}})} + \gamma$ be small enough. Then, the mapping*

$$NR_\delta : B \to VP(\Omega) \tag{4.22}$$

*is continuous and its range $NR_\delta(B)$ is a bounded and closed set.*

*Proof.* Using the estimate (3.27) and expressing $\nabla p$ by (3.24) with the following application of (3.27), (3.29), we get the inequality

$$\|v\|_{H^1(\Omega)} + \|\nabla p\|_{H^{-1}(\Omega)} \le c\Big(\|v^{\text{in}}\|_{H_0^1(\Gamma_{\text{in}})} + \sum_{j=1}^{2} \|u^j\|_{L^2(\Gamma_j)}\Big), \tag{4.23}$$

where $c(\lambda) = c_1(\lambda + \lambda^2)$. The bounds (4.23), (4.2) imply the boundedness of the operator $NR_\delta(B)$ in (4.22). Let us prove the closedness of $NR_\delta(B)$. We only prove the closedness of $NR_\delta(B)|_{\Omega_\delta}$ in $W^2_{3/2}(\Omega_\delta)^2 \times W^1_{3/2}(\Omega_\delta)$ because the closedness of $NR_\delta(B)$ in $\Phi \times L^2(\Omega)$ can be established in the same way. Let $(v_k, p_k) \in NR_\delta(B)$ with

$$(v_k, p_k)|_{\Omega_\delta} \to (\hat{v}, \hat{p})|_{\Omega_\delta} \quad \text{in } W^2_{3/2}(\Omega_\delta)^2 \times W^1_{3/2}(\Omega_\delta) \quad (k \to \infty). \tag{4.24}$$

Inclusion $(v_k, p_k) \in NR_\delta(B)$ implies relation $(v_k, p_k) = NR_\delta(u_k^1, u_k^2)$ for some $(u_k^1, u_k^2) \in B$. Since $B$ is a bounded set, passing if necessary to a subsequence, we can assume that $(u_k^1, u_k^2) \rightharpoonup (\hat{u}^1, \hat{u}^2)$ weakly in $L^2(\Gamma_1) \times L^2(\Gamma_2)$. Hence, $(\hat{u}^1, \hat{u}^2) \in B$ because the set $B$ is convex. Now, we substitute $(v_k, u_k^1, u_k^2)$ into (3.25). Evidently one can pass to the limit in (3.25). Since $\nabla p_k = \Delta v_k - v_k \cdot \nabla v_k$, then, $\nabla p_k \to \nabla \hat{p}$ weakly in the space $\nabla W^1_{3/2}(\Omega_\delta) = \{\nabla p \,|\, p \in W^1_{3/2}(\Omega_\delta)\}$. By virtue of (4.2) the functions $p_k$ are bounded with respect to $k$. That is why, passing if necessary to a subsequence, we get that $p_k \to \hat{p}$ in $W^1_{3/2}(\Omega_\delta)$ weakly. So $(\hat{v}, \hat{p}) = NR_\delta(\hat{u}^1, \hat{u}^2)$ and the set $NR_\delta(B)$ is closed in $W^2_{3/2}(\Omega_\delta) \times W^1_{3/2}(\Omega_\delta)$. $\square$

Since $\omega \subset \Omega$ in (2.7) is a given closed subset of domain $\Omega$, there exists $\delta > 0$ so small that

$$\omega \subset \Omega_\delta \subset \Omega. \tag{4.25}$$

We choose $\delta > 0$ such that (4.25) holds and from now on assume it as fixed.

**Definition 4.1.** Let $v^{\text{in}} \in H_0^1(\Gamma_{\text{in}})$ be fixed. The collection $(v, p, u^1, u^2) \in VP(\Omega) \times B$ is called "admissible" for problem (2.1)–(2.5), (2.7), (2.8) if $(v, p) = NR_\delta(u^1, u^2)$ and inequality (2.7) is fulfilled.

Notice that the equality $(v, p) = NR_\delta(u^1, u^2)$ means that $(v, p)$ is a generalized solution of the boundary value problem (2.1)–(2.3), (2.5). Besides, the integral in (2.4) is well defined because by virtue of the inclusion $(v, p) \in W^2_{3/2}(\Omega_\delta)^2 \times W^1_{3/2}(\Omega_\delta)$ all traces used in (2.4) are well defined. An inequality for each $x \in \omega$, as in (2.7), is well defined for $(v, p) = NR_\delta(u^1, u^2)$ because such $v = (v_1, v_2)$ belong to $W^2_{3/2}(\Omega_\delta)$ that is embedded into $C(\overline{\Omega}_\delta)$ by the Sobolev embedding theorem. The set of all admissible collections, i.e., the admissible set for the extremal problem (2.1)–(2.5), (2.7), (2.8), is denoted by $\mathfrak{A}$. We impose the following important condition.

*Condition* 1. The admissible set of the extremal problem (2.1)–(2.5), (2.7), (2.8) is not empty,

$$\mathfrak{A} \neq \emptyset. \tag{4.26}$$

*Remark* 4.2. The situation with Condition 1 is not trivial at all. Calculations show that this condition is fulfilled rather often. Indeed, if $\|v^{\text{in}}\|_{\widehat{H}_0^1}$ and $\gamma^2$ in (2.8) are sufficiently small, and the part $S$ of the boundary is convex, then, as numerical calculations show (see Section 8), $v_1 \geq 0$ on certain subdomains $\omega \subset \Omega$ (see also [VD]). Moreover, the calculated steady flow is stable and therefore this is the case of small Reynolds number which we consider in this paper.

Recall that by definition the collection $(\hat{v}, \hat{p}, \hat{u}^1, \hat{u}^2)$ is the solution of problem (2.1)–(2.5), (2.7), (2.8) if $(\hat{v}, \hat{p}, \hat{u}^1, \hat{u}^2) \in \mathfrak{A}$ and

$$J(\hat{v}, \hat{p}, \hat{u}^1, \hat{u}^2) = \inf_{(v,p,u^1,u^2) \in \mathfrak{A}} J(v, p, u^1, u^2), \tag{4.27}$$

where $J(v, p, u^1, u^2)$ is the functional in (2.4), (2.6). Notice that the dependence of $J$ on $(u^1, u^2)$ is implicit and connected with the domain $\mathfrak{A}$ for the functional $J$.

**Theorem 4.3.** *If* $\|v^{in}\|_{H_0^1(\Gamma_{in})}$ *and* $\gamma^2$ *in* (2.8) *are small enough, then there exists a solution* $(\hat{v}, \hat{p}, \hat{u}^1, \hat{u}^2)$ *of problem* (2.1)–(2.8).

*Proof.* i) First, we prove that the projection $\Pi\mathfrak{A}$ of the admissible set $\mathfrak{A} \subset PV_1(\Omega) \times B$ into $W^2_{3/2}(\Omega_\delta{}^2) \times W^1_{3/2}(\Omega_\delta)$ is closed in this space. Since

$$\mathfrak{A} = \big\{(v, p, u^1, u^2) \,\big|\, (v, p) = NR_\delta(u^1, u^2),\ v_1(x) \geq 0,\ x \in \omega\big\},$$

by virtue of Lemma 4.2, it is enough to prove that if $(v_k, p_k) \to (\hat{v}, \hat{p})$ as $k \to \infty$ in $W^2_{3/2}(\Omega_\delta)^2 \times W^1_{3/2}(\Omega_\delta)$ and $v_1^k \geq 0$ on $\omega$ for each $k$, then $\hat{v}_1 \geq 0$ on $\omega$. But this assertion immediately follows from the embedding $W^2_{3/2}(\Omega_\delta) \subset C(\overline{\Omega}_\delta)$ and the inclusion $\omega \subset \Omega_\delta$.

ii) Next, we consider the direct product of the Besov spaces

$$W^{11/6}_{3/2}(\Omega_\delta)^2 \times W^{5/6}_{3/2}(\Omega_\delta) \quad ^5$$

and introduce the trace operator $\hat{\gamma}_S(v,p) = n\cdot\sigma_{|S} := (-np+2\mathcal{D}(v)n)_{|S}$ (see (2.4), (2.6)). Then, the well-known Besov theorem ([BIN]) implies that the operator

$$\hat{\gamma}_S : W^{11/6}_{3/2}(\Omega_\delta)^2 \times W^{5/6}_{3/2}(\Omega_\delta) \to W^{1/6}_{3/2}(S)^2 \tag{4.28}$$

is continuous. Since the embedding $W^{1/6}_{3/2}(S) \subset L^1(S)$ is continuous, the functional in (2.4),

$$J(v,p) = \int_S n\cdot\sigma\cdot e_1\,ds = \int_S \hat{\gamma}_S(v,p)\cdot e_1\,ds, \tag{4.29}$$

is continuous on the space $W^{11/6}_{3/2}(\Omega_\delta)^2 \times W^{5/6}_{3/2}(\Omega_\delta)$. As is well known, the embedding $W^{2}_{3/2}(\Omega_\delta)^2 \times W^{1}_{3/2}(\Omega_\delta) \subset W^{11/6}_{3/2}(\Omega_\delta)^2 \times W^{5/6}_{3/2}(\Omega_\delta)$ is compact. Therefore, by virtue of part i) of this proof, the set $\Pi\mathfrak{A}$ is a compact subset of the space $W^{11/6}_{3/2}(\Omega_\delta)^2 \times W^{5/6}_{3/2}(\Omega_\delta)$. Evidently the extremal problem (2.1)–(2.8) is equivalent to the problem

$$J = \int_S \hat{\gamma}_S(v,p)\cdot e_1\,ds \to \inf, \quad (v,p)\in\Pi\mathfrak{A}. \tag{4.30}$$

Problem (4.30) is a minimization problem for a continuous function on a compact set. Therefore it possesses a solution, which completes the proof. □

## 5. Abstract Lagrange principle

To derive the optimality system for problem (2.1)–(2.8), we use the abstract Lagrange principle. For problems without phase constraints one can recall the Lagrange principle from [ATF, F]. The essential peculiarity of the extremal problem studied here is just the phase constraint (2.7). For such extremal problems the Lagrange principle has been established as well [DM, G, MDO]. We recall some abstract notion (for details we refer to [MDO]).

### 5.1. Sub-linear functionals

Let $Y$ be a Banach space. A functional $\varphi : Y \to \mathbb{R}$ is called "sub-linear" if it satisfies

a) $\varphi(\lambda y) = \lambda\varphi(y)$, $\forall y\in Y$, $\lambda > 0$ (positive homogeneity),
b) $\varphi(x+y) \le \varphi(x)+\varphi(y)$ (subadditivity).

Notice that for a functional satisfying a) condition b) is equivalent to a convexity condition. The sub-linear functional $\varphi$ is called "bounded" if there exists a constant $c>0$ such that

c) $|\varphi(y)| \le c\|y\| \quad \forall y\in Y$.

[5]When the upper index is not integer, the Besov space coincides with the corresponding Sobolev space. Therefore, we use the notation of Sobolev spaces. We use the Besov spaces because the trace theorem is not always true for Sobolev spaces.

**Lemma 5.1.** *If the sub-linear functional* $\varphi$ *satisfies* $\varphi(y) \le c\|y\| \; \forall y \in Y$ *with a certain* $c > 0$*, then* $|\varphi(y)| \le c\|y\|$ *and* $|\varphi(y_1) - \varphi(y_2)| \le c\|y_1 - y_2\|$*, i.e.,* $\varphi$ *is a Lipschitz functional with the same constant* $c$*.*

*Proof.* The proof can be found in [MDO], p. 75. □

A linear functional $l \in Y^*$ is called "supported by a sub-linear functional $\varphi(y)$" if $l(y) \le \varphi(y) \; \forall y \in Y$. The set of all functionals supported by $\varphi$ is called a "subdifferential of $\varphi$" (at zero) and is denoted by $\partial\varphi$ (at zero). If $\varphi$ is a bounded sub-linear functional, then $\partial\varphi$ is a non-empty convex closed set and $\forall l \in \partial\varphi : \|l\| \le c$. Let $f : Y \to \mathbb{R}$ be a functional. If for $y_0, y_1 \in Y$ there exists the limit

$$f'(y_0, y_1) := \lim_{\lambda \to 0} \frac{f(y_0 + \lambda y_1) - f(y_0)}{\lambda},$$

then $f'(y_0, y_1)$ is called the "derivative" of $f$ at the point $y_0$ in the direction $y_1$.

## 5.2. Formulation of the Lagrange principle

Consider an abstract extremal problem of the form of problem (2.1)–(2.8),

$$f_0(y) \to \inf, \quad F(y) = 0, \quad f_1(y) \le 0, \quad G(y) \le 0, \tag{5.1}$$

where $f_i : Y \to \mathbb{R}$, $i = 0, 1$, $G : Y \to \mathbb{R}$ are functionals defined on a Banach space $Y$, and $F : Y \to Z$ is a map to another Banach space $Z$. Suppose that there exists a solution $\hat{y} \in Y$ of problem (5.1) and the mappings $f_i : Y \to \mathbb{R}$, $i = 0, 1$, and $F : Y \to Z$ are continuously differentiable in a neighborhood of $\hat{y}$. Assume also that:

a) The image $F'(\hat{y})Y$ of $Y$ is closed in $Z$.

b) $G(y)$ possesses a derivative $G'(\hat{y}, y_1)$ at $\hat{y}$ in each direction $y_1 \in Y$, and the map $Y \ni y \to G'(\hat{y}, y)$ is a bounded sub-linear functional on $Y$.

The Lagrange function for problem (5.1) has the form

$$\mathcal{L}(\hat{y}, \lambda_0, \lambda_1, z^*, \alpha) = \sum_{j=0}^{1} \lambda_j f_j(\hat{y}) + \langle F(\hat{y}), z^* \rangle + \alpha G(\hat{y}). \tag{5.2}$$

The following theorem holds (see [DM, G, MDO]).

**Theorem 5.2.** *Let the conditions formulated above be fulfilled. Then, there exist Lagrange multipliers* $(\lambda_0, \lambda_1, z^*, \alpha) \in \mathbb{R}^2 \times Z^* \times \mathbb{R}$ *satisfying the following conditions.*

i) *Non-triviality condition:*

$$|\lambda_0| + |\lambda_1| + \|z^*\| + |\alpha| > 0. \tag{5.3}$$

ii) *Condition of sign concordance:*

$$\lambda_0 \ge 0, \quad \lambda_1 \ge 0, \quad \alpha \ge 0. \tag{5.4}$$

iii) *Condition of complementary slackness:*

$$\lambda_i f_i(\hat{y}) = 0,\ i = 0, 1; \quad \alpha G(\hat{y}) = 0. \tag{5.5}$$

iv) *Euler-Lagrange equation: there exists* $\mu^* \in \partial G'(\hat{y}, \cdot)$ *such that*

$$\begin{aligned} \big\langle \mathcal{L}'_y(\hat{y}, \lambda_0, \lambda_1, z^*, \alpha, \mu^*), h \big\rangle &= \sum_{j=0}^{1} \lambda_j \big\langle f'_j(\hat{y}), h \big\rangle + \big\langle F'(\hat{y}), h \big\rangle + \alpha \big\langle \mu^*, h \big\rangle \\ &= 0, \end{aligned} \tag{5.6}$$

*for all* $h \in Y$.

In the remainder of this section, we briefly recall some properties of concrete functionals used for defining the phase constraints. Details can be found in [MDO]. These properties will be used in the derivation of the optimality system for the extremal problem (2.1)–(2.8).

### 5.3. The functional $\max_{x \in M} y(x)$ and its support functionals

Let $M \subset \Omega$ be an arbitrary closed subset. We consider the functional $\Theta : C(\overline{\Omega}) \to \mathbb{R}$:

$$\Theta(y) = \max_{x \in M} y(x), \quad y \in C(\overline{\Omega}), \tag{5.7}$$

where $C(\overline{\Omega})$ is the space of continuous functions defined on $\overline{\Omega}$. This is a Lipschitz functional with constant 1 because

$$|\Theta(y_1) - \Theta(y_2)| \le \max_{x \in M} |y_1(x) - y_2(x)| \le \|y_1 - y_2\|_{C(\overline{\Omega})}.$$

By the Riesz Theorem $C(\overline{\Omega})^*$ consists of functionals of the form

$$\lambda(y) = \int_\Omega y(x)\, \mu(dx), \tag{5.8}$$

where $\mu(dx)$ is a measure that can have positive as well as negative values. Evidently the functional in (5.7) is sub-linear (see subsection 5.1).

**Lemma 5.3.** *The functional* $\lambda(y) = \int_\Omega y(x)\, \mu(dx)$ *from* $C(\overline{\Omega})^*$ *is supported by the functional* $\Theta(y)$ *from* (5.7) *if and only if* $\mu(dx)$ *satisfies:*

1) $\mu(dx)$ *is supported on* $M$, *i.e.,* $\forall\, y \in C(\overline{\Omega}) : y(x) = 0,$ *for* $x \in M$, *we have* $\lambda(y) = \int_\Omega y(x)\, \mu(dx) = 0$.
2) $\mu(dx) \ge 0$.
3) $\int_\Omega \mu(dx) = 1$.

*Proof.* For the proof see [MDO], p. 95. □

### 5.4. Directional derivatives of $\max y(x)$

Let $y_0 \in C(\overline{\Omega})$, $y_1 \in C(\overline{\Omega})$ and $\Theta(y)$ be the functional in (5.7). We calculate the derivative $\Theta'(y_0, y_1)$ in the direction $y_1$. Without loss of generality, we suppose that $\Theta(y_0) = 0$. Then, the closed set

$$M_0 = \{\, x \in M \mid y_0(x) = 0 \,\}$$

is not empty.

**Lemma 5.4.** *For all $y_1 \in C(\overline{\Omega})$ the functional in* (5.7) *possesses a derivative in the direction $y_1$ at $y_0(x) \in C(\overline{\Omega})$, which is defined by the equality*

$$\Theta'(y_0, y_1) = \max_{x \in M_0} y_1(x). \tag{5.9}$$

Evidently (5.9) defines a sub-linear functional. By Lemma 5.3 the set $\partial\Theta'(y_0, y_1)$ of linear functionals supported by $\Theta'(y_0, y_1)$ consists of all probability measures $\mu(dx)$ concentrated (supported) on the set $M_0$.

### 5.5. Directional derivative of the functional $\max \Phi(x, y)$

Let $\Phi(x, y) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. On $C(\overline{\Omega})$, we consider the functional

$$G(y) = \max_{x \in M} \Phi(x, y(x)), \tag{5.10}$$

where $M$ is a closed subset of $\Omega$. Let $y_0 \in C(\overline{\Omega})$ be such that $G(y_0) = 0$ and

$$M_0 := \left\{ x \in \overline{\Omega} \,\middle|\, \Phi(x, y_0(x)) \right\} = 0. \tag{5.11}$$

Evidently $G(y) = \Theta(N(y))$ where $\Theta$ is the functional in (5.7) and

$$N \,:\, C(\overline{\Omega}) \to C(\overline{\Omega}) \quad y \to \Phi(x, y(x))$$

is a Nemytskiy operator. Since $N$ is differentiable in the Fréchet sense and $N'(y)h = \Phi'_y(x, y(x))h(x)$ (see [ATF]), and $\Theta$ possesses a derivative at $N(y_0)$ in an arbitrary direction $y_1 \in C(\overline{\Omega})$, then, by the theorem on the derivative of the superposition of functions the functional $G(y) = \Theta(N(y))$ at $y_0$ also possesses a derivative in an arbitrary direction $y_1$ that is defined by the equality

$$G'(y_0, y_1) = \max_{x \in M_0} \Big( \Phi'_y(x, y_0(x)) y_1(x) \Big). \tag{5.12}$$

Evidently, the functional in (5.12) is sub-linear in $y_1$.

**Lemma 5.5.** *The set of linear functionals supported by a sub-linear functional $y_1 \to G'(y_0, y_1)$ consists of the functionals $l \in C(\overline{\Omega})^*$, which have the representation*

$$l(y_1) = \int_{\Omega} \Phi'_y\big(x, y_0(x)\big) y_1(x)\, \mu(dx), \tag{5.13}$$

*where $\mu(dx)$ is a probability measure concentrated on the set $M_0$* (*see* (5.11)).

## 6. Application of the abstract Lagrange principle

After some preliminaries related to checking the condition a) in Subsection 5.2, we check that all conditions of the Lagrange principle are satisfied for problem (2.1)–(2.8) and apply the Lagrange principle to this situation.

### 6.1. On the smoothness of solutions for the Oseen problem

Let $(\hat{v}, \hat{p}) \in \Phi \times L^2(\Omega)$ be the solution of the extremal problem (2.1)–(2.8) constructed in Section 4. We consider the Oseen problem, i.e., the linearization of the Navier-Stokes problem at $(\hat{v}, \hat{p})$:

$$-\Delta v + \hat{v} \cdot \nabla v + v \cdot \nabla \hat{v} + \nabla p = f, \quad \nabla \cdot v = 0 \quad \text{in } \Omega, \tag{6.1}$$

$$v|_{\Gamma_0} = 0, \quad (\partial_n v - pn)|_{\Gamma} = g, \tag{6.2}$$

where $\Omega$ is the domain introduced in Section 2. We have

$$\Gamma_0 = \Gamma_{\text{in}} \cup S' \cup S \cup A \cup H, \quad \Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_{\text{out}}, \tag{6.3}$$

and therefore $g$ consists of three components $g_i$ on $\Gamma_i$, $i = 1, 2$, and $g_{\text{out}}$ on $\Gamma_{\text{out}}$. We suppose that $g_{\text{out}} \equiv 0$. Since $\|\hat{v}\|_{H^1}$ is small enough one can prove, as in Section 3, that for each $f \in L^2(\Omega)$, $g \in H^{1/2}(\Gamma)$ there exists a unique generalized solution $v \in \Phi_0$ of problem (6.1), (6.2) where $\Phi_0$ is the space in (3.7). Since problem (6.1), (6.2) is elliptic in the Agmon-Douglas sense, the solution possesses additional smoothness: in each subdomain $\Omega_0$ of $\Omega$, such that $\overline{\Omega_0} \subset \Omega$, $v \in H^2(\Omega_0)$ and the pressure $p$ exists and belongs to $H^1(\Omega_0)$. Moreover $(v, p)$ are smooth up to $\partial\Omega$ except at the corner points $A$, $D$, $E$, $H$ and the points $B$, $C$, $F$, $G$: if $B(\epsilon)$ is the union of the circles with radius $\epsilon$ centered at the indicated points, then $v \in H^2(\Omega \setminus B(\epsilon))$, $p \in H^1(\Omega \setminus B(\epsilon))$ for each $\epsilon > 0$. Actually, the solution $(v, p)$ is smooth in a neighborhood of the corner points $A$, $D$, $E$, $H$, as well.

**Lemma 6.1.** *Let $B_1(\epsilon)$ be the union of the circles with radius $\epsilon$ and centers at $A, D, E, H$. Then, $(v, p) \in H^2(\Omega \cap B_1(\epsilon)) \times H^1(\Omega \cap B_1(\epsilon))$.*

*Proof.* Recall that the point $H$ is the origin, the interval $(A, H)$ belongs to the axis $x_2$ with $x_2(A) > 0$, and the interval $(EH)$ belongs to the axis $x_1$ with $x_1(H) > 0$. Consider a neighborhood of the point $H$ and extend the solution $(v, p)$ on the domain $\{x_1 < 0,\, x_2 > 0\}$ in the odd sense:

$$\begin{aligned} \text{for } x_1 < 0,\, x_2 > 0: \quad & v(x_1, x_2) = -v(-x_1, x_2),\, p(x_1, x_2) = -p(-x_1, x_2), \\ & \hat{v}(x_1, x_2) = -\hat{v}(-x_1, x_2),\, f(x_1, x_2) = -f(-x_1, x_2). \end{aligned}$$

It is easy to check that this extension satisfies (6.1) not only for the set $\{x_1 < 0,\, x_2 > 0\}$ but for $\{|x_1| < \epsilon,\, x_2 > 0\}$, as well. Since the boundary of the extended domain is smooth in a neighborhood of $H$, the extended pair $(v, p)$ belongs to $H^2 \times H^1$ in a neighborhood of $H$. Near the point $E$, we use the same arguments but apply the extension on the domain $\{x_2 < 0\}$. Our arguments near the points $A$, $D$ are analogous: additionally we need only to do appropriate changing of the variables $(x_1, x_2)$. □

The situation of the smoothness of the solution $(v, p)$ in a neighborhood of the points $B$, $C$, $F$, $G$ where the type of boundary condition changes is different. At these points the solution $(v, p)$ can possess a singularity. Therefore there is a reason to study problem (6.1), (6.2) in function spaces with weights near these points.

Let this weight be defined as a function $\rho(x_1, x_2) \in C^\infty(\Omega)$, $\rho(x_1, x_2) > 0, \forall(x_1, x_2) \in \overline{\Omega} \setminus B_2(\epsilon)$ for a certain $\epsilon > 0$, where $B_2(\epsilon)$ is the union of the circles with radius $\epsilon$ and centers at $B$, $C$, $F$, $G$, and in $\Omega \cap B_2(\epsilon)$, the weight $\rho(x_1, x_2)$ is equal to the distance of the closest points among $B$, $C$, $F$, $G$.

We introduce the following Sobolev spaces with weights. Let $k$ be a natural number or zero and $\alpha \in \mathbb{R}$. Then,

$$H^k_\alpha(\Omega) := \Big\{ u(x),\, x \in \Omega \,\Big|\, \|u\|^2_{H^k_\alpha} = \sum_{j=0}^k \int_\Omega \rho^{2(\alpha+j)}(x) \sum_{|\beta|=j} |D^\beta u(x)|^2 \, dx < \infty \Big\},$$

where $\beta = (\beta_1, \beta_2)$, $\beta_i \geq 0$, are integer, $|\beta| := \beta_1 + \beta_2$. For $k \geq 1$, let

$$\Psi^k_\alpha(\Omega) = \big\{ v \in H^k_\alpha(\Omega)^2 \,|\, \nabla \cdot v = 0,\, v|_{\Gamma_0} = 0 \big\}. \tag{6.4}$$

We need also the spaces $H^k_\alpha(B, C)$, $H^k_\alpha(G, F)$ of functions defined on intervals $(B, C)$ or $(G, F)$ with non-integer $k$. For this, we first define the space $H^k_\alpha(\mathbb{R}_+)$. Using the sign $\sim$ for notation of norm equivalence, we get for integers $k$ and $\alpha \in \mathbb{R}$:

$$\begin{aligned} \|u\|^2_{H^k_\alpha(\mathbb{R}_+)} &= \int_0^\infty \sum_{j=0}^k x^{2(\alpha+j)} |\partial^j_x u\,(x)\,|^2 \, dx \\ &\approx \int_0^\infty \sum_{j=0}^k \big| (x \cdot \partial_x)^j (x^\alpha u(x)) \big|^2 \, dx \\ &\approx \int_{-\infty}^\infty \sum_{j=0}^k \big| \partial^j_t (e^{(\alpha+1/2)t}) u(e^t)) \big|^2 \, dt, \end{aligned} \tag{6.5}$$

where in the last step, we made the change of variable $x := e^t$. Applying the Mellin transform

$$\hat{u}(\xi) = \int_0^\infty u(x) e^{i\xi \ln x} \frac{dx}{x} = \int_{-\infty}^\infty u\,(e^t)\, e^{it\xi} \, dt,$$

(which in fact is the Fourier transform of $u(e^t)$), to the function $x^\alpha u(x)$ and taking into account the Plancherel theorem, we get

$$\begin{aligned} &\int_{-\infty}^\infty \sum_{j=0}^k \big| \partial^j_t (e^{(\alpha+1/2)t} u(e^t)) \big|^2 \, dt \\ &\qquad = \frac{1}{2\pi} \int_{-\infty}^\infty \sum_{j=1}^k |\xi + i(\alpha + 1/2)|^{2j} |\hat{u}(\xi + i(\alpha + 1/2))|^2 \, d\xi. \end{aligned} \tag{6.6}$$

By virtue of (6.5), (6.6), we can introduce the following equivalent norm for $H^k_\alpha(\mathbb{R}_+)$:

$$\|u\|^2_{H^k_\alpha(\mathbb{R}_+)} := \int_{-\infty}^{\infty} (1+|\xi+i(\alpha+1/2)|^2)k|\hat{u}(\xi+i(\alpha+1/2))|^2\, d\xi. \tag{6.7}$$

But the norm in (6.7) is well defined for arbitrary $k \in \mathbb{R}$. To define $H^k_\alpha(a,b)$ with $k \in \mathbb{R}$, $\alpha \in R$, we define a decomposition of unity, i.e., $\varphi_i(x) \in C^\infty(a,b)$, $i=1,2$, $\varphi_i(x) \geq 0$, $\varphi_1(x)+\varphi_2(x) \equiv 1$, $\varphi_1(x)=1$ for $x$ close to $a$, and $\varphi_2(x)=1$ for $x$ close to $b$. Then, by definition,

$$\|u\|^2_{H^k_\alpha(a,b)} = \|\widetilde{\varphi_1 u}\|^2_{H^k_\alpha(\mathbb{R}_+)} + \|\widetilde{\varphi_2 u}\|^2_{H^k_\alpha(\mathbb{R}_+)}, \tag{6.8}$$

where by definition $\widetilde{\varphi_1 u}(y) = (\varphi_1 u)(a+y)$, $y \in \mathbb{R}_+$, and $\widetilde{\varphi_2 u}(y) = (\varphi_2 u)(b-y)$, $y \in \mathbb{R}_+$.

Now, we are in the position to formulate the main theorem of this subsection.

**Theorem 6.2.** *Let* $\|\hat{v}\|_\Phi$ *be small enough (i.e., there exists a unique generalized solution of* (6.1), (6.2)*). Then, there exists a discrete set* $\{\alpha_i\} = \mathfrak{a} \subset \mathbb{R}$ *such that for each* $\alpha \notin \mathfrak{a}$ *and for every* $f \in H^0_\alpha(\Omega)^2$, $g \in H^{1/2}_\alpha(\Gamma)^2$ *(we suppose that* $g^{out} \equiv 0$*) there exists a unique solution* $(v,p) \in \Psi^2_\alpha(\Omega) \times H^1_\alpha(\Omega)$ *of problem* (6.1), (6.2)*, and the following a priori estimates hold true:*

$$\|v\|^2_{\Psi^2_\alpha} + \|p\|^2_{H^1_\alpha(\Omega)} \leq c\big(\|f\|^2_{H^0_\alpha(\Omega)^2} + \|g\|^2_{H^{1/2}_\alpha(\Gamma)}\big). \tag{6.9}$$

*Proof.* This theorem can be proved using the Mellin transform method of Kondrat'ev [Kon1, Kon2] (see also [BR]). □

*Remark* 6.1. The considerations of this subsection can be extended to the case when the assumption that the solution $(\hat{v},\hat{p})$ of the extremal problem (2.1)–(2.8) has a sufficiently small norm is not fulfilled. In this case for $\alpha \notin \mathfrak{a}$ one can prove an analog of Theorem 6.2 in which the solvability of (6.1), (6.2) is true for $(f,g) \in \mathcal{F}$ where $\mathcal{F}$ is a subspace of $H^0_\alpha(\Omega)^2 \times H^{1/2}_\alpha(\Gamma)^2$ of finite codimension. This assertion is sufficient for the application of the Lagrange principle.

### 6.2. First reduction of the problem

First of all, in problem (2.1)–(2.8), we remove the unknown functions $u_1$, $u_2$ (controls) together with relations (2.5) and change condition (2.8) to

$$\sum_{i=1}^{2} \|\partial_n v - pn\|^2_{\Gamma_i} \leq \gamma^2. \tag{6.10}$$

Evidently, the new problem is equivalent to the old one. In problem (2.1)–(2.3), (2.4), (2.7), (6.10), we make the following change of the dependent variables:

$$v = w + \hat{v}, \quad p = t + \hat{p}, \tag{6.11}$$

where $(\hat{v}, \hat{p})$ is a solution of the original extremal problem (2.1)–(2.8). As a result, we obtain the following extremal problem: Minimize the functional

$$J_0(w,t) = \int_S (\partial_n w - tn) \cdot e_1 \, ds \to \inf \tag{6.12}$$

on the set of pairs $(w,t)$ satisfying

$$-\Delta w + \hat{v} \cdot \nabla w + w \cdot \nabla \hat{v} + w \cdot \nabla w + \nabla t = 0, \quad \nabla \cdot w = 0 \quad \text{in } \Omega, \tag{6.13}$$

$$w|_{\Gamma_0} = 0, \quad (\partial_n w - tn)|_{\Gamma_{\text{out}}} = 0, \tag{6.14}$$

$$J_1(w,t) = \sum_{i=1}^{2} \|\partial_n \hat{v} - \hat{p}n + \partial_n w - tn\|_{\Gamma_i}^2 \le \gamma^2, \tag{6.15}$$

$$w_1(x) + \hat{v}(x) \ge 0 \quad x \in \omega. \tag{6.16}$$

Evidently, problems (6.12)–(6.16) and (2.1)–(2.8) are equivalent; moreover since $(\hat{v}, \hat{p})$ is a solution of (2.1)–(2.8), then by (6.11) the solution $(\hat{w}, \hat{t})$ of problem (6.12)–(6.16) is as follows:

$$\hat{w}(x) \equiv 0, \quad \hat{t}(x) \equiv 0. \tag{6.17}$$

### 6.3. Second reduction of the problem

We take

$$Y := \{y = (w,t) \in \Psi_\alpha^2(\Omega) \times H_\alpha^1(\Omega) \,|\, (\partial_n w - tn)|_{\Gamma_{\text{out}}} = 0\}, \tag{6.18}$$

$$Z = H_\alpha^0(\Omega)^2, \tag{6.19}$$

and define the map $F(y) : Y \to Z$ by the following formula:

$$F(y) = F(w,t) = -\Delta w + \hat{v} \cdot \nabla w + w \cdot \nabla \hat{v} + w \cdot \nabla w + \nabla t. \tag{6.20}$$

The functionals $f_0$, $f_1$ and $G$ in (5.1) are defined as follows:

$$f_0(y) = J_0(w,t), \quad f_1(y) = J_1(w,t), \quad G(y) = \max_{x \in \omega} \big( - \hat{v}(x) - w(x) \big), \tag{6.21}$$

where $J_0$, $J_1$ are defined in (6.12), (6.15). With the help of (6.18)–(6.21), we reduced problem (6.12)–(6.16) to the abstract problem (5.1). We check now that all conditions of Theorem 5.2 are fulfilled. Let $\Omega_S$ be a neighborhood of $S$. Then, evidently $H_\alpha^k(\Omega_S \cap \Omega) = H^k(\Omega_S \cap \Omega)$ and therefore the trace operators $\gamma_1 w = \partial_n w_{|S}$, $\gamma_0 p = p_{|S}$ are well defined and continuous in the sense

$$\gamma_1 : \Psi_\alpha^2(\Omega) \to H^{1/2}(S)^2, \quad \gamma_0 : H_\alpha^1(\Omega) \to H^{1/2}(S).$$

Hence, the functional in (6.12) is bounded on $\Psi_\alpha^2(\Omega) \times H^1(\Omega)$, and being linear, it is continuously differentiable on this space and therefore also on $Y$. It is well known [Kon1] that the operators $\gamma_1 w = \partial_n w_{|\Gamma}$, $\gamma_0 p = p_{|\Gamma}$ are well defined and continuous in the sense

$$\gamma_1 : H_\alpha^2(\Omega) \to H_\alpha^{1/2}(\Gamma)^2, \quad \gamma_0 : H_\alpha^1(\Omega) \to H_\alpha^{1/2}(\Gamma). \tag{6.22}$$

Therefore, if $\alpha \le 0$, then

$$\begin{aligned}
\sum_{i=1}^{2} \|\partial_n \hat v - \hat p n + \partial_n w - tn\|^2_{\Gamma_i} &\le c + 2\sum_{i=1}^{2} \|\partial_n w + tn\|^2_{\Gamma_i} \\
&\le c + 2\sup_{x\in\Gamma}(\rho^{-2\alpha}) \sum_{i=1}^{2} \|\rho^{\alpha}(\partial_n w + tn)\|^2_{\Gamma_i} \\
&\le c_1\big(1 + \|\partial_n w + tn\|^2_{H^{1/2}_{\alpha}(\Gamma)}\big) \\
&\le c_2\big(1 + \|w\|^2_{\Psi^2_{\alpha}(\Omega)} + \|t\|^2_{H^1_{\alpha}(\Omega)}\big).
\end{aligned} \tag{6.23}$$

Relations (6.22), (6.23) imply the continuous differentiability of the functional in (6.15) on the space $\Psi^2_\alpha(\Omega) \times H^1_\alpha(\Omega)$, for $\alpha \le 0$.

The operator in (6.20) is evidently continuous and continuously differentiable in the spaces (6.18), (6.19), for each $\alpha \le 0$. By virtue of (6.17) the derivative $F'(\hat w, \hat t)$ at the solution $(\hat w, \hat t)$ is defined by the left part of equation (6.1). To check the property a) in subsection 5.2, we have to prove that the boundary value problem (6.1), (6.2) with $g \equiv 0$ for each $f \in H^0_\alpha(\Omega)^2$ possesses a solution $(v,p) \in Y$. For this, we use Theorem 6.2. We choose the parameter $\alpha$ in the spaces in (6.18), (6.19) as follows: if $0 \notin \mathfrak{a}$, we take $\alpha = 0$, if $0 \in \mathfrak{a}$, we take $\alpha < 0$ close enough to zero (there are no points from $\mathfrak{a}$ in the semi-interval $[\alpha, 0)$). By Theorem 6.2 property a) in Subsection 5.2 is fulfilled.

By virtue of definition (6.21) of the functional $G(y)$, property b) in Subsection 5.2 is true because of Lemmas 5.1, 5.3. Hence all conditions of Theorem 5.2 are fulfilled, and we can apply this theorem to problem (6.12)–(6.16).

### 6.4. Application of the Lagrange principle

The Lagrange function for the extremal problem (6.12)–(6.16) has the following form:

$$\begin{aligned}
\mathcal{L}(w,t,\lambda_0,\lambda_1,\alpha,z) = \lambda_0 &\int_S (\partial_n w - tn)e_1\, ds \\
&+ \frac{\lambda_1}{2}\int_{\Gamma_1\cup\Gamma_2} |\partial_n(\hat v + w) - (\hat p + t)n)|^2\, dx_1 \\
&+ \alpha \sup_{x\in\omega}(-w_1(x) - \hat v_1(x)) \\
&+ \int_\Omega (-\Delta w + \hat v\cdot\nabla w + w\cdot\nabla\hat v + w\cdot\nabla w + \nabla t) z\, dx,
\end{aligned} \tag{6.24}$$

where $(\lambda_0, \lambda_1, \alpha, z) \in \mathbb{R}^3 \times H^0_{-\alpha}(\Omega)^2$ are Lagrange multipliers. By virtue of Theorem 5.2 there exists Lagrange multipliers satisfying (5.3)–(5.5) (these conditions will be discussed later). Condition (5.6) being applied to function (6.24) at

$(\hat{w}, \hat{t}) = (0,0)$ leads to the relation

$$\begin{aligned} \lambda_0 \int_S (\partial_n h - \tau n) e_1 \, ds \; + \; \lambda_1 \int_{\Gamma_1 \cup \Gamma_2} (\partial_n \hat{v} - \hat{p} n) \cdot (\partial_n h - \tau n) \, dx_1 \\ -\alpha \int_\omega h_1(x) \, \mu(dx) + \int_\Omega (-\Delta h + \hat{v} \cdot \nabla h + h \cdot \nabla \hat{v} + \nabla \tau) z \, dx = \; 0, \end{aligned} \tag{6.25}$$

which is true for every $(h, \tau) \in Y$ (see (6.18)). In (6.25) $\mu(dx)$ is a measure on $\omega$. In the case of problem (6.12)–(6.16),

$$G(w) = \max_{x \in \omega} (-\hat{v}_1(x) - w_1(x)), \tag{6.26}$$

and we have to find the derivative of this functional at the point $w = (w_1, w_2) = (0,0)$ in the direction $h = (h_1, h_2)$. Recall that all functions in (6.26) and below belong to $\Psi^2_\alpha(\Omega)$ and by the Sobolev embedding theorem the restriction to $\omega$ of all these functions belong to $C(\omega)$. By virtue of (5.10), (5.12), the derivative of the functional (6.26) at zero in the direction $h$ has the form

$$G'(0, h) = \max_{x \in M_0} -h_1(x), \tag{6.27}$$

where $M_0 = \left\{ x \in \omega | - \hat{v}_1(x) = 0 \right\}$. By Lemma 5.4 the sub-differential $\partial G'(0, \cdot)$ consists of the functional

$$l(h) = - \int_\Omega h_1(x) \, \mu(dx),$$

where $\mu(dx)$ is a probability measure supported on the set $M_0$. Just this measure is written in equation (6.25).

## 7. The optimality system

In this section, we obtain the main result of this paper, the optimality system for problem (2.1)–(2.8).

### 7.1. Derivation of the optimality system

At first, we take $h \in \Psi^2_\alpha(\Omega) \cap C^\infty_0(\Omega)^2$, $\tau \in C^\infty_0(\Omega)$ in (6.25). In this way, we get

$$\int_\Omega (-\Delta h + \hat{v} \cdot \nabla h + h \cdot \nabla \hat{v} + \nabla \tau) \cdot z \, dx = \alpha \int_\omega h_1(x) \, \mu(dx). \tag{7.1}$$

If we take $h = 0$ in (7.1), the resulting equality yields

$$\nabla \cdot z = 0 \quad \text{in } \Omega, \tag{7.2}$$

which is to be understood in the distributional sense. Accordingly, taking $\tau \equiv 0$ in (7.1), we get

$$\int_\Omega (-\Delta z - \hat{v} \cdot \nabla z + \nabla \hat{v}^* z) \cdot h \, dx = \alpha \int_\omega h_1 \, \mu \, (dx) \,, \tag{7.3}$$

for all $h \in \Psi^2_\alpha(\Omega) \cap C^\infty_0(\Omega)^2$ , where $(\nabla \hat{v})^* z = (\partial_1 \hat{v} \cdot z, \partial_2 \hat{v} \cdot z)$.

This equality and the De Rham Theorem (see [T]) imply that there exists a distribution $\sigma(x)$ such that

$$-\Delta z - \hat{v}\cdot\nabla z + \nabla\hat{v}^* z - \nabla\sigma = \alpha e_1 \mu(dx) \tag{7.4}$$

where $e_1 = (1,0)$.

System (7.4), (7.2) is elliptic in the sense of Douglas-Nirenberg. Therefore for each subdomain $\Omega_1$ of $\Omega$ compactly enclosed in $\Omega\setminus\omega$, i.e. $\overline{\Omega}_1 \subset \Omega\setminus\omega$, we have $z\in H^2(\Omega_1)$, $\nabla\sigma\in L^2(\Omega_1)$.

Moreover $(z,\nabla\sigma)$ possesses enough smoothness near $\partial\Omega$ in order to define the traces of these functions on $\partial\Omega$. To prove this one has to use methods of ([F] Chapter 2.5, [LM]).

Now, we take an arbitrary $(h,\tau)\in Y$ in (6.25) and integrate by parts. Then, taking into account (7.2), (7.4), we get for all $(h,\tau)\in Y$:

$$\begin{aligned}\lambda_0 \int_S (-\tau n + \partial_n h)\cdot e_1\,dx + \lambda_1 \int_{\Gamma_1\cup\Gamma_2} (\partial_n\hat{v} - \hat{p}n)(\partial_n h - \tau n)dx_1 \\ + \int_{\partial\Omega} \big\{(-\partial_n h + \tau n)z + \partial_n z\cdot h + (\hat{v}\cdot n)(h\cdot z)\big\}\,dx + \int_\Omega \nabla\sigma\cdot h\,dx = 0.\end{aligned} \tag{7.5}$$

Suppose that $(h,\tau)\in Y$ and $(h,\tau)$ equals zero in a neighborhood of $(\partial\Omega\setminus S)$. Then, recalling that $h_{|S}=0$, we obtain from (7.5) that

$$\int_S (-\tau n + \partial_n h)(\lambda_0 e_1 - z)\,dx = 0. \tag{7.6}$$

Since $\nabla\cdot h = 0$ and $h_{|S} = 0$, then $(-\tau n + \partial_n h)_{|S} = (-\tau n + \partial_n h_T)_{|S}$ where $h_T$ is the component of vector field $h$ that is tangent to $S$. Evidently the set of $(-\tau n + \partial_n h_T)_{|S}$ is dense in $L^2(S)^2$. Therefore (7.6) implies

$$z_{|S} = \lambda_0 e_1. \tag{7.7}$$

Analogously, if we take $(h,\tau)\in Y$ that equals zero in a neighborhood of $\partial\Omega\setminus\{\Gamma_{\rm in}\cup S'\}$), we obtain from (7.5)

$$\int_{\Gamma_{\rm in}\cup S'} (\tau n - \partial_n h) z\,dx = 0.$$

This implies the equality

$$z_{|\Gamma_{\rm in}\cup S'} = 0. \tag{7.8}$$

Taking $(h,\tau)\in Y$, $(h,\tau)=0$ in a neighborhood of $\partial\Omega\setminus\Gamma_{\rm out}$ and using that $(-\partial_n h + \tau n)|_{\Gamma_{\rm out}} = 0$, we get from (7.5) that

$$\int_{\Gamma_{\rm out}} \big(\partial_n z + n\sigma + (\hat{v}\cdot n)z\big) h\,dx_2 = 0,$$

and therefore, since $\int_{\Gamma_{\rm out}} n\cdot h\,dx_2 = 0$, we get

$$\big(\partial_n z + n\sigma + (\hat{v}\cdot n)z\big)_{|\Gamma_{\rm out}} = nc, \tag{7.9}$$

where $c$ is a constant.

At last, for $(h,\tau)\in Y$, $(h,\tau)=0$ in a neighborhood of $\partial\Omega\setminus(\Gamma_1\cup\Gamma_2)$, we obtain from (7.5)

$$\int_{\Gamma_1\cup\Gamma_2}\big(-\partial_n h+\tau n)z+\big(\partial_n z+n\sigma+(\hat v\cdot n)z\big)\cdot h \\ +\lambda_1(\partial_n\hat v-\hat p n)(\partial_n h-\tau n)\,dx_1=0. \tag{7.10}$$

Taking in (7.10) $h=0$ and $-\partial_n h+\tau n$ running through the dense set in $\big(L^2(\Gamma_1\cup\Gamma_2)\big)^2$, we get

$$z|_{\Gamma_1\cup\Gamma_2}=\lambda_1\left(\partial_n\hat v-\hat p n\right)\big|_{\Gamma_1\cup\Gamma_2}. \tag{7.11}$$

If we take $-\partial_n h+\tau n=0$ and $h$ is arbitrary, then we obtain

$$\big(\partial_n z+n\sigma+(\hat v\cdot n)z\big)\big|_{\Gamma_1\cup\Gamma_2}=nc, \tag{7.12}$$

where $c$ is a constant. Notice that the constant $c$ in (7.9) and the constant $c$ in (7.12) corresponding to $\Gamma_1$ and $\Gamma_2$ are equal. Indeed, for determining (7.9), (7.12), we can take $h=0$ on $\partial\Omega\setminus(\Gamma_{\text{out})\cup\Gamma_1\cup\Gamma_2}$ and $h$ arbitrary on $\Gamma_{\text{out}}\cup\Gamma_1\cup\Gamma_2$ (Compare with (3.11)). Adding this $c$ to $\sigma$ we can take $c=0$.

### 7.2. The final form of the optimality system

Let $(\hat v,\hat p,\hat u_1,\hat u_2)$ be the solution of problem (2.1)–(2.8). Then, the optimality system for this problem consists of equations (2.1), (2.2), (7.4), (7.2) and the boundary conditions (2.3), (7.7)–(7.9), (7.11), (7.12). We rewrite these equations in the following form:

$$-\Delta\hat v+\hat v\cdot\nabla\hat v+\nabla\hat p=0,\quad \nabla\cdot\hat v=0,\, x\in\Omega, \tag{7.13}$$

$$-\Delta z-\hat v\cdot\nabla z+\nabla\hat v^* z-\nabla\sigma=e_1\alpha\mu(dx),\quad \nabla\cdot z=0,\, x\in\Omega, \tag{7.14}$$

$$\hat v_{|\Gamma_{\text{in}}}=v^{\text{in}},\quad (\partial_n\hat v-\hat p n)_{|\Gamma_{\text{out}}}=0,\quad v_{|S\cup S'}=0, \tag{7.15}$$

$$z_{|S}=\lambda_0 e_1,\quad z_{|\Gamma_{\text{in}}\cup S'}=0,\quad z_{|\Gamma_1\cup\Gamma_2}=\lambda_1\left(\partial_n\hat v-\hat p n\right)_{|\Gamma_1\cup\Gamma_2}, \tag{7.16}$$

$$\big(\partial_n z+n\sigma+(\hat v\cdot n)z\big)_{|\Gamma\text{out}}=0,\quad \big(\partial_n z+n\sigma+(\hat v\cdot n)z\big)_{|\Gamma_1\cup\Gamma_2}=0. \tag{7.17}$$

By virtue of (5.3), (5.5) the optimality system should be supplemented by the following condition:

1) Conditions of signs concordance:

$$\lambda_0\ge 0,\quad \lambda_1\ge 0,\quad \alpha\ge 0. \tag{7.18}$$

2) Conditions of complementary slackness

$$\lambda_1(J_1-\gamma^2)=0,\quad \alpha\min_{x\in\omega}\hat v(x)=0, \tag{7.19}$$

where $J_1=J_1(0,0)$ and $J_1(w,t)$ is defined in (6.15).

## 8. Numerical calculations

The validity of the crucial Condition 1 for a given subset $\omega \subset \Omega$ (see (4.26)) can hardly be shown analytically but rather requires computational confirmation. Figure 2 shows a series of plots of the velocity component $v_1$ for increasing strength of the control (positive pressure drop). Figure 3 shows corresponding plots for symmetric and asymmetric action of the control. The latter results demonstrate that for a large class of sub-domains $\omega \subset \Omega$ the property $v_{1|\omega} \geq 0$ can be achieved by applying appropriate controls.

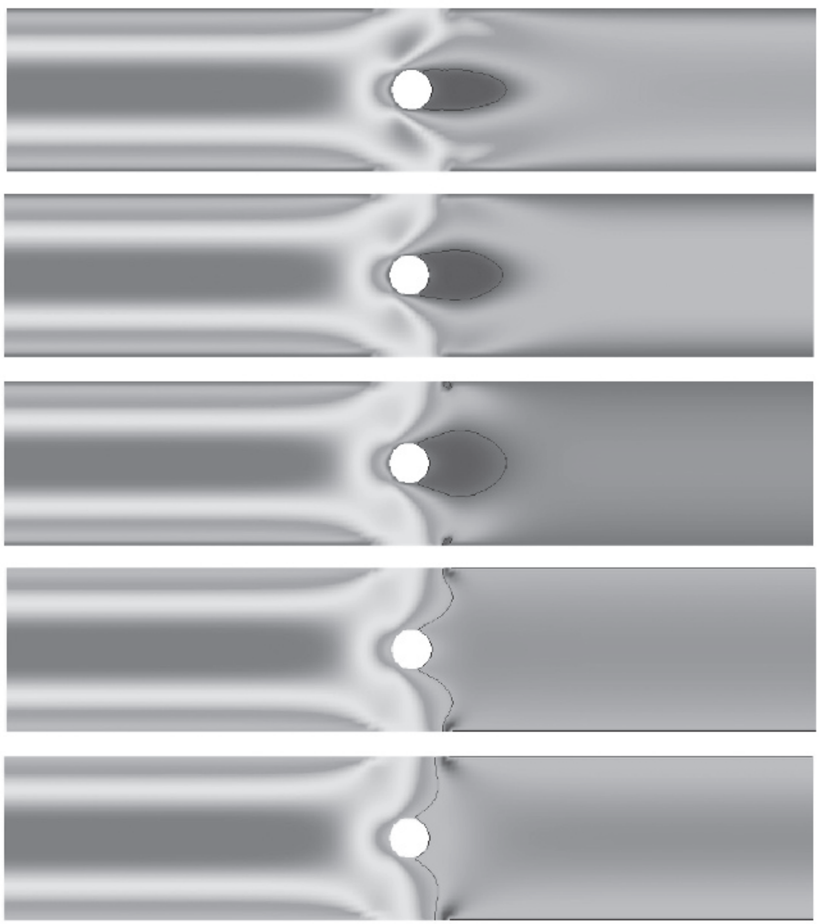

FIGURE 2. Velocity component $v_1$ for increasing strength of control pressure; area of $v_1 < 0$ dark grey

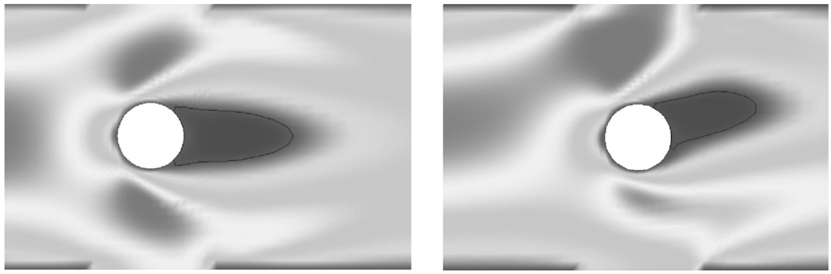

FIGURE 3. Velocity component $v_1$ for symmetric (left) and asymmetric (right) control pressure drop; area of $v_1 < 0$ dark grey

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# On Some Boundary Value Problem for the Stokes Equations with a Parameter in an Infinite Sector

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*In Memory of Alexandre Vasil′evich Kazhikhov*

**Abstract.** As a model problem of the nonstationary free boundary problem for the Navier-Stokes equations in a vessel whose wall has a contact with a free surface, we are concerned in this paper with the boundary value problem for the stationary Stokes equations with a parameter in an infinite sector with the slip and the stress boundary conditions. Existence of the unique solution is proved in weighted Sobolev spaces.

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**Keywords.** Stokes equations with a parameter, weighted Sobolev spaces, infinite sector

## 1. Introduction

The present paper is considered as a continuation of our article [4]. Let $d_\theta$ be a plane angle of opening $\theta$ in the polar coordinates $(r, \varphi)$,

$$d_\theta = \left\{ x = (r\cos\varphi, r\sin\varphi) \in \mathbb{R}^2 \;\middle|\; r > 0,\ 0 < \varphi < \theta \right\};$$

let $\gamma_0 = \{\varphi = 0,\ r > 0\}$, $\gamma_\theta = \{\varphi = \theta,\ r > 0\}$ be the sides of the angle.

In this paper we consider the problem of determining the velocity vector field $\mathbf{u} = \mathbf{u}(x, s) = (u_1(x, s), u_2(x, s))$ and the pressure $q = q(x, s)$ satisfying in $d_\theta$ the Stokes equations with a parameter $s \in \mathbb{C}$

$$\begin{cases} s\mathbf{u} - \nu\Delta\mathbf{u} + \nabla q = \mathbf{f}, \quad \nabla\cdot\mathbf{u} = \rho \quad \text{in } d_\theta, \\ u_2|_{\gamma_0} = 0, \quad 2\nu D_{12}(\mathbf{u})\big|_{\gamma_0} = b_0, \quad \mathbb{P}(\mathbf{u}, q)\mathbf{n}_\theta|_{\gamma_\theta} = \mathbf{b}_\theta. \end{cases} \tag{1.1}$$

Here, $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$, $\mathbb{P}(\mathbf{u}, q) = -q\mathbb{I} + 2\nu\mathbb{D}(\mathbf{u})$ is the stress tensor, $\mathbb{D}(\mathbf{u})$ is the velocity deformation tensor with elements $D_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ $(i, j = 1, 2)$, $\mathbb{I}$ is the unit tensor of degree 2, $\mathbf{n}_\theta$ is the unit vector of the outward normal to $\gamma_\theta$, $(\mathbf{f}, \rho)$, $b_0$ and $\mathbf{b}_\theta$ are given functions on $d_\theta$, $\gamma_0$ and $\gamma_\theta$, respectively, and $\nu$ is a coefficient of viscosity, assumed to be a positive constant.

The problem (1.1) is related to the time-dependent problem with free boundary (see [4]).

We consider (1.1) in weighted Sobolev spaces ([6], [3]). Let $k \in \{0\} \cup \mathbb{N}$ and $\mu \in \mathbb{R}$. We define a space $\mathrm{H}^k_\mu(d_\theta)$ as a completion of the set of infinitely differentiable functions with compact support vanishing near the vertex of the angle equipped with the norm

$$\begin{aligned}\|u\|^2_{\mathrm{H}^k_\mu(d_\theta)} &= \sum_{|\alpha|\le k} \int_{d_\theta} r^{2(\mu-k+|\alpha|)} |\mathrm{D}^\alpha_x u(x)|^2 \,\mathrm{d}x \\ &\equiv \sum_{|\alpha|\le k} \|\mathrm{D}^\alpha_x u\|^2_{\mathrm{L}_{2,\mu-k+|\alpha|}(d_\theta)},\end{aligned}$$

where $r = |x| = \sqrt{x_1^2 + x_2^2}$, $\alpha = (\alpha_1, \alpha_2)$ is a multi-index, $|\alpha| = \alpha_1 + \alpha_2$,

$$\mathrm{D}^\alpha_x u = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} u.$$

The space of traces of functions in $\mathrm{H}^{k+1}_\mu(d_\theta)$ on the side $\gamma_0$ or $\gamma_\theta$ is the space $\mathrm{H}^{k+1/2}_\mu(\gamma)$ $(\gamma = \gamma_0$ or $\gamma_\theta)$ with the norm

$$\begin{aligned}\|b\|^2_{\mathrm{H}^{k+1/2}_\mu(\gamma)} &= \sum_{j=0}^{k} \int_0^\infty r^{2(\mu-k-1/2+j)} |\mathrm{D}^j_r b(r)|^2 \,\mathrm{d}r \\ &\quad + \int_0^\infty r^{2\mu} \,\mathrm{d}r \int_0^r \frac{|\mathrm{D}^k_r b(r+\rho) - \mathrm{D}^k_r b(r)|^2}{\rho^2} \,\mathrm{d}\rho \\ &\equiv \sum_{j=0}^{k} \|\mathrm{D}^j_r b\|^2_{\mathrm{L}_{2,\mu-k-1/2+j}(\gamma)} + \|b\|^2_{\mathrm{L}^{k+1/2}_\mu(\gamma)},\end{aligned}$$

where $\mathrm{D}_r = \mathrm{d}/\mathrm{d}r$. We also use the notation $\|b\|^2_{\mathrm{L}^l_\mu(\gamma)} = \sum_{|\alpha|=l} \|\mathrm{D}^\alpha_x b\|^2_{\mathrm{L}_{2,\mu}(\gamma)}$.

Two-dimensional vector-valued functions are denoted by bold-faced letters such as $\mathbf{u} = (u_1, u_2)$. Similarly, we use bold-faced letters to denote the function spaces of two-dimensional vector-valued functions.

Our main theorem is as follows.

**Theorem 1.1.** *Let* $\Re s = h \ge \nu$ *and* $k \in \{0\} \cup \mathbb{N}$, $\theta \in (0, \pi/2)$, $\mu \in (0, 1/2)$ *be numbers satisfying* (2.3), (2.4) *below. Suppose that* $\rho = \nabla \cdot \mathbf{g}$ *with* $\mathbf{g} = (g_1, g_2)$,

$g_2|_{\gamma_0} = 0$, *and* $\mathbf{f}$, $\rho$, $\mathbf{g}$, $b_0$, *and* $\mathbf{b}_\theta$ *satisfy*

$$\begin{aligned} F_k \equiv & \sum_{l=0}^{k} |s|^{k-l} \|\mathbf{f}(\cdot, s)\|^2_{\mathbf{H}^l_\mu(d_\theta)} \\ & + \sum_{l=0}^{k} \Big( |s|^{k-l} \|b_0(\cdot, s)\|^2_{\mathrm{H}^{l+1/2}_\mu(\gamma_0)} + |s|^{k+1/2-l} \|b_0(\cdot, s)\|^2_{\mathrm{L}^l_\mu(\gamma_0)} \Big) \\ & + \sum_{l=0}^{k} \Big( |s|^{k-l} \|\mathbf{b}_\theta(\cdot, s)\|^2_{\mathbf{H}^{l+1/2}_\mu(\gamma_\theta)} + |s|^{k+1/2-l} \|\mathbf{b}_\theta(\cdot, s)\|^2_{\mathbf{L}^l_\mu(\gamma_\theta)} \Big) < \infty, \\ G_k \equiv & \sum_{l=0}^{k+1} |s|^{k+1-l} \|\rho(\cdot, s)\|^2_{\mathrm{H}^l_\mu(d_\theta)} + \sum_{l=0}^{k} |s|^{2+l} \|\mathbf{g}(\cdot, s)\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} < \infty. \end{aligned}$$

*Then there exists a unique solution*

$$\mathbf{u}(\cdot, s) \in \mathbf{H}^{k+2}_\mu(d_\theta), \quad \nabla q(\cdot, s) \in \mathbf{H}^k_\mu(d_\theta), \quad q(\cdot, s)|_{\gamma_\theta} \in \mathrm{H}^{k+3/2}_\mu(\gamma_\theta)$$

*to problem* (1.1) *satisfying the inequality*

$$\begin{aligned} & \sum_{l=0}^{k+2} |s|^{2+k-l} \|\mathbf{u}(\cdot, s)\|^2_{\mathbf{H}^l_\mu(d_\theta)} + \sum_{l=0}^{k} |s|^{k-l} \|\nabla q(\cdot, s)\|^2_{\mathbf{H}^l_\mu(d_\theta)} \\ & \quad + \sum_{l=0}^{k} \Big( |s|^{k-l} \|q(\cdot, s)\|^2_{\mathrm{H}^{l+1/2}_\mu(\gamma_\theta)} + |s|^{k+1/2-l} \|q(\cdot, s)\|^2_{\mathrm{L}^l_\mu(\gamma_\theta)} \Big) \\ & \leq c(F_k + G_k), \end{aligned} \tag{1.2}$$

*where* $c$ *is a constant independent of* $|s|$.

## 2. Auxiliary propositions

In this section we describe the necessary results for the later discussions.

### 2.1. Stationary problem

In this subsection we are concerned mainly with the problem

$$\begin{cases} -\nu \Delta \mathbf{v} + \nabla p = \mathbf{f}, \quad \nabla \cdot \mathbf{v} = \rho \quad \text{in } d_\theta, \\ v_2|_{\gamma_0} = b_1, \quad 2\nu D_{12}(\mathbf{v})\big|_{\gamma_0} = b_2, \quad \mathbb{P}(\mathbf{v}, p)\mathbf{n}_\theta|_{\gamma_\theta} = \mathbf{b}_3. \end{cases} \tag{2.1}$$

It was shown in [4] that, after transforming the problem (2.1) by the polar coordinates $(r, \phi)$ and making use of the Mellin transform with respect to $r$, the system of ordinary differential equations thus obtained concerning $\phi \in (0, \theta)$ has a solution for arbitrary $(\mathbf{f}, \rho, b_1, b_2, \mathbf{b}_3)$ provided that $\lambda$ is not equal to the non-zero solution of the equation

$$\mathcal{P}(\lambda) \equiv \sin(2\lambda\theta) + \lambda \sin(2\theta) = 0, \quad \lambda \in \mathbb{C} \backslash \{0\} \tag{2.2}$$

on the complex plane $\mathbb{C}$. (It was also shown in [4] that $\lambda = 0$ is an eigenvalue of the problem.)

Let us confirm that for arbitrary fixed $\theta \in (0, \pi/2)$ and $k \in \{0\} \cup \mathbb{N}$, we can find a number $\mu \in (0, 1/2)$ satisfying

$$l + 1 - \mu \neq \left(m + \frac{1}{2}\right)\frac{\pi}{\theta} \qquad (l = 0, 1, \ldots, k;\ m \in \mathbb{Z}), \tag{2.3}$$

$$\mathcal{P}(l + 1 - \mu + \mathrm{i}\beta) \neq 0 \qquad (l = 0, 1, \ldots, k;\ \beta \in \mathbb{R}) \tag{2.4}$$

simultaneously. Indeed, it is easy to see that (2.4) is equivalent to

$$\begin{cases} \sin\left((l+1-\mu)2\theta\right)\cosh(2\theta\beta) + (l+1-\mu)\sin(2\theta) \neq 0, \\ \cos\left((l+1-\mu)2\theta\right)\sinh(2\theta\beta) + \beta\sin(2\theta) \neq 0. \end{cases} \tag{2.5}$$

Therefore, if $\beta = 0$, then we have

$$\frac{\sin\big((l+1-\mu)2\theta\big)}{(l+1-\mu)2\theta} \neq -\frac{\sin(2\theta)}{2\theta}.$$

On the other hand, if $\beta \neq 0$, then we have

$$\frac{\tan\left((l+1-\mu)2\theta\right)}{(l+1-\mu)2\theta} \neq \frac{\tanh(2\theta\beta)}{2\theta\beta} \qquad (l = 0, 1, \ldots, k;\ \beta \in \mathbb{R}\backslash\{0\}),$$

since $\cos\left((l+1-\mu)2\theta\right) \neq 0$ by (2.3).

For the solvability of problem (2.1) we have already established the following theorem:

**Theorem 2.1** ([4])**.** *Suppose that $k \in \{0\} \cup \mathbb{N}$, $\theta \in (0, 2\pi]$ and the line $\Re\lambda = k + 1 - \mu$ is free from the solution of the equation (2.2). Then for any $\mathbf{f} = (f_1, f_2) \in \mathbf{H}^k_\mu(d_\theta)$, $\rho \in \mathrm{H}^{1+k}_\mu(d_\theta)$, $b_1 \in \mathrm{H}^{k+3/2}_\mu(\gamma_0)$, $b_2 \in \mathrm{H}^{k+1/2}_\mu(\gamma_0)$, $\mathbf{b}_3 = (b_{31}, b_{32}) \in \mathbf{H}^{k+1/2}_\mu(\gamma_\theta)$ satisfying the relation*

$$\int_{d_\theta}\left(f_1 - \nu\frac{\partial\rho}{\partial x_1}\right)\,\mathrm{d}x = \int_{\gamma_0} b_2\,\mathrm{d}r + \int_{\gamma_\theta} b_{31}\,\mathrm{d}r,$$

*problem (2.1) has a unique solution $\mathbf{v} \in \mathbf{H}^{k+2}_\mu(d_\theta)$, $p \in \mathrm{H}^{k+1}_\mu(d_\theta)$. Moreover this solution satisfies the inequality*

$$\begin{aligned}\|\mathbf{v}\|_{\mathbf{H}^{k+2}_\mu(d_\theta)} + \|p\|_{\mathrm{H}^{k+1}_\mu(d_\theta)} \leq c_{1,k}\Big(&\|\mathbf{f}\|_{\mathbf{H}^k_\mu(d_\theta)} + \|\rho\|_{\mathrm{H}^{k+1}_\mu(d_\theta)} + \|b_1\|_{\mathrm{H}^{k+3/2}_\mu(\gamma_0)} \\ &+ \|b_2\|_{\mathrm{H}^{k+1/2}_\mu(\gamma_0)} + \|\mathbf{b}_3\|_{\mathbf{H}^{k+1/2}_\mu(\gamma_\theta)}\Big),\end{aligned}$$

*where $c_{1,k}$ is a positive constant independent of $\mathbf{f}$, $\rho$, $b_1$, $b_2$ and $\mathbf{b}_3$.*

We also use the following theorem due to Kondrat′ev.

**Theorem 2.2** ([6])**.** *Let $k \in \{0\} \cup \mathbb{N}$, $\mu \in \mathbb{R}$ and $\theta \in (0, 2\pi]$ satisfy $k + 1 - \mu \neq (m + 1/2)\pi/\theta$, $m \in \mathbb{Z}$. For arbitrary $g \in \mathrm{H}^k_\mu(d_\theta)$, $a_1 \in \mathrm{H}^{k+1/2}_\mu(\gamma_0)$, $a_2 \in \mathrm{H}^{k+3/2}_\mu(\gamma_\theta)$, problem*

$$\Delta\phi = g \quad \textit{in}\ d_\theta, \qquad \frac{\partial\phi}{\partial x_2}\bigg|_{\gamma_0} = a_1, \quad \phi\big|_{\gamma_\theta} = a_2$$

*has a unique solution* $\phi \in \mathrm{H}^{k+2}_\mu(d_\theta)$ *satisfying the inequality*

$$\|\phi\|_{\mathrm{H}^{k+2}_\mu(d_\theta)} \leq \frac{c}{\cos((k+1-\mu)\theta)} \left( \|g\|_{\mathrm{H}^k_\mu(d_\theta)} + \|a_1\|_{\mathrm{H}^{k+1/2}_\mu(\gamma_0)} + \|a_2\|_{\mathrm{H}^{k+3/2}_\mu(\gamma_\theta)} \right).$$

### 2.2. Some inequalities

**Lemma 2.1.** *Let* $u$ *be an infinitely differentiable function defined in* $d_\theta$ *vanishing near the origin and infinity. Then the following inequalities are true:*

$$\begin{aligned} \|u\|_{\mathrm{L}_{2,\mu_1}(d_\theta)} &\leq \left| \frac{2}{\mu_1+\mu+1} \right|^{\mu-\mu_1} \|\nabla u\|^{\mu-\mu_1}_{\mathbf{L}_{2,\mu}(d_\theta)} \|u\|^{1-(\mu-\mu_1)}_{\mathrm{L}_{2,\mu}(d_\theta)} \\ &\equiv C_I \|\nabla u\|^{\mu-\mu_1}_{\mathbf{L}_{2,\mu}(d_\theta)} \|u\|^{1-(\mu-\mu_1)}_{\mathrm{L}_{2,\mu}(d_\theta)} \end{aligned} \tag{2.6}$$

*for* $\mu_1 \in [\mu-1, \mu],\ \mu_1+\mu+1 \neq 0,$

$$\begin{aligned} \|u\|_{\mathrm{L}_{2,\mu_1}(\gamma)} &\leq \left( \left| \frac{2}{(1+2\mu_1)\theta} \right| + 2 \right)^{1/2} \left| \frac{2}{1+2\mu_1} \right|^{\mu-\mu_1} \|\nabla u\|^{1/2+\mu-\mu_1}_{\mathbf{L}_{2,\mu}(d_\theta)} \|u\|^{1/2-(\mu-\mu_1)}_{\mathrm{L}_{2,\mu}(d_\theta)} \\ &\equiv C_{II} \|\nabla u\|^{1/2+\mu-\mu_1}_{\mathbf{L}_{2,\mu}(d_\theta)} \|u\|^{1/2-(\mu-\mu_1)}_{\mathrm{L}_{2,\mu}(d_\theta)} \end{aligned} \tag{2.7}$$

*for* $\mu_1 \in [\mu-1/2, \mu],\ 1+2\mu_1 \neq 0.$

This result can be found in [2], [3]. In the Appendix we give a detailed proof of it for the sake of reader's convenience.

**Lemma 2.2** ([8])**.** *Let* $\mathbf{v} \in \mathbf{W}^1_2(d_\theta)$ *and its support be contained in a circle* $B_R(0)$, $R>0$. *Then Korn's second inequality*

$$\|\nabla \mathbf{v}\|^2_{\mathbf{L}_2(d_\theta)} \leq C_K \left( \|\mathbb{D}(\mathbf{v})\|^2_{\mathbf{L}_2(d_\theta)} + \|\mathbf{v}\|^2_{\mathbf{L}_2(d_\theta)} \right) \tag{2.8}$$

*holds true, where the constant* $C_K$ *is independent of* $R$.

## 3. Proof of Theorem 1.1

The proof is divided into several steps.

*Step* 1. We assume $\rho = 0$ and decompose $\mathbf{f} \in \mathbf{L}_{2,\mu}(d_\theta)$ into the form $\mathbf{f} = \mathbf{f}^* + \nabla\psi$, where $\psi$ is a solution of

$$\Delta\psi = \nabla\cdot\mathbf{f} \quad \text{in } d_\theta, \quad \left.\frac{\partial\psi}{\partial x_2}\right|_{\gamma_0} = f_2|_{\gamma_0}, \quad \psi|_{\gamma_\theta} = 0. \tag{3.1}$$

Hence $\mathbf{f}^*$ satisfies

$$\nabla\cdot\mathbf{f}^* = 0 \quad \text{in } d_\theta, \quad f^*_2|_{\gamma_0} = 0.$$

It follows from the result in [5] that

$$\|\nabla\psi\|_{\mathbf{L}_{2,\mu}(d_\theta)} \leq c\|\mathbf{f}\|_{\mathbf{L}_{2,\mu}(d_\theta)}.$$

Putting $q^* = q - \psi$, we consider the problem

$$\begin{cases} s\mathbf{u} - \nabla\cdot\mathbb{P}(\mathbf{u}, q^*) = \mathbf{f}^*, \quad \nabla\cdot\mathbf{u} = 0 \quad \text{in } d_\theta, \\ u_2|_{\gamma_0} = 0, \quad 2\nu D_{12}(\mathbf{u})|_{\gamma_0} = b_0, \quad \mathbb{P}(\mathbf{u}, q^*)\mathbf{n}_\theta|_{\gamma_\theta} = \mathbf{b}_\theta. \end{cases} \tag{3.2}$$

*Step* 1-1. First of all we show the inequality

$$\|\mathbf{u}\|^2_{\mathbf{H}^2_\mu(d_\theta)} + |s|\|\mathbf{u}\|^2_{\mathbf{H}^1_\mu(d_\theta)} + |s|^2\|\mathbf{u}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} + \|\nabla q^*\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} \\ + \|q^*\|^2_{\mathrm{H}^{1/2}_\mu(\gamma_\theta)} + |s|^{1/2}\|q^*\|^2_{\mathrm{L}_{2,\mu}(\gamma_\theta)} \le cF_0. \tag{3.3}$$

Inequality (3.3) is proved by similar calculations as Proposition 3.1 in [3]. Indeed, we multiply equation (3.2) by $(1-\mathrm{i}\,\mathrm{sgn}\,\Im s)\bar{\mathbf{u}}$ and integrate over $d_\theta$. After integration by parts, we obtain

$$\begin{aligned}(1-\mathrm{i}\,\mathrm{sgn}\,\Im s)s\int_{d_\theta}|\mathbf{u}|^2\,\mathrm{d}x &+ (1-\mathrm{i}\,\mathrm{sgn}\,\Im s)2\nu\int_{d_\theta}|\mathbb{D}(\mathbf{u})|^2\,\mathrm{d}x \\ &= (1-\mathrm{i}\,\mathrm{sgn}\,\Im s)\left[\int_{d_\theta}\mathbf{f}^*\cdot\bar{\mathbf{u}}\,\mathrm{d}x - \int_{\gamma_0} b_0\bar{u}_1\,\mathrm{d}r + \int_{\gamma_\theta}\mathbf{b}_\theta\cdot\bar{\mathbf{u}}\,\mathrm{d}r\right] \\ &\equiv (1-\mathrm{i}\,\mathrm{sgn}\,\Im s)\,l(\bar{\mathbf{u}}).\end{aligned}$$

Taking the real part of this equation and multiplying it by $|s|^{1-\mu}$ yields

$$\begin{aligned}|s|^{1-\mu}&\left((h+|\Im s|)\|\mathbf{u}\|^2_{\mathbf{L}_2(d_\theta)} + 2\nu\|\mathbb{D}(\mathbf{u})\|^2_{\mathbf{L}_2(d_\theta)}\right) \\ &= |s|^{1-\mu}\Re\left((1-\mathrm{i}\,\mathrm{sgn}\,\Im s)l(\bar{\mathbf{u}})\right),\end{aligned} \tag{3.4}$$

since $(1-\mathrm{i}\,\mathrm{sgn}\,\Im s)s = s + |\Im s| - h(\mathrm{sgn}\,\Im s)\mathrm{i}$. Let us estimate the right-hand side of (3.4). The first term is estimated as follows.

$$\begin{aligned}|s|^{1-\mu}&\left|\int_{d_\theta}\mathbf{f}^*\cdot\bar{\mathbf{u}}\,\mathrm{d}x\right| \\ &\le |s|^{1-\mu}\|\mathbf{f}^*\|_{\mathbf{L}_{2,\mu}(d_\theta)}\|\mathbf{u}\|_{\mathbf{L}_{2,-\mu}(d_\theta)} \\ &\le |s|^{1-\mu}\|\mathbf{f}^*\|_{\mathbf{L}_{2,\mu}(d_\theta)}C_I\|\nabla\mathbf{u}\|^\mu_{\mathbf{L}_2(d_\theta)}\|\mathbf{u}\|^{1-\mu}_{\mathbf{L}_2(d_\theta)} \\ &\le \varepsilon\left(|s|^{1-\mu}\|\nabla\mathbf{u}\|^2_{\mathbf{L}_2(d_\theta)}\right)^\mu\left(|s|^{2-\mu}\|\mathbf{u}\|^2_{\mathbf{L}_2(d_\theta)}\right)^{1-\mu} + C_\varepsilon\|\mathbf{f}^*\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} \\ &\le \varepsilon\left(|s|^{1-\mu}\|\nabla\mathbf{u}\|^2_{\mathbf{L}_2(d_\theta)} + |s|^{2-\mu}\|\mathbf{u}\|^2_{\mathbf{L}_2(d_\theta)}\right) + C_\varepsilon\|\mathbf{f}^*\|^2_{\mathbf{L}_{2,\mu}(d_\theta)}.\end{aligned} \tag{3.5}$$

Here and in what follows, $\varepsilon$ denotes an arbitrarily small positive number. For the second term the inequality

$$|s|^{1-\mu}\left|\int_{\gamma_0} b_0\bar{u}_1\mathrm{d}r\right| \le |s|^{1-\mu}\|b_0\|_{\mathrm{L}_{2,\mu_1}(\gamma_0)}\|\mathbf{u}\|_{\mathbf{L}_{2,-\mu_1}(\gamma_0)}$$

holds for any $\mu_1 \in (\mu-1/2,\mu)\cap[0,1/2)$. Since

$$\begin{aligned}\|b_0\|_{\mathrm{L}_{2,\mu_1}(\gamma_0)} &\le \|b_0\|^\alpha_{\mathrm{L}_{2,\mu}(\gamma_0)}\|b_0\|^{1-\alpha}_{\mathrm{L}_{2,\mu-1/2}(\gamma_0)} && \text{for } \alpha = 2\mu_1 - 2\mu + 1 \in (0,1), \\ \|\mathbf{u}\|_{\mathbf{L}_{2,-\mu_1}(\gamma_0)} &\le C_{II}\|\nabla\mathbf{u}\|^{1/2+\mu_1}_{\mathbf{L}_2(d_\theta)}\|\mathbf{u}\|^{1/2-\mu_1}_{\mathbf{L}_2(d_\theta)} && \text{for } \mu_1 \in [0,1/2),\end{aligned}$$

one can find

$$
\begin{aligned}
&|s|^{1-\mu}\|b_0\|_{\mathrm{L}_{2,\mu_1}(\gamma_0)}\|\mathbf{u}\|_{\mathbf{L}_{2,-\mu_1}(\gamma_0)} \\
&\quad \le C_{II}\left(|s|^{1/4}\|b_0\|_{\mathrm{L}_{2,\mu}(\gamma_0)}\right)^{\alpha}\|b_0\|^{1-\alpha}_{\mathrm{L}_{2,\mu-1/2}(\gamma_0)} \\
&\qquad\qquad \times\left(|s|^{(1-\mu)/2}\|\nabla\mathbf{u}\|_{\mathbf{L}_2(d_\theta)}\right)^{1/2+\mu_1}\left(|s|^{(2-\mu)/2}\|\mathbf{u}\|_{\mathbf{L}_2(d_\theta)}\right)^{1/2-\mu_1} \\
&\quad \le \varepsilon\left(|s|^{1-\mu}\|\nabla\mathbf{u}\|^2_{\mathbf{L}_2(d_\theta)}+|s|^{2-\mu}\|\mathbf{u}\|^2_{\mathbf{L}_2(d_\theta)}\right) \\
&\qquad\qquad + C_\varepsilon\left(|s|^{1/2}\|b_0\|^2_{\mathrm{L}_{2,\mu}(\gamma_0)}+\|b_0\|^2_{\mathrm{L}_{2,\mu-1/2}(\gamma_0)}\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&|s|^{1-\mu}\left(\left|\int_{\gamma_0} b_0\bar{u}_1 \mathrm{d}r\right|+\left|\int_{\gamma_\theta}\mathbf{b}_\theta\cdot\bar{\mathbf{u}}\ \mathrm{d}r\right|\right) \\
&\quad \le \varepsilon\left(|s|^{1-\mu}\|\nabla\mathbf{u}\|^2_{\mathbf{L}_2(d_\theta)}+|s|^{2-\mu}\|\mathbf{u}\|^2_{\mathbf{L}_2(d_\theta)}\right)+C_\varepsilon\left(|s|^{1/2}\|b_0\|^2_{\mathrm{L}_{2,\mu}(\gamma_0)}\right. \\
&\qquad \left. +\|b_0\|^2_{\mathrm{L}_{2,\mu-1/2}(\gamma_0)}+|s|^{1/2}\|\mathbf{b}_\theta\|^2_{\mathbf{L}_{2,\mu}(\gamma_\theta)}+\|\mathbf{b}_\theta\|^2_{\mathbf{L}_{2,\mu-1/2}(\gamma_\theta)}\right).
\end{aligned}
\tag{3.6}
$$

Substituting (3.5)–(3.6) into (3.4), we obtain

$$
\begin{aligned}
&|s|^{1-\mu}\left((h+|\Im s|)\|\mathbf{u}\|^2_{\mathbf{L}_2(d_\theta)}+2\nu\|\mathbb{D}(\mathbf{u})\|^2_{\mathbf{L}_2(d_\theta)}\right) \\
&\quad \le \varepsilon\left(|s|^{1-\mu}\|\nabla\mathbf{u}\|^2_{\mathbf{L}_2(d_\theta)}+|s|^{2-\mu}\|\mathbf{u}\|^2_{\mathbf{L}_2(d_\theta)}\right)+C_\varepsilon F_0.
\end{aligned}
\tag{3.7}
$$

By Korn's second inequality (2.8) and the assumption $\Re s = h \ge \nu$ we find

$$
\begin{aligned}
&(h+|\Im s|)\|\mathbf{u}\|^2_{\mathbf{L}_2(d_\theta)}+2\nu\|\mathbb{D}(\mathbf{u})\|^2_{\mathbf{L}_2(d_\theta)} \\
&\quad \ge \frac{1}{2}\left(|s|\|\mathbf{u}\|^2_{\mathbf{L}_2(d_\theta)}+\nu\left(\|\mathbf{u}\|^2_{\mathbf{L}_2(d_\theta)}+\|\mathbb{D}(\mathbf{u})\|^2_{\mathbf{L}_2(d_\theta)}\right)\right) \\
&\quad \ge \frac{1}{2}\left(|s|\|\mathbf{u}\|^2_{\mathbf{L}_2(d_\theta)}+\frac{\nu}{C_K}\|\nabla\mathbf{u}\|^2_{\mathbf{L}_2(d_\theta)}\right).
\end{aligned}
$$

This together with (3.7) yields

$$
|s|^{1-\mu}\|\nabla\mathbf{u}\|^2_{\mathbf{L}_2(d_\theta)}+|s|^{2-\mu}\|\mathbf{u}\|^2_{\mathbf{L}_2(d_\theta)} \le cF_0. \tag{3.8}
$$

Next we multiply equation (3.2) by $(1-\mathrm{i}\,\mathrm{sgn}\,\Im s)\bar{\mathbf{u}}|x|^{2\mu}$ and integrate over $d_\theta$. After integrating by parts, we take a real part of the resulting equation and multiply it by $|s|$. Then, we find

$$
\begin{aligned}
&|s|\left((h+|\Im s|)\|\mathbf{u}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)}+2\nu\|\mathbb{D}(\mathbf{u})\|^2_{\mathbf{L}_{2,\mu}(d_\theta)}\right) \\
&\quad = |s|\Re\left((1-\mathrm{i}\,\mathrm{sgn}\,\Im s)\, l(\bar{\mathbf{u}}|x|^{2\mu})\right) \\
&\qquad - |s|\Re\left((1-\mathrm{i}\,\mathrm{sgn}\,\Im s)\int_{d_\theta}\mathbb{P}(\mathbf{u},q^*)(\nabla|x|^{2\mu})\cdot\bar{\mathbf{u}}\ \mathrm{d}x\right).
\end{aligned}
\tag{3.9}
$$

The first term of the right-hand side of (3.9) is estimated as follows:

$$|s|\left|\int_{d_\theta}\mathbf{f}^*\cdot\bar{\mathbf{u}}|x|^{2\mu}\,\mathrm{d}x\right|\le\varepsilon|s|^2\|\mathbf{u}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)}+C_\varepsilon\|\mathbf{f}^*\|^2_{\mathbf{L}_{2,\mu}(d_\theta)},\tag{3.10}$$

$$\begin{aligned}&|s|\left(\left|\int_{\gamma_0}b_0\bar{u}_1\,r^{2\mu}\,\mathrm{d}r\right|+\left|\int_{\gamma_\theta}\mathbf{b}_\theta\cdot\bar{\mathbf{u}}\,r^{2\mu}\,\mathrm{d}r\right|\right)\\&\quad\le|s|\left(\|b_0\|_{\mathrm{L}_{2,\mu}(\gamma_0)}\|\mathbf{u}\|_{\mathbf{L}_{2,\mu}(\gamma_0)}+\|\mathbf{b}_\theta\|_{\mathbf{L}_{2,\mu}(\gamma_\theta)}\|\mathbf{u}\|_{\mathbf{L}_{2,\mu}(\gamma_\theta)}\right)\\&\quad\le|s|\left(\|b_0\|_{\mathrm{L}_{2,\mu}(\gamma_0)}+\|\mathbf{b}_\theta\|_{\mathbf{L}_{2,\mu}(\gamma_\theta)}\right)C_{II}\|\nabla\mathbf{u}\|^{1/2}_{\mathbf{L}_{2,\mu}(d_\theta)}\|\mathbf{u}\|^{1/2}_{\mathbf{L}_{2,\mu}(d_\theta)}\\&\quad\le\varepsilon\left(|s|\|\nabla\mathbf{u}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)}+|s|^2\|\mathbf{u}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)}\right)\\&\qquad+C_\varepsilon\left(|s|^{1/2}\|b_0\|^2_{\mathrm{L}_{2,\mu}(\gamma_0)}+|s|^{1/2}\|\mathbf{b}_\theta\|^2_{\mathbf{L}_{2,\mu}(\gamma_\theta)}\right).\end{aligned}\tag{3.11}$$

For the second term of the right-hand side of (3.9) since

$$|s|\left|\int_{d_\theta}\mathbb{P}(\mathbf{u},q^*)\left(\nabla|x|^{2\mu}\right)\cdot\bar{\mathbf{u}}\,\mathrm{d}x\right|\le c(2\mu)|s|\int_{d_\theta}\left(|q^*|+|\nabla\mathbf{u}|\right)|\mathbf{u}||x|^{2\mu-1}\,\mathrm{d}x,$$

$$\begin{aligned}|s|\int_{d_\theta}|q^*|\,|\mathbf{u}||x|^{2\mu-1}\,\mathrm{d}x&\le|s|\|q^*\|_{\mathrm{L}_{2,\mu-1/2}(d_\theta)}\|\mathbf{u}\|_{\mathbf{L}_{2,\mu-1/2}(d_\theta)}\\&\le\varepsilon|s|^{1/2}\|q^*\|^2_{\mathrm{L}_{2,\mu-1/2}(d_\theta)}+C_\varepsilon|s|^{3/2}\|\mathbf{u}\|^2_{\mathbf{L}_{2,\mu-1/2}(d_\theta)},\end{aligned}$$

$$\begin{aligned}|s|\int_{d_\theta}|\nabla\mathbf{u}||\mathbf{u}||x|^{2\mu-1}\,\mathrm{d}x&\le|s|\|\nabla\mathbf{u}\|_{\mathbf{L}_{2,\mu}(d_\theta)}\|\mathbf{u}\|_{\mathbf{L}_{2,\mu-1}(d_\theta)}\\&\le\varepsilon|s|\|\nabla\mathbf{u}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)}+C_\varepsilon|s|\|\mathbf{u}\|^2_{\mathbf{L}_{2,\mu-1}(d_\theta)},\end{aligned}$$

we have by virtue of (2.6) and (3.8)

$$\begin{aligned}|s|^{3/2}\|\mathbf{u}\|^2_{\mathbf{L}_{2,\mu-1/2}(d_\theta)}&\le|s|^{3/2}\left(C_I\|\nabla\mathbf{u}\|^{1/2-\mu}_{\mathbf{L}_2(d_\theta)}\|\mathbf{u}\|^{1/2+\mu}_{\mathbf{L}_2(d_\theta)}\right)^2\\&=C_I^2\left(|s|^{1-\mu}\|\nabla\mathbf{u}\|^2_{\mathbf{L}_2(d_\theta)}\right)^{1/2-\mu}\left(|s|^{2-\mu}\|\mathbf{u}\|^2_{\mathbf{L}_2(d_\theta)}\right)^{1/2+\mu}\\&\le C_I^2\left(|s|^{1-\mu}\|\nabla\mathbf{u}\|^2_{\mathbf{L}_2(d_\theta)}+|s|^{2-\mu}\|\mathbf{u}\|^2_{\mathbf{L}_2(d_\theta)}\right)\\&\le cF_0,\end{aligned}\tag{3.12}$$

$$\begin{aligned}|s|\|\mathbf{u}\|^2_{\mathbf{L}_{2,\mu-1}(d_\theta)}&\le|s|\left(C_I\|\nabla\mathbf{u}\|^{1-\mu}_{\mathbf{L}_2(d_\theta)}\|\mathbf{u}\|^{\mu}_{\mathbf{L}_2(d_\theta)}\right)^2\\&=C_I^2\left(|s|^{1-\mu}\|\nabla\mathbf{u}\|^2_{\mathbf{L}_2(d_\theta)}\right)^{1-\mu}\left(|s|^{2-\mu}\|\mathbf{u}\|^2_{\mathbf{L}_2(d_\theta)}\right)^{\mu}\\&\le C_I^2\left(|s|^{1-\mu}\|\nabla\mathbf{u}\|^2_{\mathbf{L}_2(d_\theta)}+|s|^{2-\mu}\|\mathbf{u}\|^2_{\mathbf{L}_2(d_\theta)}\right)\\&\le cF_0.\end{aligned}\tag{3.13}$$

Therefore, one can get

$$|s| \left| \int_{d_\theta} \mathbb{P}(\mathbf{u}, q^*) \left(\nabla |x|^{2\mu}\right) \cdot \bar{\mathbf{u}} \, \mathrm{d}x \right| \\ \leq \varepsilon \left( |s|^{1/2} \|q^*\|^2_{\mathrm{L}_{2,\mu-1/2}(d_\theta)} + |s| \|\nabla \mathbf{u}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} \right) + C_\varepsilon F_0. \tag{3.14}$$

Substituting (3.10)–(3.14) into (3.9), we obtain

$$|s| \left( (h + |\Im s|) \|\mathbf{u}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} + 2\nu \|\mathbb{D}(\mathbf{u})\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} \right) \\ \leq \varepsilon \left( |s|^{1/2} \|q^*\|^2_{\mathrm{L}_{2,\mu-1/2}(d_\theta)} + |s| \|\nabla \mathbf{u}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} \right) + C_\varepsilon F_0. \tag{3.15}$$

On the other hand, applying Korn's inequality (2.8) to the function $\mathbf{u}|x|^\mu$, we find

$$\|\nabla \mathbf{u}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} \leq C'_K \left( \|\mathbb{D}(\mathbf{u})\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} + \|\mathbf{u}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} + \|\mathbf{u}\|^2_{\mathbf{L}_{2,\mu-1}(d_\theta)} \right).$$

Consequently, we get

$$(h + |\Im s|) \|\mathbf{u}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} + 2\nu \|\mathbb{D}(\mathbf{u})\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} \\ \geq \frac{1}{2} \left( |s| \|\mathbf{u}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} + \nu \left( \frac{1}{C'_K} \|\nabla \mathbf{u}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} - \|\mathbf{u}\|^2_{\mathbf{L}_{2,\mu-1}(d_\theta)} \right) \right). \tag{3.16}$$

Now we choose $\varepsilon > 0$ so small that (3.12), (3.13) and (3.16) yield

$$|s| \|\nabla \mathbf{u}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} + |s|^2 \|\mathbf{u}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} \leq c\varepsilon |s|^{1/2} \|q^*\|^2_{\mathrm{L}_{2,\mu-1/2}(d_\theta)} + F_0.$$

It is easily derived from (3.2) that

$$\begin{cases} \Delta q^* = 0 \quad \text{in } d_\theta, \\ \left.\dfrac{\partial q^*}{\partial x_2}\right|_{\gamma_0} = -\dfrac{\partial b_0}{\partial x_1}, \quad q^*\big|_{\gamma_\theta} = 2\nu \mathbb{D}(\mathbf{u}) \mathbf{n}_\theta \cdot \mathbf{n}_\theta\big|_{\gamma_\theta} - \mathbf{b}_\theta \cdot \mathbf{n}_\theta. \end{cases} \tag{3.17}$$

According to the result in [5], we have

$$|s|^{1/2} \|q^*\|^2_{\mathrm{L}_{2,\mu-1/2}(d_\theta)} \leq \frac{c}{\cos^2\left(\left(\mu + \frac{1}{2}\right)\theta\right)} \Big( |s|^{1/2} \|b_0\|^2_{\mathrm{L}_{2,\mu}(\gamma_0)} \\ + |s|^{1/2} \|\mathbf{b}_\theta\|^2_{\mathbf{L}_{2,\mu}(\gamma_\theta)} + \nu |s|^{1/2} \|\nabla \mathbf{u}\|^2_{\mathbf{L}_{2,\mu}(\gamma_\theta)} \Big) \\ \leq \frac{c}{\cos^2\left(\left(\mu + \frac{1}{2}\right)\theta\right)} \left( F_0 + \nu |s|^{1/2} \|\nabla \mathbf{u}\|^2_{\mathbf{L}_{2,\mu}(\gamma_\theta)} \right).$$

Note that $\theta \in (0, \pi/2)$ and $\mu \in (0, 1/2)$ imply $\cos\left((\mu + 1/2)\theta\right) \neq 0$. Using the inequality

$$|s|^{1/2} \|\nabla \mathbf{u}\|^2_{\mathbf{L}_{2,\mu}(\gamma_\theta)} \leq C^2_{II} \|\mathrm{D}^2 \mathbf{u}\|_{\mathbf{L}_{2,\mu}(d_\theta)} |s|^{1/2} \|\nabla \mathbf{u}\|_{\mathbf{L}_{2,\mu}(d_\theta)} \\ \leq c \left( \|\mathrm{D}^2 \mathbf{u}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} + |s| \|\nabla \mathbf{u}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} \right), \tag{3.18}$$

we arrive at the estimate

$$|s| \|\nabla \mathbf{u}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} + |s|^2 \|\mathbf{u}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} \leq \varepsilon \|\mathrm{D}^2 \mathbf{u}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} + cF_0. \tag{3.19}$$

Since $(\mathbf{u}, q^*)$ can be considered as a solution of problem

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla q^* = \mathbf{f}^* - s\mathbf{u}, \quad \nabla\cdot\mathbf{u} = 0 \quad \text{in } d_\theta, \\ u_2|_{\gamma_0} = 0, \quad 2\nu D_{12}(\mathbf{u})|_{\gamma_0} = b_0, \quad \mathbb{P}(\mathbf{u}, q^*)\mathbf{n}_\theta\big|_{\gamma_\theta} = \mathbf{b}_\theta, \end{cases} \tag{3.20}$$

Theorem 2.1 for $k = 0$ is applicable to problem (3.20), so that if the line $\Re\lambda = 1-\mu$ is free from the eigenvalues, then

$$\begin{aligned} \|\mathbf{u}\|^2_{\mathbf{H}^2_\mu(d_\theta)} + \|\nabla q^*\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} &\le c_{1,0}\left(\|\mathbf{f}^* - s\mathbf{u}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} + \|b_0\|^2_{\mathrm{H}^{1/2}_\mu(\gamma_0)} + \|\mathbf{b}_\theta\|^2_{\mathbf{H}^{1/2}_\mu(\gamma_\theta)}\right) \\ &\le c_{1,0}\left(|s|^2\|\mathbf{u}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} + F_0\right). \end{aligned} \tag{3.21}$$

We conclude from (3.19) and (3.21) that

$$\|\mathbf{u}\|^2_{\mathbf{H}^2_\mu(d_\theta)} + |s|\|\nabla\mathbf{u}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} + |s|^2\|\mathbf{u}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} + \|\nabla q^*\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} \le cF_0. \tag{3.22}$$

Furthermore, (3.12) and (3.22) imply

$$|s|\|\mathbf{u}\|^2_{\mathbf{H}^1_\mu(d_\theta)} = |s|\left(\|\mathbf{u}\|^2_{\mathbf{L}_{2,\mu-1}(d_\theta)} + \|\nabla\mathbf{u}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)}\right) \le cF_0,$$

and hence

$$\|\mathbf{u}\|^2_{\mathbf{H}^2_\mu(d_\theta)} + |s|\|\mathbf{u}\|^2_{\mathbf{H}^1_\mu(d_\theta)} + |s|^2\|\mathbf{u}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} + \|\nabla q^*\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} \le cF_0. \tag{3.23}$$

Finally let us proceed to estimate the pressure $q^*$ on $\gamma_\theta$. The boundary condition leads to

$$q^*\big|_{\gamma_\theta} = 2\nu\mathbb{D}(\mathbf{u})\mathbf{n}_\theta\cdot\mathbf{n}_\theta\big|_{\gamma_\theta} - \mathbf{b}_\theta\cdot\mathbf{n}_\theta. \tag{3.24}$$

From the trace theorem and (3.18) it follows that

$$\|\mathbb{D}(\mathbf{u})\mathbf{n}_\theta\cdot\mathbf{n}_\theta\|^2_{\mathbf{H}^{1/2}_\mu(\gamma_\theta)} + |s|^{1/2}\|\mathbb{D}(\mathbf{u})\mathbf{n}_\theta\cdot\mathbf{n}_\theta\|^2_{\mathbf{L}_{2,\mu}(\gamma_\theta)} \le c\left(\|\mathbf{u}\|^2_{\mathbf{H}^2_\mu(d_\theta)} + |s|\|\mathbf{u}\|^2_{\mathbf{H}^1_\mu(d_\theta)}\right).$$

Hence

$$\|q^*\|^2_{\mathrm{H}^{1/2}_\mu(\gamma_\theta)} + |s|^{1/2}\|q^*\|^2_{\mathrm{L}_{2,\mu}(\gamma_\theta)} \le cF_0. \tag{3.25}$$

Consequently, (3.3) is proved thanks to the inequalities (3.23) and (3.25).

*Step* 1-2. Next we discuss the existence of a solution. First let us find a weak solution of it (see [1], [9], [10]). By a weak solution we mean $\mathbf{u} \in J(d_\theta) \equiv \{\mathbf{u} \in \mathbf{W}^1_2(d_\theta) \mid \nabla\cdot\mathbf{u} = 0 \text{ in } d_\theta,\ u_2|_{\gamma_0} = 0\}$ satisfying the integral identity $Q[\mathbf{u}, \boldsymbol{\varphi}] = l(\boldsymbol{\varphi})$ for all $\boldsymbol{\varphi} \in J(d_\theta)$, where

$$\begin{aligned} Q[\mathbf{u}, \boldsymbol{\varphi}] &= s\int_{d_\theta}\mathbf{u}\cdot\bar{\boldsymbol{\varphi}}\,\mathrm{d}x + 2\nu\int_{d_\theta}\mathbb{D}(\mathbf{u}):\mathbb{D}(\bar{\boldsymbol{\varphi}})\,\mathrm{d}x, \\ l(\boldsymbol{\varphi}) &= \int_{d_\theta}\mathbf{f}^*\cdot\bar{\boldsymbol{\varphi}}\,\mathrm{d}x - \int_{\gamma_0}b_0\bar{\varphi}_1\,\mathrm{d}r + \int_{\gamma_\theta}\mathbf{b}_\theta\cdot\bar{\boldsymbol{\varphi}}\,\mathrm{d}r. \end{aligned}$$

**Lemma 3.1.** *Suppose that* $\mathbf{f}^*$, $b_0$ *and* $\mathbf{b}_\theta$ *satisfy*

$$F_0^* \equiv \|\mathbf{f}^*\|_{\mathbf{L}_{2,\mu}(d_\theta)} + \|b_0\|_{\mathrm{H}^{1/2}_\mu(\gamma_0)} + \|b_0\|_{\mathrm{L}_{2,\mu}(\gamma_0)} + \|\mathbf{b}_\theta\|_{\mathbf{H}^{1/2}_\mu(\gamma_\theta)} + \|\mathbf{b}_\theta\|_{\mathbf{L}_{2,\mu}(\gamma_\theta)} < \infty.$$

*Then for each* $s \in \mathbb{C}$, $\Re s = h \geq \nu$, *there exists a unique weak solution* $\mathbf{u} \in J(d_\theta)$ *such that*

$$\|\mathbf{u}\|_{\mathbf{W}^1_2(d_\theta)} \leq cF_0^*.$$

*Proof.* Since

$$\begin{aligned} |Q[\mathbf{u},\boldsymbol{\varphi}]| &\leq (|s|+2\nu)\|\mathbf{u}\|_{\mathbf{W}^1_2(d_\theta)}\|\boldsymbol{\varphi}\|_{\mathbf{W}^1_2(d_\theta)}, \\ |\Re Q[\mathbf{u},\mathbf{u}]| &\geq \frac{\nu}{2}\|\mathbf{u}\|^2_{\mathbf{L}_2(d_\theta)} + \frac{\nu}{2}\left(\|\mathbf{u}\|^2_{\mathbf{L}_2(d_\theta)} + \|\mathbb{D}(\mathbf{u})\|^2_{\mathbf{L}_2(d_\theta)}\right) \\ &\geq \frac{\nu}{2}\min\left\{1, \frac{1}{C_K}\right\}\|\mathbf{u}\|^2_{\mathbf{W}^1_2(d_\theta)}, \end{aligned}$$

the sesqui-linear form $Q[\cdot,\cdot]$ is bounded and coercive. While it is obvious from (3.5) and (3.6) that $l\,(\boldsymbol{\varphi})$ is a linear continuous functional on $J(d_\theta)$. Thus the Lax-Milgram Theorem ([7]) leads to the assertion of Lemma 3.1. □

**Remark**. From (2.6) with $\mu_1 = 0$ one can deduce

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{W}^1_2(d_\theta)} &\leq C_I\left(\|\nabla\mathbf{u}\|^\mu_{\mathbf{L}_{2,\mu}(d_\theta)}\|\mathbf{u}\|^{1-\mu}_{\mathbf{L}_{2,\mu}(d_\theta)} + \|\mathrm{D}^2\mathbf{u}\|^\mu_{\mathbf{L}_{2,\mu}(d_\theta)}\|\nabla\mathbf{u}\|^{1-\mu}_{\mathbf{L}_{2,\mu}(d_\theta)}\right) \\ &\leq c\left(\|\mathbf{u}\|_{\mathbf{H}^2_\mu(d_\theta)} + \left(\frac{|s|}{\nu}\right)^{1/2}\|\mathbf{u}\|_{\mathbf{H}^1_\mu(d_\theta)} + \frac{|s|}{\nu}\|\mathbf{u}\|_{\mathbf{L}_{2,\mu}(d_\theta)}\right). \end{aligned}$$

Hence the weak solution obtained in Lemma 3.1 obeys the estimate (3.3).

Since the pressure $q^*$ satisfies (3.17), the existence and the estimate of $\nabla q^* \in \mathbf{H}^1_\mu(d_\theta)$ follow from Theorem 2.2 provided that $1-\mu \neq (m+1/2)\pi/\theta,\ m \in \mathbb{Z}$.

*Step* 1-3. We derive the inequality (1.2) with $\rho = 0$. Firstly inequality (3.3) multiplied by $|s|$ yields

$$\begin{aligned} |s|\|\mathbf{u}\|^2_{\mathbf{H}^2_\mu(d_\theta)} + |s|^2\|\mathbf{u}\|^2_{\mathbf{H}^1_\mu(d_\theta)} + |s|^3\|\mathbf{u}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} + |s|\|\nabla q^*\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} \\ + |s|\|q^*\|^2_{\mathrm{H}^{1/2}_\mu(\gamma_\theta)} + |s|^{3/2}\|q^*\|^2_{\mathrm{L}_{2,\mu}(\gamma_\theta)} \leq cF_1. \end{aligned} \tag{3.26}$$

Secondly, applying Theorem 2.1 for $k=1$ to problem (3.20), we have

$$\|\mathbf{u}\|^2_{\mathbf{H}^3_\mu(d_\theta)} + \|\nabla q^*\|^2_{\mathrm{H}^1_\mu(d_\theta)} \leq c_{1,1}\left(|s|^2\|\mathbf{u}\|^2_{\mathbf{H}^1_\mu(d_\theta)} + F_1\right) \leq cF_1 \tag{3.27}$$

under the condition that there are no eigenvalues on the line $\Re\lambda = 2-\mu$. Estimating $\|q^*\|^2_{\mathrm{H}^{3/2}_\mu(\gamma_\theta)}$ and $|s|^{1/2}\|q^*\|^2_{\mathrm{L}^1_\mu(\gamma_\theta)}$ from (3.24) together with (3.26), (3.27) implies (1.2) with $k=1$ and $\rho=0$. Repeating this procedure, we obtain the desired inequality.

*Step* 2. We discuss problem (1.1). Let us seek a solution of (1.1) in the form $(\mathbf{u}, q) = (\nabla\phi + \mathbf{u}^{**}, -s\phi + q^{**})$, where $\phi$ satisfies

$$\Delta\phi = \rho \quad \text{in } d_\theta, \quad \frac{\partial\phi}{\partial x_2}\bigg|_{\gamma_0} = 0, \quad \phi\big|_{\gamma_\theta} = 0, \tag{3.28}$$

and $(\mathbf{u}^{**}, q^{**})$ solves problem (3.2) with $(\mathbf{f}^*, b_0, \mathbf{b}_\theta)$ replaced by $(\mathbf{f}^* + \nu\nabla\rho, b_0 - 2\nu D_{12}(\nabla\phi)|_{\gamma_0}, \mathbf{b}_\theta - 2\nu\mathbb{D}(\nabla\phi)\mathbf{n}_\theta|_{\gamma_\theta})$.

As for problem (3.28), we have the following result.

**Lemma 3.2.** *Assume that* $k + 1 - \mu \neq (m + 1/2)\pi/\theta$ $(k = 0, 1;\ m \in \mathbb{Z})$ *and* $\rho = \nabla\cdot\mathbf{g}$ *be as in Theorem* 1.1 *with* $G_0 < \infty$. *Then there exists a unique solution* $\nabla\phi(\cdot, s) \in \mathbf{H}^2_\mu(d_\theta)$ *of* (3.28) *which obeys the estimate*

$$\|\nabla\phi(\cdot, s)\|^2_{\mathbf{H}^2_\mu(d_\theta)} + |s|\|\nabla\phi(\cdot, s)\|^2_{\mathbf{H}^1_\mu(d_\theta)} + |s|^2\|\nabla\phi(\cdot, s)\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} \leq c\,G_0, \tag{3.29}$$

*where* $c$ *is a constant independent of* $|s|$.

*Proof.* Theorem 2.2 for $k = 0, 1$ implies the existence of a solution and the estimates of $\|\nabla\phi\|^2_{\mathbf{H}^2_\mu(d_\theta)}$ and $|s|\|\nabla\phi\|^2_{\mathbf{H}^1_\mu(d_\theta)}$. Moreover, it follows from the result in [5] that

$$|s|^2\|\nabla\phi\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} \leq c|s|^2\|\mathbf{g}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)},$$

which completes the proof of (3.29). □

The estimate of $\nabla\phi$ for general $k$ follows from the similar arguments as those in *Step* 1.3 with the aid of Theorem 2.2. Therefore Theorem 1.1 is completely proved.

We would like to thank Professor T. Iguchi for helpful discussions especially on the proof of Lemma 2.1.

## Appendix: Proof of Lemma 2.1.

Integrating by parts with respect to $r$, we have

$$\begin{aligned}
\|u\|^2_{\mathrm{L}_{2,(\mu_1+\mu-1)/2}(d_\theta)} &= \int_{d_\theta} |u|^2\, r^{\mu_1+\mu-1}\, \mathrm{d}x = \int_0^\theta \int_0^\infty |u|^2\, r^{\mu_1+\mu}\, \mathrm{d}r\, \mathrm{d}\varphi \\
&= -\frac{1}{\mu_1+\mu+1}\int_0^\theta \int_0^\infty 2uu_r r^{\mu_1+\mu+1}\, \mathrm{d}r\, \mathrm{d}\varphi \\
&\leq \left|\frac{2}{\mu_1+\mu+1}\right| \int_{d_\theta} |u||\nabla u|\, r^{\mu_1+\mu}\, \mathrm{d}x \\
&\leq \left|\frac{2}{\mu_1+\mu+1}\right| \|u\|_{\mathrm{L}_{2,\mu_1}(d_\theta)}\|\nabla u\|_{\mathbf{L}_{2,\mu}(d_\theta)}
\end{aligned} \tag{A.1}$$

provided $\mu_1 + \mu + 1 \neq 0$. When $\mu - \mu_1 = 1$, (A.1) means the desired inequality (2.6) itself. The case $\mu - \mu_1 = 0$ is obvious, so that it is sufficient to consider the case $0 < \mu - \mu_1 < 1$. By Hölder's inequality we get

$$\begin{aligned}\|u\|^2_{\mathrm{L}_{2,\mu_1}(d_\theta)} &= \int_{d_\theta} |u|^{2\alpha}\, r^{2(\mu_1-\mu)+2\mu\alpha}\, |u|^{2(1-\alpha)}\, r^{2\mu(1-\alpha)}\ \mathrm{d}x \\ &\le \left(\int_{d_\theta} |u|^2\, r^{\frac{2(\mu_1-\mu)}{\alpha}+2\mu}\ \mathrm{d}x\right)^{\alpha} \left(\int_{d_\theta} |u|^2\, r^{2\mu}\ \mathrm{d}x\right)^{1-\alpha}\end{aligned}$$

for $\alpha \in (0,1)$. Now we choose $\alpha = 2(\mu-\mu_1)/(\mu-\mu_1+1)$. It is obvious that this $\alpha$ really belongs to $(0,1)$ for $0 < \mu - \mu_1 < 1$. Then we have

$$\|u\|^2_{\mathrm{L}_{2,\mu_1}(d_\theta)} \le \|u\|^{2\alpha}_{\mathrm{L}_{2,(\mu_1+\mu-1)/2}(d_\theta)} \|u\|^{2(1-\alpha)}_{\mathrm{L}_{2,\mu}(d_\theta)}. \tag{A.2}$$

Substituting (A.1) into (A.2), we get

$$\|u\|^2_{\mathrm{L}_{2,\mu_1}(d_\theta)} \le \left|\frac{2}{\mu_1+\mu+1}\right|^{\alpha} \|u\|^{\alpha}_{\mathrm{L}_{2,\mu_1}(d_\theta)} \|\nabla u\|^{\alpha}_{\mathbf{L}_{2,\mu}(d_\theta)} \|u\|^{2(1-\alpha)}_{\mathrm{L}_{2,\mu}(d_\theta)},$$

and hence

$$\|u\|_{\mathrm{L}_{2,\mu_1}(d_\theta)} \le \left|\frac{2}{\mu_1+\mu+1}\right|^{\frac{\alpha}{2-\alpha}} \|\nabla u\|^{\frac{\alpha}{2-\alpha}}_{\mathbf{L}_{2,\mu}(d_\theta)} \|u\|^{\frac{2(1-\alpha)}{2-\alpha}}_{\mathrm{L}_{2,\mu}(d_\theta)}.$$

This inequality coincides with (2.6).

For (2.7) we start with the equality

$$u^2(r,\varphi) = u^2(r,0) + \int_0^{\varphi} \frac{\partial}{\partial\varphi'} u^2(r,\varphi')\ \mathrm{d}\varphi', \quad \varphi \in (0,\theta).$$

This gives

$$u^2(r,0) \le u^2(r,\varphi) + 2\int_0^{\theta} |u||u_\varphi|\ \mathrm{d}\varphi.$$

Multiplying both sides by $r^{2\mu_1}$ and integrating with respect to $r \in (0,\infty)$ and $\varphi \in (0,\theta)$ yields

$$\begin{aligned}\theta\|u\|^2_{\mathrm{L}_{2,\mu_1}(\gamma)} &\le \int_0^{\theta}\int_0^{\infty} |u|^2\, r^{2\mu_1}\ \mathrm{d}r\,\mathrm{d}\varphi + 2\theta\int_0^{\theta}\int_0^{\infty} |u||u_\varphi|\, r^{2\mu_1}\ \mathrm{d}r\,\mathrm{d}\varphi \\ &\le \int_{d_\theta} |u|^2\, r^{2\mu_1-1}\ \mathrm{d}x + 2\theta\int_{d_\theta} |u||\nabla u|\, r^{2\mu_1}\ \mathrm{d}x.\end{aligned}$$

For the first term in the right-most-hand side we integrate by parts with respect to $r$ as in the proof of (A.1), so that

$$\int_{d_\theta} |u|^2\, r^{2\mu_1-1}\ \mathrm{d}x \le \left|\frac{2}{1+2\mu_1}\right| \int_{d_\theta} |u||\nabla u|\, r^{2\mu_1}\ \mathrm{d}x.$$

Therefore, we have

$$\begin{aligned}\|u\|^2_{\mathrm{L}_{2,\mu_1}(\gamma)} &\le \left(\left|\frac{2}{(1+2\mu_1)\theta}\right| + 2\right) \int_{d_\theta} |u||\nabla u|\, r^{2\mu_1}\ \mathrm{d}x \\ &\le \left(\left|\frac{2}{(1+2\mu_1)\theta}\right| + 2\right) \|u\|_{\mathrm{L}_{2,2\mu_1-\mu}(d_\theta)} \|\nabla u\|_{\mathbf{L}_{2,\mu}(d_\theta)}.\end{aligned} \tag{A.3}$$

It remains only to make use of (2.6) with $\mu_1$ replaced by $2\mu_1 - \mu$. Then we arrive at

$$\|u\|_{\mathrm{L}_{2,2\mu_1-\mu}(d_\theta)} \le \left|\frac{2}{2\mu_1+1}\right|^{2(\mu-\mu_1)} \|\nabla u\|^{2(\mu-\mu_1)}_{\mathbf{L}_{2,\mu}(d_\theta)} \|u\|^{1-2(\mu-\mu_1)}_{\mathrm{L}_{2,\mu}(d_\theta)}. \tag{A.4}$$

Substituting (A.4) into (A.3) gives (2.7).

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# Unilateral Contact Problems Between an Elastic Plate and a Beam

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**Abstract.** It is well known that crack problems are formulated in domains with cuts. Since the beginning of 1990, the crack theory with inequality type boundary conditions, imposed at the crack faces, has been under active study. These boundary conditions describe a mutual non-penetration between crack faces. The models obtained are non-linear. From a mechanical standpoint, the non-linear models are more preferable as compared to linear ones. It turned out that contact problems for bodies of different dimensions are also described in non-smooth domains with inequality type boundary conditions imposed on sets of small dimensions. In particular, to describe a unilateral contact between elastic plates and beams we have to consider a cracked domain or a domain with a removed point. In both cases the main difficulties are related to non-smoothness of the domain. In the present paper we discuss two problems describing a unilateral contact between an elastic plate and a beam.

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## 1. Case of inclined beam

In this section we consider a contact problem between an elastic plate and an inclined elastic beam. Let $\Omega \subset R^2$ be a bounded domain with smooth boundary $\Gamma$. We assume that $\Omega$ corresponds to the middle surface of the plate. A unit external normal vector to $\Gamma$ is denoted by $q = (q_1, q_2)$. The beam is situated at angle $\alpha \in (0, \frac{\pi}{2}]$ with respect to the plate. The plane $x_1 0 x_2$ is orthogonal to the plane $y_1 0 y_2$. The beam has both vertical $v$ and tangential $u$ displacements along axes $x_2, x_1$, respectively. The plate has only the vertical displacement $w$. Let $(0, 0) \in \Omega$ be a point of possible contact between the plate and the beam. A middle line of the beam is denoted by $\gamma$. We assume that $\gamma$ is the interval $(0, 1)$, and the point $x = 0$ is a contact one for the beam. Here and in what follows $x_1$ is denoted by $x$.

The end point $x = 1$ of the beam is clamped. The boundary $\Gamma$ of the plate is also clamped.

We write

$$m(v) = v_{,ij}q_jq_i, \;\; t^q(v) = v_{,ijk}s_ks_jq_i + v_{,ijj}q_i, \;\; (s_1, s_2) = (-q_2, q_1),$$
$$v_{,i} = \frac{\partial v}{\partial y_i}, \; i = 1, 2, \; (y_1, y_2) \in \Omega.$$

First of all we recall Green's formula. To this end, introduce the space

$$V = \{u \in H^2(\Omega) \mid \Delta^2 u \in L^2(\Omega)\}.$$

Then for $u \in V$ we can define $m(u) \in H^{-1/2}(\Gamma)$, $t^q(u) \in H^{-3/2}(\Gamma)$, and the following formula holds ([5], [8])

$$\int\limits_\Omega \psi \Delta^2 u = \int\limits_\Omega \psi_{,ij}u_{,ij} + \langle t^q(u), \psi\rangle_{3/2,\Gamma} - \langle m(u), \psi_q\rangle_{1/2,\Gamma}, \;\; \forall \psi \in H^2(\Omega). \tag{1}$$

Here $\langle\cdot,\cdot\rangle_{i/2,\Gamma}$ means the duality pairing between the space $H^{-i/2}(\Gamma)$ and its dual $H^{i/2}(\Gamma)$, $i = 1, 3$.

We start with a variational formulation of the problem. Consider the Sobolev spaces

$$\tilde{H}^1(\gamma) = \{u \in H^1(\gamma) \mid u = 0 \;\text{ at }\; x = 1\},$$
$$\tilde{H}^2(\gamma) = \{v \in H^2(\gamma) \mid v = v_x = 0 \;\text{ at }\; x = 1\}$$

and introduce the energy functional on the space $H = \tilde{H}^1(\gamma) \times \tilde{H}^2(\gamma) \times H_0^2(\Omega)$,

$$\Pi(u, v, w) = \frac{1}{2}\int\limits_\gamma u_x^2 - \int\limits_\gamma hu + \frac{1}{2}\int\limits_\gamma v_{xx}^2 - \int\limits_\gamma gv + \int\limits_\Omega w_{,ij}w_{,ij} - \int\limits_\Omega fw,$$

where $f \in L^2(\Omega), h, g \in L^2(\gamma)$ are given functions. Consider the set of admissible displacements

$$K = \{(u, v, w) \in H \mid u(0)\sin\alpha + v(0)\cos\alpha \geq w(0)\}.$$

We can find a unique solution of the minimization problem

$$\inf_{(u,v,w)\in K} \Pi(u, v, w). \tag{2}$$

The solution of this problem satisfies the variational inequality

$$(u, v, w) \in K, \tag{3}$$

$$\int\limits_\gamma \{u_x(\bar{u}_x - u_x) - h(\bar{u} - u)\} + \int\limits_\gamma \{v_{xx}(\bar{v}_{xx} - v_{xx}) - g(\bar{v} - v)\} \tag{4}$$
$$+ \int\limits_\Omega w_{,ij}(\bar{w}_{,ij} - w_{,ij}) - \int\limits_\Omega f(\bar{w} - w) \geq 0, \quad \forall (\bar{u}, \bar{v}, \bar{w}) \in K.$$

It is possible to give a differential formulation of the problem (3), (4). To this end, choose a closed curve $\Sigma$ of the class $C^{1,1}, \Sigma \subset \Omega$, such that $0 \in \Sigma$. Denote by $\nu = (\nu_1, \nu_2)$ a unit normal vector to the curve $\Sigma$. In this case the domain $\Omega$ is divided into two subdomains $\Omega_1$ and $\Omega_2$ with boundaries $\Sigma$ and $\Sigma \cup \Gamma$, respectively. Assume that $\nu$ is oriented towards $\Omega_2$. Denote $\Omega_0 = \Omega \setminus \{0\}$. We have to find functions $u(x), v(x), w(y), x \in \gamma, y = (y_1, y_2) \in \Omega_0$, such that

$$-u_{xx} = h \text{ in } \gamma, \tag{5}$$

$$v_{xxxx} = g \text{ in } \gamma, \tag{6}$$

$$\Delta^2 w = f \text{ in } \Omega_0, \tag{7}$$

$$w = w_q = 0 \text{ on } \Gamma, \tag{8}$$

$$u = v = v_x = 0 \text{ at } x = 1, \; v_{xx} = 0 \text{ at } x = 0, \tag{9}$$

$$u(0) \sin\alpha + v(0) \cos\alpha \geq w(0), \; u_x(0) \cos\alpha = -v_{xxx}(0) \sin\alpha, \tag{10}$$

$$u_x(0)(w(0) - u(0)\sin\alpha - v(0)\cos\alpha) = 0, \; u_x(0) \leq 0, \tag{11}$$

$$[m(w)] = 0, \; [t^\nu(w)] = \frac{1}{\sin\alpha} u_x(0) \delta_\Sigma \quad \text{on } \Sigma. \tag{12}$$

Here $\delta_\Sigma$ is a distribution on $\Sigma$ defined by the formula $\delta_\Sigma(\xi) = \xi(0)$. It is important that $\Sigma$ is an arbitrary curve with the above properties.

Now we derive relations (5)–(12) from the variational inequality (3), (4) and demonstrate in what sense boundary conditions (12) are fulfilled. First note that equilibrium equations (5)–(7) follow from (3), (4) in the distributional sense.

We can next take test functions $(\bar{u}, \bar{v}, \bar{w}) = (u, v, w + \varphi)$ in (4), $\varphi \in H_0^2(\Omega)$, $\varphi(0) \leq 0$. This provides the inequality

$$\int_\Omega w_{,ij} \varphi_{,ij} - \int_\Omega f\varphi \geq 0.$$

Applying the Green's formula like (1) for the subdomains $\Omega_1, \Omega_2$, and taking into account (5)–(8), we derive

$$-\langle [m(w)], \varphi_\nu \rangle_{1/2,\Sigma} + \langle [t^\nu(w)], \varphi \rangle_{3/2,\Sigma} \geq 0, \quad \forall \varphi \in H_0^2(\Omega), \; \varphi(0) \leq 0$$

which gives

$$[m(w)] = 0 \text{ in the sense of } H^{-1/2}(\Sigma), \tag{13}$$

$$\langle [t^\nu(w)], \varphi \rangle_{3/2,\Sigma} \geq 0, \quad \forall \varphi \in H_0^2(\Omega), \quad \varphi(0) \leq 0. \tag{14}$$

From (14) it follows that $\langle [t^\nu(w)], \varphi \rangle_{3/2,\Sigma} = 0$ provided that $\varphi(0) = 0$. Hence the value $\langle [t^\nu(w)], \varphi \rangle_{3/2,\Sigma}$ depends only on $\varphi(0)$, and consequently

$$\langle [t^\nu(w)], \varphi \rangle_{3/2,\Sigma} = k\varphi(0), \; k = \text{const.} \tag{15}$$

The constant $k$ will be found below.

Substitute next in (4) test functions of the form $(\bar{u}, \bar{v}, \bar{w}) = (u+\psi, v, w)$, where $\psi \in \tilde{H}^1(\gamma), \psi(0) \geq 0$. We obtain

$$\int_\gamma u_x \psi_x - \int_\gamma h\psi \geq 0.$$

Whence, by (5), the inequality $u_x(0) \leq 0$ follows.

Now we substitute in (4) test functions of the form $(\bar{u}, \bar{v}, \bar{w}) = (u, v+\xi, w)$, $\xi \in \tilde{H}^2(\gamma), \xi(0) \geq 0$. It implies

$$\int_\gamma v_{xx}\xi_{xx} - \int_\gamma g\xi \geq 0,$$

hence, by (6), the inequality

$$v_{xxx}(0)\xi(0) - v_{xx}(0)\xi_x(0) \geq 0$$

follows which, in its own turn, implies $v_{xxx}(0) \geq 0,\ v_{xx}(0) = 0$.

The next step consists in substituting $(\bar{u}, \bar{v}, \bar{w}) = (u, v, w) \pm (\psi, \xi, \varphi)$ as test functions in (4), where $(\psi, \xi, \varphi) \in H,\ \psi(0)\sin\alpha + \xi(0)\cos\alpha = \varphi(0)$. This substitution provides the identity

$$\int_\gamma u_x\psi_x - \int_\gamma h\psi + \int_\gamma v_{xx}\xi_{xx} - \int_\gamma g\xi + \int_\Omega w_{,ij}\varphi_{,ij} - \int_\Omega f\varphi = 0.$$

Integrating by parts here, in view of (1), (13), (5)–(8), we derive

$$\langle [t^\nu(w)], \varphi\rangle_{3/2,\Sigma} + u_x\psi|_0^1 - v_{xxx}\xi|_0^1 = 0$$

or, by (15),

$$k\varphi(0) - u_x(0)\psi(0) + v_{xxx}(0)\xi(0) = 0. \tag{16}$$

This relation holds for all $(\psi, \xi, \varphi) \in H, \psi(0)\sin\alpha + \xi(0)\cos\alpha = \varphi(0)$. Take $\psi(0) = 0$ in (16). In this case $\xi(0)\cos\alpha = \varphi(0)$. It gives $k\varphi(0) + v_{xxx}(0)\xi(0) = 0$, hence

$$k\cos\alpha = -v_{xxx}(0). \tag{17}$$

If we take $\xi(0) = 0$, i.e., $\psi(0)\sin\alpha = \varphi(0)$, the equality (16) implies

$$k = \frac{1}{\sin\alpha} u_x(0). \tag{18}$$

Relations (17), (18) prove the second equality (10). Moreover, by (15), we have

$$[t^\nu(w)] = \frac{1}{\sin\alpha} u_x(0)\delta_\Sigma.$$

Now substitute $(\bar{u}, \bar{v}, \bar{w}) = (u, v, w) + (\psi, \xi, \varphi)$ as test functions in (4), $(\psi, \xi, \varphi) \in K$. It implies the inequality

$$\int_\gamma u_x\psi_x - \int_\gamma h\psi + \int_\gamma v_{xx}\xi_{xx} - \int_\gamma g\xi + \int_\Omega w_{,ij}\varphi_{,ij} - \int_\Omega f\varphi \geq 0 \tag{19}$$

which, by (1), (13), (5)–(8), can be rewritten as

$$\langle [t^\nu(w)], \varphi\rangle_{3/2,\Sigma} - u_x(0)\psi(0) + v_{xxx}(0)\xi(0) \geq 0, \ \ \forall(\psi,\xi,\varphi) \in K. \tag{20}$$

Inequality (20) together with (13) and (22) below contains full information on boundary conditions (9)–(12).

Substitutions of $(\bar{u},\bar{v},\bar{w}) = (0,0,0), (\bar{u},\bar{v},\bar{w}) = 2(u,v,w)$ in (4) yield

$$kw(0) - u_x(0)u(0) + v_{xxx}(0)v(0) = 0. \tag{21}$$

The constant $k$ can be taken from (18) and substituted in (21). This gives

$$u_x(0)w(0) - \sin\alpha u_x(0)u(0) + \sin\alpha v_{xxx}(0)v(0) = 0.$$

Meanwhile, we have $u_x(0)\cos\alpha = -v_{xxx}(0)\sin\alpha$, whence,

$$u_x(0)(w(0) - u(0)\sin\alpha - v(0)\cos\alpha) = 0 \tag{22}$$

which coincides with the first relation of (11). To summarize, we see that all relations (5)–(12) are derived from (3), (4).

In fact, we can give one more differential formulation of the problem (3), (4) equivalent to (5)–(12). In this case instead of $\Omega_0$ the smooth domain $\Omega$ is used. Namely, we have to find functions $u(x), v(x),\ w(y),\ x \in \gamma, y = (y_1,y_2) \in \Omega$, such that

$$-u_{xx} = h \ \text{ in } \ \gamma, \tag{23}$$

$$v_{xxxx} = g \ \text{ in } \ \gamma, \tag{24}$$

$$\Delta^2 w = f + \frac{1}{\sin\alpha}u_x(0)\delta_0 \ \text{ in } \ \Omega, \tag{25}$$

$$w = w_q = 0 \ \text{ on } \ \Gamma, \tag{26}$$

$$u = v = v_x = 0 \ \text{ at } \ x = 1, \ v_{xx} = 0 \ \text{ at } \ x = 0, \tag{27}$$

$$u(0)\sin\alpha + v(0)\cos\alpha \geq w(0), \ u_x(0)\cos\alpha = -v_{xxx}(0)\sin\alpha, \tag{28}$$

$$u_x(0)(w(0) - u(0)\sin\alpha - v(0)\cos\alpha) = 0, \ u_x(0) \leq 0, \tag{29}$$

where $\delta_0$ is the Dirac measure, ie., $\delta_0(\xi) = \xi(0)$, $\xi \in C_0^\infty(\Omega)$.

To derive (25) we can take $\psi(0)\sin\alpha + \xi(0)\cos\alpha = \varphi(0)$ in (19), $(\psi,\xi,\varphi) \in H$. This implies

$$\int_\gamma u_x\psi_x - \int_\gamma h\psi + \int_\gamma v_{xx}\xi_{xx} - \int_\gamma g\xi + \int_\Omega w_{,ij}\varphi_{,ij} - \int_\Omega f\varphi = 0. \tag{30}$$

Integrating by parts in (30) we obtain

$$\int_\Omega w_{,ij}\varphi_{,ij} - \int_\Omega f\varphi - u_x(0)\psi(0) + v_{xxx}(0)\xi(0) = 0. \tag{31}$$

Since

$$\psi(0) = \frac{\varphi(0) - \xi(0)\cos\alpha}{\sin\alpha}, \ \ v_{xxx}(0) = -\frac{u_x(0)\cos\alpha}{\sin\alpha},$$

we arrive at the relation

$$\int_\Omega w_{,ij}\varphi_{,ij} - \int_\Omega f\varphi = \frac{u_x(0)\varphi(0)}{\sin\alpha}, \quad \forall \varphi \in C_0^\infty(\Omega)$$

which means a fulfillment of (25).

On the other hand, variational inequality (3)–(4) can be derived from (23)–(29) as well as from (5)–(12).

In fact, the model (5)–(12) should contain a number of parameters. In what follows we consider a passage to the limit when a rigidity of a contacting body goes to infinity. To specify a case, assume that a rigidity of the inclined beam is increasing. In this case instead of (23), (24) the following equilibrium equations are assumed to be fulfilled:

$$-\frac{1}{\lambda}u_{xx} = h, \quad \frac{1}{\lambda}v_{xxxx} = g \text{ in } \gamma$$

with a positive parameter $\lambda$. We are interesting in a passage to the limit as $\lambda \to 0$.

First, consider the variational formulation of this problem. We have to find functions $u^\lambda, v^\lambda, w^\lambda$ such that

$$(u^\lambda, v^\lambda, w^\lambda) \in K, \tag{32}$$

$$\int_\gamma \Big\{\frac{1}{\lambda}u_x^\lambda(\bar u_x - u_x^\lambda) - h(\bar u - u^\lambda)\Big\} + \int_\gamma \Big\{\frac{1}{\lambda}v_{xx}^\lambda(\bar v_{xx} - v_{xx}^\lambda) - g(\bar v - v^\lambda)\Big\}$$

$$+ \int_\Omega w_{,ij}^\lambda(\bar w_{,ij} - w_{,ij}^\lambda) - \int_\Omega f(\bar w - w^\lambda) \ge 0, \quad \forall (\bar u, \bar v, \bar w) \in K. \tag{33}$$

Taking in (33) test functions of the form $\bar u,\bar v,\bar w) = (0,0,0), (\bar u,\bar v,\bar w) = 2(u^\lambda, v^\lambda, w^\lambda)$ we derive

$$\frac{1}{\lambda}\int_\gamma (u_x^\lambda)^2 - \int_\gamma hu^\lambda + \frac{1}{\lambda}\int_\gamma (v_{xx}^\lambda)^2 - \int_\gamma gv^\lambda + \int_\Omega w_{,ij}^\lambda w_{,ij}^\lambda - \int_\Omega fw^\lambda = 0. \tag{34}$$

Hence the following estimate holds:

$$\frac{1}{\lambda}\|u^\lambda\|^2_{\tilde H^1(\gamma)} + \frac{1}{\lambda}\|v^\lambda\|^2_{\tilde H^2(\gamma)} + \|w^\lambda\|^2_{H_0^2(\Omega)} \le c$$

with a constant $c$ independent of $\lambda$. We can assume that as $\lambda \to 0$ the following convergence takes place:

$$w^\lambda \to w^0 \text{ weakly in } H_0^2(\Omega), \tag{35}$$

$$v^\lambda \to 0 \text{ strongly in } \tilde H^2(\gamma), \tag{36}$$

$$u^\lambda \to 0 \text{ strongly in } \tilde H^1(\gamma). \tag{37}$$

Note that the limit function $w^0$ satisfies the condition

$$w^0(0) \le 0.$$

Let us take test functions of the form $(0, 0, \bar{w})$ in (33), $\bar{w} \in H_0^2(\Omega), \bar{w}(0) \le 0$. We obtain

$$\int_\Omega w^\lambda_{,ij}(\bar{w}_{,ij} - w^\lambda_{,ij}) - \int_\Omega f(\bar{w} - w^\lambda)$$
$$\ge \frac{1}{\lambda}\int_\gamma (u^\lambda_x)^2 - \int_\gamma hu^\lambda + \frac{1}{\lambda}\int_\gamma (v^\lambda_{xx})^2 - \int_\gamma gv^\lambda.$$

Passing to the lower limit here we derive a variational inequality for finding the function $w^0$. Namely,

$$w^0 \in H_0^2(\Omega),\ w^0(0) \le 0, \tag{38}$$

$$\int_\Omega w^0_{,ij}(\bar{w}_{,ij} - w^0_{,ij}) - \int_\Omega f(\bar{w} - w^0) \ge 0,\ \ \forall \bar{w} \in H_0^2(\Omega), \bar{w}(0) \le 0. \tag{39}$$

The problem (38), (39) describes a contact of the plate with a rigid punch at the point $y = 0$.

It is possible to improve (35)–(37). Namely, the following convergence takes place

$$w^\lambda \to w^0 \ \text{ strongly in } \ H_0^2(\Omega),$$
$$\frac{1}{\sqrt{\lambda}} v^\lambda \to 0 \ \text{ strongly in } \ \tilde{H}^2(\gamma),$$
$$\frac{1}{\sqrt{\lambda}} u^\lambda \to 0 \ \text{ strongly in } \ \tilde{H}^1(\gamma).$$

## 2. Case of horizontal beam

In this section we analyze the beam which is parallel with respect to the plate. In this case the beam can be seen as an elastic thin obstacle for the plate. As for obstacle problems for the biharmonic operator we refer the reader to [1], [2], [3], [6]. Thin rigid obstacles for plates were analyzed in [7]. To avoid a mutual penetration between the plate and the beam a restriction of the Signorini type is imposed (see the Signorini problem in [4]).

Let $\Omega \subset R^2$ be a bounded domain with smooth boundary $\Gamma$ such that $\sigma = (0, 1) \times \{0\} \subset \Omega$. Consider the Sobolev spaces $H_0^2(\Omega)$, $H_0^2(\sigma)$ and the energy functional on $H_0^2(\Omega) \times H_0^2(\sigma)$,

$$\Pi(w, u) = \frac{1}{2}\int_\Omega w_{,ij}w_{,ij} - \int_\Omega fw + \frac{1}{2}\int_\sigma au^2_{xx} - \int_\sigma gu, \tag{40}$$

where $w_{,i} = \frac{\partial w}{\partial x_i}, i = 1, 2, (x_1, x_2) \in \Omega; u_x = \frac{du}{dx}, x = x_1$; $f \in L^2(\Omega), g \in L^2(\sigma)$; and $a \in L^\infty(\sigma)$ are given functions, and $a \ge c_0 > 0$, $c_0 = \text{const}$. We identify the functions given only on $\sigma$ with functions of one variable $x$.

Consider the set of admissible displacements

$$P = \{(w,u) \in H_0^2(\Omega) \times H_0^2(\sigma) \mid w - u \geq 0 \ \text{ on } \ \sigma\}$$

and the minimization problem

$$\inf_{(w,u)\in P} \Pi(w,u).$$

It is clear that this problem has a solution satisfying the variational inequality

$$(w,u) \in P, \tag{41}$$

$$\int_\Omega w_{,ij}(\bar{w}_{,ij} - w_{,ij}) - \int_\Omega f(\bar{w} - w) \tag{42}$$
$$+ \int_\sigma a u_{xx}(\bar{u}_{xx} - u_{xx}) - \int_\sigma g(\bar{u} - u) \geq 0, \quad \forall (\bar{w}, \bar{u}) \in P.$$

The functions $w(x_1,x_2)$ and $u(x)$ describe the vertical displacements of the plate and the beam. The domain $\Omega$ corresponds to the middle surface of the plate and $\sigma$ corresponds to the beam.

Extend the curve $\sigma$ up to a closed curve $\Sigma$ of the class $C^{1,1}$ such that $\Sigma \subset \Omega$. In this case the domain $\Omega$ is divided into two subdomains $\Omega_1$ and $\Omega_2$ with boundaries $\Sigma$ and $\Sigma \cup \Gamma$, respectively. Denote by $\nu = (\nu_1, \nu_2)$ a unit normal vector to the curve $\Sigma$ directed to the domain $\Omega_2$.

We can give a differential formulation of the problem (41), (42). The normal vector to $\Gamma$ is denoted by $n = (n_1, n_2)$. Write $\Omega_\sigma = \Omega \setminus \bar{\sigma}$. We have to find the functions $w, u$, on $\Omega_\sigma$ and $\sigma$ such that

$$\Delta^2 w = f \ \text{ in } \ \Omega_\sigma, \tag{43}$$

$$w = w_n = 0 \ \text{ on } \ \Gamma, \tag{44}$$

$$w - u \geq 0, \ [w] = [w_\nu] = 0, \ [m(w)] = 0 \ \text{ on } \ \sigma, \tag{45}$$

$$[t^\nu(w)] \geq 0, \ [t^\nu(w)](w-u) = 0 \ \text{ on } \ \sigma, \tag{46}$$

$$[t^\nu(w)] = -(a u_{xx})_{xx} + g \ \text{ on } \ \sigma, \tag{47}$$

$$u = u_x = 0 \ \text{ on } \ \partial\sigma. \tag{48}$$

It is possible to describe in what sense boundary conditions (45)–(47) are fulfilled. Problem formulations (43)–(48) and (41), (42) are equivalent.

Moreover, one more formulation of the problem (41), (42) (or the problem (43)–(48)), the so-called mixed formulation, can be provided. Let $m = \{m_{ij}\}, i,j = 1,2$. Write $\nabla\nabla m = m_{ij,ij}$ and define boundary operators on $\Sigma$,

$$m_\nu = m_{ij}\nu_j\nu_i, \ T^\nu(m) = m_{ij,k}s_k s_j \nu_i + m_{ij,j}\nu_i.$$

If $\varphi$ is a scalar function defined in $\Omega_\sigma$, we put

$$\nabla\nabla\varphi = \{\varphi_{,ij}\}, i,j = 1,2.$$

Consider additional functions $m = \{m_{ij}\}$, $m_{ij} = w_{,ij},\ i,j = 1,2,\ M = au_{xx} - G$, where $G$ is the solution of the problem

$$G_{xx} = g \ \text{ on } \ \sigma,\ G = 0 \ \text{ on } \ \partial\sigma.$$

We can rewrite the problem (43)–(48) in the following equivalent form:

$$\nabla\nabla m = f \ \text{ in } \ \Omega_\sigma, \tag{49}$$

$$m = \nabla\nabla w \ \text{ in } \ \Omega_\sigma, \tag{50}$$

$$w = w_n = 0 \ \text{ on } \ \Gamma, \tag{51}$$

$$w - u \geq 0,\ [w] = [w_\nu] = 0,\ [m_\nu] = 0 \ \text{ on } \ \sigma, \tag{52}$$

$$[T^\nu(m)] \geq 0,\ [T^\nu(m)](w - u) = 0 \ \text{ on } \ \sigma, \tag{53}$$

$$[T^\nu(m)] = -M_{xx} \ \text{ on } \ \sigma, \tag{54}$$

$$(M + G)a^{-1} = u_{xx} \ \text{ on } \ \sigma, \tag{55}$$

$$u = u_x = 0 \ \text{ on } \ \partial\sigma. \tag{56}$$

Now we are able to give a mixed formulation of the problem. Introduce the so-called set of admissible moments

$$L = \{(\bar m, \bar M) \mid \ \bar m = \{\bar m_{ij}\}, i,j = 1,2;\ \bar m, \nabla\nabla\bar m \in L^2(\Omega_\sigma),\ \bar M \in L^2(\sigma);$$
$$[\bar m_\nu] = 0,\ [T^\nu(\bar m)] \geq 0,\ [T^\nu(\bar m)] = -\bar M_{xx} \text{ on } \sigma\}.$$

Mixed formulation of the problem (41), (42) is as follows. Find functions $w, m = \{m_{ij}\}, i,j = 1,2, M$, such that

$$w \in L^2(\Omega_\sigma), (m, M) \in L, \tag{57}$$

$$\nabla\nabla m = f \ \text{ in } \ \Omega_\sigma, \tag{58}$$

$$\int\limits_{\Omega_\sigma} m(\bar m - m) - \int\limits_{\Omega_\sigma} w(\nabla\nabla\bar m - \nabla\nabla m) \tag{59}$$
$$+ \int\limits_\sigma a^{-1}(M + G)(\bar M - M) \geq 0, \ \ \forall(\bar m, \bar M) \in L.$$

Problem formulations (43)–(48) and (57)–(59) are also equivalent. This means that we can derive (57)–(59) from (43)–(48) and conversely, relations (43)–(48) follow from (57)–(59). Comparing formulations (57)–(59) and (41)–(42) we see that the set $P$ contains a restriction on displacements $u, w$. All the other boundary conditions (45)–(47) follow from (41)–(42). On the other hand, the problem formulation (57)–(59) contains a restriction on the moments. Meanwhile, boundary conditions (44)–(48) can be recovered from (57)–(59). The function $u$ can be recovered from the problem

$$(M + G)_{xx} = (au_{xx})_{xx} \ \text{ on } \ \sigma,$$
$$u = u_x = 0 \ \text{ on } \ \partial\sigma.$$

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# On Lighthill's Acoustic Analogy for Low Mach Number Flows

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*Dedicated to A.V. Kazhikov*

**Abstract.** Most predictions of the noise generated by a turbulent flow are done using a model due to Lighthill from the 1950's (the Lighthill analogy). In a large region of a fluid at rest surrounding a small region containing a small Mach number, high Reynolds number turbulent flow, this is

Step 1: Solve the *incompressible* Navier-Stokes equations with the constant density $\varrho_\infty$ for the velocity $\mathbf{u}$.

Step 2: Compute $\operatorname{div}(\operatorname{div}(\rho_\infty \mathbf{u} \otimes \mathbf{u}))$ and solve the inhomogeneous acoustic equation in both regions for the acoustic density fluctuations $R$:

$$\partial_t^2 R - \omega \triangle R = \operatorname{div}(\operatorname{div}(\rho_\infty \mathbf{u} \otimes \mathbf{u})),$$

where $\sqrt{\omega}$ is the speed of sound.

Current understanding of the derivation of the Lighthill analogy seems to be a variation on Lighthill's original reasoning and has resisted elaboration by the tools of both formal asymptotics and rigorous mathematics. In this report we give a rigorous derivation of Lighthill's acoustic analogy (including the sound source $\operatorname{div}(\operatorname{div}(\rho_\infty \mathbf{u} \otimes \mathbf{u}))$ being derived from an *incompressible* flow simulation. from the compressible Navier-Stokes and energy equation as $\mathrm{Ma} \to 0$.

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# 1. Introduction

## 1.1. The Lighthill equation and Lighthill's acoustic analogy

The mathematical simulation of aeroacoustic sound presents many technical problems related to modeling of its generation and propagation. Its importance for diverse industrial applications is without any doubt in view of the demands of user comfort and environmental regulations. A few examples where aeroacoustic noise is a critical effect include the sounds produced by jet engines of an airliner, the noise produced in high speed trains and cars, wind noise around buildings, ventilator noise in various household appliances, ....

The departure point of most methods of acoustic simulations (at least those called hybrid methods) is the Lighthill theory [36], [37]. The starting point in the Lighthill approach in the simplest case is the system of Navier-Stokes-Poisson equations describing motion of a viscous compressible gas in isentropic regime, for unknown functions density $\varrho$ and velocity $\mathbf{u}$. They read

$$\begin{aligned} \partial_t \varrho + \mathrm{div} \varrho \mathbf{u} &= 0, \\ \partial_t(\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p &= \varrho \mathbf{f} + \mathrm{div} \mathbb{S}, \end{aligned} \tag{1.1}$$

where $p = p(\varrho)$ is the pressure and $\mathbb{S} = \mathbb{S}(\nabla(\mathbf{u}))$ is the viscous stress tensor; $p$ and $\mathbb{S}$ are given functions characterizing the gas and will be specified later. In this case we can rewrite the system as

$$\begin{aligned} \partial_t R + \mathrm{div} \mathbf{Q} &= \sigma \equiv \partial_t \Sigma, \\ \partial_t \mathbf{Q} + \omega \nabla_x R &= \mathbf{F} - \mathrm{div} \mathbb{T} \end{aligned} \tag{1.2}$$

where we have denoted

$$\mathbf{Q} = \varrho \mathbf{u}, \quad R = \varrho - \varrho_\infty \tag{1.3}$$

as the momentum and the density fluctuations from the basic density distribution $\varrho_\infty$ of the background flow, and where we have set

$$\begin{aligned} \Sigma = -\varrho_\infty, \quad \omega = p'(\varrho_\infty) > 0, \quad \mathbf{F} = \varrho \mathbf{f}, \\ \mathbb{T} = \varrho \mathbf{u} \otimes \mathbf{u} + \big(p - \omega(\varrho - \varrho_\infty)\big)\mathbb{I} - \mathbb{S}. \end{aligned} \tag{1.4}$$

Taking the time derivative of the first equation in (1.2) and the divergence of the second one, we obtain

$$\partial_t^2 R - \omega \Delta R = \partial_t \sigma - \mathrm{div} \mathbf{F} + \mathrm{div}(\mathrm{div}(\mathbb{T}). \tag{1.5}$$

Lighthill reasoned that, because of the large differences in energy, there is very little feedback from acoustics to the flow. Thus, according to Lighthill's interpretation, equation (1.5) (or equivalently the system of equations (1.2)) is a non-homogenous wave equation describing the acoustic waves (fluctuations of density), where the terms at the right-hand side correspond to the monopolar ($\partial_t \sigma$), bipolar ($-\mathrm{div}\mathbf{F}$) and quadrupolar ($\mathrm{div}(\mathrm{div}(\mathbb{T}))$ acoustic sources respectively, and are considered as known and calculable from the background fluid flow field. In the sequel, we shall deal rather with the formulation (1.2) and will refer to it as to the Lighthill equation or to the Lighthill acoustic analogy.

The physical sense of the terms at the right-hand side of equation (1.5) is the following.

The first term $\partial_t \sigma$ represents the acoustic sources created by the changes of control volumes due to changes of pressure or displacements of a rigid surface: this source can be schematically described via a particle whose diameter changes (pulsates) creating acoustic waves (density perturbations). It may be interpreted as well as an instationary injection of a fluid mass $\sigma$ per unit volume. The acoustic noise of a gun shot is a typical example.

The second term div$\mathbf{F}$ describes the acoustic sources due to external forces (usually resulting from the action of a solid surface on the fluid). These sources are responsible for most of the acoustic noise in the machines and ventilators.

The third term div(div($\mathbb{T}$) is the acoustic source due to the turbulence and viscous effects in the background fluid flow which supports the density oscillations (acoustic waves). The noise of steady or non-steady jets in aero-acoustics is the typical example.

The tensor $\mathbb{T}$ is called the Lighthill tensor. It is composed from three tensors whose physical interpretation is the following: the first term is the Reynolds tensor with components $\varrho u_i u_j$ describing the (nonlinear) turbulence effects, the term $(p - \omega(\varrho - \varrho_\infty))\mathbb{I}$ expresses the entropy fluctuations and the third one is the viscous stress tensor $\mathbb{S}$.

The method for predicting noise using Lighthill's equation is usually referred to as a hybrid method since noise generation and propagation are treated separately. The first step consists in using data provided by numerical simulations to form the sound sources. The second step then consists in solving the wave equation forced by these source terms to determine the sound radiation. The main advantage of this approach is that most of the conventional flow simulations can be used in the first step.

In its simplest form of a large region of a fluid at rest surrounding a small region containing a small Mach number, high Reynolds number turbulent flow, the, so-called, hybrid method based upon Lighthill's theory is, e.g., Wagner, Huttl and Sagaut [46],

Step 1: Solve the *incompressible* Navier-Stokes equations for the velocity $\mathbf{u}$.

Step 2: Compute $\mathrm{div}(\mathrm{div}(\rho_\infty \mathbf{u} \otimes \mathbf{u}))$ (possibly plus more terms that are often dropped) and solve the inhomogeneous acoustic equation (1.5) in both regions for the acoustic density fluctuations $R$.

In practical numerical simulations, the Lighthill tensor is calculated from the velocity and density fields obtained by using various direct numerical methods and solvers for compressible Navier-Stokes equations. Then the acoustic effects are evaluated from the Lighthill equation by using diverse direct numerical methods for solving the non-homogenous wave equations (see, e.g., Colonius [8], Mitchell et al. [35], Freud et al. [23], among others). For flows in the low Mach number regimes the direct simulations are often costly, unstable, inefficient and unreliable, essentially due to the presence of rapidly oscillating acoustic waves (with periods proportional to the Mach number) in the equations themselves. In the low Mach number regimes the acoustic analogies such as the Lighthill equation, in combination with the incompressible flow solvers, give more reliable results, see [23].

Indeed, if the Mach number is small, the background flow can be considered as incompressible, implying negligible entropy fluctuations for non-heated or isentropic flows [36], Bogey et al. [4], Freud et al. [23]; thus the Lighthill tensor reduces to

$$\mathbb{T} = \varrho \mathbf{u} \otimes \mathbf{u} - \mathbb{S}, \quad \text{where} \quad \varrho_\infty = \text{const.}, \quad \text{div}\mathbf{u} = 0. \tag{1.6}$$

Moreover, due to the latter condition, for newtonian fluids, $\text{div}(\text{div}\mathbb{S}) = 0$, and the only relevant part of the Lighthill tensor in the Lighthill equation is the Reynolds tensor $\varrho \mathbf{u} \otimes \mathbf{u}$, cf. Lighthill [36].

The comparison of numerical simulations using compressible solvers on one hand and incompressible solvers on the other hand at low Mach number regimes show no noticeable difference in the evaluated acoustic fields and a good agreement with experiments up to $\text{Ma} = 0.6$, see Boersma [3] and references quoted there.

For a complete review of numerical methods, evaluation and approximation of various sound sources in the Lighthill equation from the point of view of mathematical modeling and acoustic simulations see Freud et al. [23].

The Lighthill acoustic analogy as described above involves the interaction of two motions of different time scales: the slow variables describe the background fluid flow governed by the Navier-Stokes equations; the fast variables describe the sound propagation and are governed by a non-homogenous wave equation.

The main goal of the present paper is to establish a link between the recent theory of low Mach number limits in various models describing viscous compressible fluids (which started with the pioneering paper of Lions, Masmoudi [30]) on one hand, and Lighthill's acoustic analogy ([36], [37]) as well as underlying hybrid methods used by numerical analysts in acoustics (see, e.g., Boersma [3] or Feud et al. [23]). The point of view presented in this paper should be compared and combined with other interpretations and results such as [22] or [18].

We shall prove rigorously, that the Lighthill equation (1.2) with the right-hand side calculated from incompressible Navier-Stokes equations can be obtained as a particular low Mach number limit of the Navier-Stokes-Poisson system describing viscous compressible gas in isentropic regime, more precisely, as a superposition of slow variables being governed by the incompressible Navier-Stokes equations and fast time variables solving a homogenous wave equation.

This result is obtained in the context of weak solutions, on an arbitrary large time interval and for the ill prepared initial data. It is formulated in Theorem 2.1.

We also prove, under certain assumptions on initial data, that the right hand side of the Lighthill equation is independent of time and can be calculated from the steady incompressible Navier-Stokes equations. This result is formulated in Theorem 2.2.

All these results can be reformulated and proved for the complete Navier-Stokes-Fourier system describing the motion of viscous heat conducting gasses modulo overcoming additional technical difficulties in the underlying mathematical analysis.

One may anticipate that the future development of mathematical fluid mechanics as well as capabilities of the related numerical simulations will depend on understanding not only the asymptotic models (the Lighthill equation being an example) but the way they can be rigorously derived. We therefore believe that the theorems itself are as important as the methods leading to their proofs. These methods have their mathematical background in physically motivated scaling analysis of the Navier-Stokes-Poisson system; the main challenge and difficulty consists in the fact that we deal with interaction of fluid motions characterized by two different time scales.

## 2. Original and target problems, main results

### 2.1. Weak formulation of the Lighthill equations

We shall investigate the Lighthill equation (1.2) with $\sigma = \partial_t \Sigma$, $\mathbf{F}$ and $\mathbb{T}$ a given scalar-, resp. vector- resp. tensor-valued functions on a sufficiently smooth (at least Lipschitz) bounded domain $\Omega$ on an arbitrary large time interval $(0,T)$, $T>0$.

Hereafter, we explain what we mean under the weak solution of system (1.2):

**Definition 1.1** *Let $\mathbb{T}$, $\mathbf{F}$ and $\Sigma$ belong to $L^1(0,T;L^1(\Omega))$. We say that a couple $(R,\mathbf{Q}) \in L^1(0,T;L^1(\Omega)) \times L^1(0,T;L^1(\Omega;R^3))$ represents a weak solution of the Lighthill equation (1.2) on $(0,T)\times\Omega$ if there exists a couple $(R_0^{(1)},\mathbf{Q}_0) \in L^1(\Omega) \times L^1(\Omega;R^3)$ (of initial conditions) such that*

$$\begin{aligned}&\int_0^T \int_\Omega \Big( (R-\Sigma)\partial_t \varphi + \mathbf{Q}\cdot\nabla_x \varphi \Big) \mathrm{d}x\mathrm{d}t \\ &\quad = -\int_\Omega R_0^{(1)} \varphi(0)\mathrm{d}x, \quad \varphi \in C_c^\infty([0,T)\times\overline{\Omega})),\end{aligned} \tag{2.1}$$

$$\begin{aligned}&\int_0^T \int_\Omega \Big( \mathbf{Q}\partial_t \varphi + \omega R \mathrm{div}\varphi \Big) \mathrm{d}x\mathrm{d}t \\ &\quad = -\int_0^T \int_\Omega \Big( \mathbf{F}\cdot\varphi + \mathbb{T} : \nabla_x \varphi \Big) \mathrm{d}x\mathrm{d}t - \int_\Omega \mathbf{Q}_0 \cdot \varphi(0,\cdot)\mathrm{d}x, \\ &\quad \varphi \in C_c^\infty([0,T)\times\overline{\Omega});R^3), \quad \varphi\cdot\mathbf{n} = 0 \quad \textit{on } [0,T)\times\partial\Omega.\end{aligned} \tag{2.2}$$

If $(R,\mathbf{Q})$ is a sufficiently smooth weak solution to the Lighthill equation in a sufficiently smooth domain $\Omega$ with sufficiently smooth external data $\mathbf{F}$, $\Sigma$ and $\mathbb{T}$, $(\mathbb{T}\mathbf{n})\times\mathbf{n} = 0$ at $\partial\Omega$, corresponding to initial conditions $(R_0^{(1)},\mathbf{Q}_0)$ then it verifies

$$\begin{gathered}\partial_t \mathbf{R} + \mathrm{div}\mathbf{Q} = \partial_t \Sigma \quad \text{in } (0,T)\times\Omega, \\ \partial_t \mathbf{Q} + \omega\nabla_x R = \mathbf{F} - \mathrm{div}\mathbb{T} \quad \text{in } (0,T)\times\Omega, \\ \mathbf{Q}\cdot\mathbf{n} = 0 \quad \text{at } (0,T)\times\partial\Omega,\end{gathered} \tag{2.3}$$

and

$$R(0,x) = R_0^{(1)}(x) - \Sigma(0,x), \quad x \in \Omega \tag{2.4}$$

in the classical sense.

### 2.2. The Navier-Stokes-Poisson system with small Mach number

The time evolution of the density $\varrho$, the velocity $\mathbf{u}$ of a viscous, compressible fluid in isentropic and low Mach number regime characterized by the Mach number, $\mathrm{Ma} = \varepsilon$, is governed by the *Navier-Stokes-Poisson system*:

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0, \tag{2.5}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho) = \operatorname{div} \mathbb{S} + \varrho \mathbf{f}. \tag{2.6}$$

The other non-dimensional parameters in this system, such as Strouhal number, Reynolds number and eventually Froude number, have been normalized to 1 (see, for instance, Klein [25] or the survey paper by Klein et al. [26] for more details about the dimensional analysis of fluid dynamics equations).

Once completed with the boundary conditions

$$\mathbf{u} \cdot \mathbf{n} = 0, \ (\mathbb{S}\mathbf{n}) \times \mathbf{n} = 0 \text{ on the boundary } \partial\Omega, \tag{2.7}$$

the conservation of total energy in $\Omega$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \Big( \frac{\varepsilon^2}{2} \varrho |\mathbf{u}|^2 + H(\varrho) \Big) \mathrm{d}x + \varepsilon^2 \int_0^\tau \int_\Omega \mathbb{S} : \nabla_x \mathbf{u} \mathrm{d}x \mathrm{d}t = \varepsilon^2 \int_\Omega \varrho \mathbf{f} \mathrm{d}x, \tag{2.8}$$

follows provided both $\varrho > 0$ and $\mathbf{u}$ are “sufficiently” smooth.

In (2.5–2.8),

$$\mathbb{S} = \mathbb{S}(\nabla_x \mathbf{u}) = \mu \Big( \nabla_x \mathbf{u} + \nabla_x^\perp \mathbf{u} - \frac{2}{3} \operatorname{div} \mathbf{u} \ \mathbb{I} \Big) + \zeta \operatorname{div} \mathbf{u} \mathbb{I}, \tag{2.9}$$

is the viscous stress tensor with shear ($\mu$) and bulk ($\zeta$) constant viscosities which satisfy

$$\mu > 0, \quad \zeta \geq 0 \tag{2.10}$$

and the so-called potential energy $H$ is given by

$$H(\varrho) = \varrho P(\varrho) \quad \text{where} \quad P(\varrho) = P(1) + \int_1^\varrho \frac{p(s)}{s^2} \mathrm{d}s. \tag{2.11}$$

Here and hereafter, the symbol $\frac{1}{2}(\nabla_x \mathbf{u} + \nabla_x^\perp \mathbf{u})$ denotes the symmetrized gradient. Since the entropy is supposed to be constant through the flow, the pressure takes the form

$$p(\varrho) = \varrho^\gamma, \quad \gamma > 1, \quad \text{yielding} \quad H(\varrho) = \frac{1}{\gamma - 1} \varrho^\gamma. \tag{2.12}$$

The physical values of the adiabatic constant $\gamma$ are given by formula $\gamma = \frac{R + c_v}{c_v}$, where $c_v$ is the specific heat at constant volume and $R$ is the universal gas constant; physically reasonable values of $\gamma$'s are in the range $(1, \frac{5}{3}]$, $\gamma = \frac{5}{3}$ being the adiabatic constant of the monoatomic gases. We notice that $p(\varrho)$ satisfies the standard thermodynamics stability condition asserting its strict monotonicity (cf.

Bechtel et al. [2]) and that function $H$ in (2.11) is strictly convex on $(0,\infty)$. Later on, we shall suppose

$$\gamma > \frac{3}{2} \tag{2.13}$$

which is the condition required both by the existence theory and low Mach number limit applied in this paper. Notice that at least adiabatic constants of monoatomic gases do enter into this range.

We shall investigate the density fluctuations around a positive constant density $\overline{\varrho}$. Conformably to this fact, the Navier-Stokes-Poisson system (2.5–2.6) will be supplemented with initial conditions

$$\varrho(0,\cdot) = \varrho_0,\ (\varrho\mathbf{u})(0,\cdot) = \varrho_0\mathbf{u}_0, \tag{2.14}$$

where the initial data

$$\varrho_0 = \varrho_{\varepsilon,0} = \overline{\varrho} + \varepsilon\varrho^{(1)}_{\varepsilon,0},\ \mathbf{u}_0 = \mathbf{u}_{\varepsilon,0}, \tag{2.15}$$

are chosen so that

$$\overline{\varrho} = \frac{1}{|\Omega|}\int_\Omega \varrho_{\varepsilon,0}\ \mathrm{d}x = \text{const.} > 0, \tag{2.16}$$

with the quantities $\varrho^{(1)}_{\varepsilon,0}$, $\mathbf{u}_{\varepsilon,0}$, bounded uniformly with respect to $\varepsilon \to 0$.

Here and in what follows, we should always keep in mind that the absence of density in the dependence of the transport coefficients $\mu$ and $\zeta$ is required by the *existence theory* and does not play any significant role in the present paper. We also should keep in mind that a more general pressure law than (2.12) could be taken into account, provided $p(\varrho) \sim \varrho^\gamma$ for large $\varrho$'s and provided the function $H$ in (2.11) remains strictly convex on $(0,\infty)$.

**Definition 1.2** *We shall say that a couple* $\{\varrho,\mathbf{u}\}$ *is a* **bounded energy weak solution** *of the Navier-Stokes-Poisson system* (2.5–2.12) *on a time interval* $(0,T)$ *if the following conditions are satisfied:*

- *the density* $\varrho$ *is a non-negative function,* $\varrho \in L^\infty(0,T;L^\gamma(\Omega))$*, the velocity field* $\mathbf{u}$ *belongs to the space* $L^2(0,T;W^{1,2}(\Omega;R^3))$*,*

$$\mathbf{u}\cdot\mathbf{n} = 0 \quad \textit{on } (0,T)\times\partial\Omega$$

*and the integral identity*

$$\begin{aligned}&\int_0^T\int_\Omega \Big(\varrho B(\varrho)\partial_t\varphi + \varrho B(\varrho)\mathbf{u}\cdot\nabla_x\varphi - b(\varrho)\ \mathrm{div}\mathbf{u}\ \varphi\Big)\ \mathrm{d}x\ \mathrm{d}t\\ &\quad = -\int_\Omega \varrho_0 B(\varrho_0)\varphi(0,\cdot)\ \mathrm{d}x\end{aligned} \tag{2.17}$$

*holds for any test function* $\varphi \in \mathcal{D}([0,T)\times\overline{\Omega}))$*, and any* $b$ *such that*

$$b \in L^\infty \cap C[0,\infty),\ B(\varrho) = B(1) + \int_1^\varrho \frac{b(z)}{z^2}\ \mathrm{d}z; \tag{2.18}$$

- *the momentum* $\varrho\mathbf{u}$ *belongs to* $L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega;R^3))$, *and the integral identity*

$$\begin{aligned}\int_0^T\int_\Omega\Big(\varrho\mathbf{u}\cdot\partial_t\varphi+(\varrho\mathbf{u}\otimes\mathbf{u}):\nabla_x\varphi+\frac{1}{\varepsilon^2}p(\varrho,\vartheta)\ \mathrm{div}\varphi\Big)\ \mathrm{d}x\mathrm{d}t\\ =\int_0^T\int_\Omega\Big(\mathbb{S}:\nabla_x\varphi-\varrho\mathbf{f}\cdot\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t-\int_\Omega\varrho_0\mathbf{u}_0\cdot\varphi(0,\cdot)\ \mathrm{d}x\end{aligned}\tag{2.19}$$

*is satisfied for any* $\varphi\in C_c^\infty([0,T)\times\overline{\Omega};R^3)$, $\varphi\cdot\mathbf{n}=0$ *on* $(0,T)\times\partial\Omega$;
- *the total energy balance*

$$\begin{aligned}\Big[\int_\Omega\Big(\frac{\varepsilon^2}{2}\varrho|\mathbf{u}|^2+H(\varrho)\Big)\ \mathrm{d}x\Big](\tau)+\varepsilon^2\int_0^\tau\int_\Omega\mathbb{S}:\nabla_x\mathbf{u}\mathrm{d}x\mathrm{d}t\\ \leq\varepsilon^2\int_0^\tau\int_\Omega\varrho\mathbf{f}\cdot\mathbf{u}\mathrm{d}x\mathrm{d}t+\int_\Omega\Big(\frac{\varepsilon^2}{2}\varrho_0|\mathbf{u}_0|^2+H(\varrho_0)\Big)\ \mathrm{d}x\end{aligned}\tag{2.20}$$

*holds for a.a.* $\tau\in(0,T)$.

It follows from (2.17) and (2.19) that $(\varrho,\varrho\mathbf{u})$ admit pointwise time values, namely $\varrho\in C_{\mathrm{weak}}([0,T];L^\gamma(\Omega)$ and $\varrho\mathbf{u}\in C_{\mathrm{weak}}([0,T];L^{\frac{2\gamma}{\gamma+1}}(\Omega))$, meaning among other things the satisfaction of initial conditions $\varrho_0$, $\varrho_0\mathbf{u}_0$ in the weak sense.

Note that (2.17) is the so-called renormalized formulation of a continuity equation introduced by DiPerna and Lions [13].

The existence of variational solutions in the sense of Definition 1.2 was established in Lions [29] for $\gamma\geq 9/5$ and in [19] for $\gamma>3/2$ for $\Omega\subset R^3$ a bounded spatial domain, where the velocity field was supposed to vanish on the boundary. The details about necessary modifications to accommodate the slip boundary conditions (2.7) can be found in [18]. More recent information about the existence results for the Navier-Stokes-Poisson system or for the Navier-Stokes-Fourier system can be found in the monographs [14], [39], [18].

### 2.3. Incompressible Navier-Stokes equations, non-steady case

In order to conclude this part, we introduce a standard concept of weak solutions to the system of Navier-Stokes equations describing incompressible Newtoninan fluid introduced more than 70 years ago by Leray [28].

In the classical framework, one is searching for a couple $(\Pi,\mathbf{U})$ representing pressure and velocity fields, $\Pi$ a scalar-valued and $\mathbf{U}$ a vector-valued functions of time $t\in[0,T)$ and space $x\in\Omega$, which satisfies

$$\begin{gathered}\overline{\varrho}\partial_t\mathbf{U}+\overline{\varrho}\mathrm{div}(\mathbf{U}\otimes\mathbf{U})+\nabla\Pi=\mathrm{div}\mathbb{S}+\overline{\varrho}\mathbf{f},\\ \mathrm{div}\mathbf{U}=0\end{gathered}\tag{2.21}$$

endowed with the initial conditions

$$\mathbf{U}(0,x)=\mathbf{U}_0(x)\tag{2.22}$$

and boundary conditions (2.7).

Multiplying the first (momentum) equation in (2.21) scalarly by $\mathbf{U}$, and integrating over $\Omega$, yields, for a classical solution $(\Pi, \mathbf{U})$, the energy identity, which reads

$$\frac{1}{2}\overline{\varrho}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega |\mathbf{U}|^2 \mathrm{d}x + \int_\Omega \mathbb{S} : \nabla_x \mathbf{U} \mathrm{d}x = \overline{\varrho}\int_\Omega \mathbf{f}\cdot\mathbf{U}\mathrm{d}x. \tag{2.23}$$

In (2.22) we have denoted by $\overline{\varrho} > 0$ the constant density of the fluid and by $\mathbb{S}$ the same viscous stress tensor as that one defined in (2.9–2.10); obviously, by virtue of the second (continuity) equation in (2.21), it simplifies to

$$\mathbb{S} = \mu\Big(\nabla_x \mathbf{U} + \nabla_x^\perp \mathbf{U}\Big). \tag{2.24}$$

**Definition 1.3**

**(i)** *We shall say that function* $\mathbf{U}$ *is a weak solution of Navier-Stokes system* (2.21)*, supplemented with the boundary conditions* (2.7) *and the initial conditions* (2.22) *if the following conditions are satisfied:*

- $$\mathbf{U} \in L^\infty(0,T; L^2(\Omega; R^3)) \cap L^2(0,T; W^{1,2}(\Omega; R^3)),$$
  $$\mathrm{div}\mathbf{U} = 0 \ \textit{a.a. on} \ (0,T)\times\Omega, \ \ \mathbf{U}\cdot\mathbf{n}|_{(0,T)\times\partial\Omega} = 0;$$

- *the integral identity*

$$\begin{aligned}&\int_0^T \int_\Omega \Big(\overline{\varrho}\mathbf{U}\cdot\partial_t\varphi + \overline{\varrho}(\mathbf{U}\otimes\mathbf{U}) : \nabla_x\varphi\Big)\ \mathrm{d}x\mathrm{d}t\\ &= -\int_0^T \int_\Omega \overline{\varrho}\mathbf{f}\cdot\varphi\mathrm{d}x\mathrm{d}t\\ &\quad + \int_0^T \int_\Omega \mu(\nabla_x\mathbf{U} + \nabla_x^\perp \mathbf{U}) : \nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t - \int_\Omega \overline{\varrho}\mathbf{U}_0\cdot\varphi(0,\cdot)\ \mathrm{d}x\end{aligned} \tag{2.25}$$

  *holds for any test function*

$$\varphi \in C_c^\infty([0,T)\times\overline{\Omega}; R^3), \ \mathrm{div}\ \varphi = 0, \ \varphi\cdot\mathbf{n}|_{[0,T)\times\partial\Omega} = 0. \tag{2.26}$$

**(ii)** *We say that* $\mathbf{U}$ *is a weak solution with bounded energy if* $\mathbf{U}$ *is a weak solution which satisfies the energy inequality*

$$\begin{aligned}\Big[\frac{1}{2}\overline{\varrho}\int_\Omega |\mathbf{U}|^2\mathrm{d}x\Big](\tau) + \int_0^\tau \int_\Omega \mathbb{S} : \nabla_x\mathbf{U}\mathrm{d}x\mathrm{d}t\\ \le \overline{\varrho}\int_0^\tau \int_\Omega \mathbf{f}\cdot\mathbf{U}\mathrm{d}x\mathrm{d}t + A\end{aligned} \tag{2.27}$$

*for a.a.* $\tau \in (0,T)$*, where* $A$ *is a positive constant depending only on initial data.*

**(iii)** *We shall say that function* $\mathbf{U}$ *is a Leray-Hopf weak solution of Navier-Stokes system* (2.21)*, supplemented with the boundary conditions* (2.7) *and the initial* *conditions* (2.22) *if it is a weak solution with bounded energy and the constant* $A$ *in* (2.27) *has the form*

$$A = \frac{1}{2}\overline{\varrho} \int_\Omega |\mathbf{U}_0|^2 \mathrm{d}x. \tag{2.28}$$

### 2.4. Incompressible Navier-Stokes equations, steady case

With the notation of the previous section the steady Navier-Stokes equations read

$$\begin{gathered} \overline{\varrho}\mathrm{div}(\mathbf{U} \otimes \mathbf{U}) + \nabla_x \Pi = \mathrm{div}\mathbb{S} + \overline{\varrho}\mathbf{f}, \\ \mathrm{div}\mathbf{U} = 0. \end{gathered} \tag{2.29}$$

They are completed with slip boundary condition (2.7). Weak solutions to this system are defined as follows.

**Definition** 1.4 *We say that a vector field* $\mathbf{U}$ *is a weak solution of the problem* (2.29), (2.7) *if*

$$\mathbf{U} \in W^{1,2}(\Omega; R^3), \quad \mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \mathrm{div}\mathbf{U} = 0 \tag{2.30}$$

*and*

$$\begin{gathered} \forall \varphi \in C_c^\infty(\overline{\Omega}; R^3) \quad \varphi \cdot \mathbf{n}|_{\partial\Omega} = 0, \\ \int_\Omega \Big(\overline{\varrho}\mathbf{U} \otimes \mathbf{U}) - \mu(\nabla \mathbf{U} + \nabla^\perp \mathbf{U})\Big) : \nabla_x \varphi \mathrm{d}x = - \int_\Omega \overline{\varrho}\mathbf{f} \cdot \varphi \mathrm{d}x. \end{gathered} \tag{2.31}$$

In the context of steady Navier-Stokes equations, we shall investigate the asymptotic limits to the Navier-Stokes-Poisson system (2.5–2.7) emanating from initial data (2.15), (2.16) which are, in addition, well prepared, meaning

$$\begin{gathered} \varrho_{\varepsilon,0}^{(1)} \to 0 \quad \text{a.e. in } L^\infty(\Omega), \\ \varrho_{\varepsilon,0}|\mathbf{u}_{\varepsilon,0}|^2 \to \overline{\varrho}|\mathbf{u}_0|^2 \quad \text{weakly in } L^1(\Omega). \end{gathered} \tag{2.32}$$

### 2.5. The main results – asymptotic limits

For the formulation of the main theorem and also throughout the proofs we shall need the Helmholtz projection on the divergence free vector fields, $\mathbf{H}$, and its orthogonal complement, projection on gradients, $\mathbf{H}^\perp$.

For $\mathbf{v} \in L^p(\Omega; R^3)$, $1 < p < \infty$, where $\Omega$ is a Lipschitz domain,

$$\mathbf{H}^\perp(\mathbf{v}) = \nabla\psi, \quad \mathbf{H}(\mathbf{v}) = \mathbf{v} - \nabla\psi, \tag{2.33}$$

where $\psi \in \overline{W^{1,p}}(\Omega) := \{z \in W^{1,p}(\Omega) \,|\, \int_\Omega z \mathrm{d}x = 0\}$ is a (unique) solution of the weak Neumann problem

$$\forall \eta \in C_c^\infty(\overline{\Omega}), \quad \int_\Omega \nabla\psi \cdot \nabla\eta \mathrm{d}x = \int_\Omega \mathbf{v} \cdot \nabla\eta \mathrm{d}x. \tag{2.34}$$

Thus

$$\begin{gathered} \mathbf{H}^\perp : L^p(\Omega; R^3) \to \mathbf{G}^p(\Omega) := \{\nabla z \,|\, z \in \overline{W^{1,p}}(\Omega)\}, \\ \mathbf{H} : L^p(\Omega; R^3) \to \dot{\mathbf{L}}^p(\Omega) := \{\mathbf{z} \in L^p(\Omega; R^3) \,|\mathrm{div}\mathbf{z} = 0\} \end{gathered} \tag{2.35}$$

are continuous linear operators from $L^p(\Omega; R^3)$ to $L^p(\Omega; R^3)$, where the spaces at the right-hand side are closed subspaces of $L^p(\Omega; R^3)$. In particular,

$$L^2(\Omega; R^3) = \mathbf{H}(L^2(\Omega)) \oplus \mathbf{H}^\perp(L^2(\Omega)) \tag{2.36}$$

where the direct sum is orthogonal.

Due to the elliptic regularity applied to (2.34), $\mathbf{H}$ and $\mathbf{H}^\perp$ are continuous linear operators from $W^{k,p}(\Omega; R^3)$ to $W^{k,p}(\Omega; R^3)$, $k \in N$.

Having introduced all the necessary material we are ready to state the main result concerning the asymptotic limit of solutions to the Navier-Poisson-Stokes system for low values of the Mach number.

**Theorem 2.1.** *Let $\Omega \subset R^3$ be a bounded domain of class $C^{2,\nu}$, $\nu \in (0,1)$. Let $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon\}_{\varepsilon>0}$ be a family of bounded energy weak solutions to the Navier-Stokes-Poisson system in the sense of Definition 1.2, with $p$, $H$ determined in terms of $\varrho_\varepsilon$ by (2.12–2.13) and $\mathbb{S}$ defined in (2.9–2.10). Furthermore, assume the solutions emanate from the initial state*

$$\varrho_{\varepsilon,0} = \overline{\varrho} + \varepsilon \varrho_{\varepsilon,0}^{(1)}, \quad \mathbf{u}_{\varepsilon,0}, \quad \text{with } \int_\Omega \varrho_{\varepsilon,0}^{(1)} \mathrm{d}x = 0,\ \overline{\varrho} = \text{const.} > 0, \tag{2.37}$$

*where*

$$\varrho_{\varepsilon,0}^{(1)} \to \varrho_0^{(1)},\ \mathbf{u}_{\varepsilon,0} \to \mathbf{u}_0, \quad \text{weakly-(*) in } L^\infty(\Omega) \tag{2.38}$$

*and from the right-hand side*

$$\mathbf{f} \in L^\infty(0,T; L^2(\Omega; R^3)). \tag{2.39}$$

*Let us introduce fast time variables*

$$\begin{aligned} r_\varepsilon(t,x) &= \varrho(\varepsilon t, x), & \mathbf{v}_\varepsilon(t,x) &= \mathbf{u}(\varepsilon t, x), \\ r_\varepsilon^{(1)}(t,x) &= \frac{r_\varepsilon(t,x) - \overline{\varrho}}{\varepsilon}, & \mathbf{q}_\varepsilon(t,x) &= (r_\varepsilon \mathbf{v}_\varepsilon)(t,x). \end{aligned} \tag{2.40}$$

*Then the following holds true:*

- $$\varrho_\varepsilon \to \overline{\varrho} \text{ in } C([0,T]; L^r(\Omega)) \cap L^\infty(0,T; L^\gamma(\Omega)),\ r \in [1,\gamma) \tag{2.41}$$

  *and, passing to a subsequence if necessary,*

  $$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ weakly in } L^2(0,T; W^{1,2}(\Omega; R^3)), \tag{2.42}$$

  $$\begin{gathered} \forall \mathcal{T} \in (0,\infty), \quad r_\varepsilon^{(1)} \to r^{(1)} \quad \text{in } C_{weak}([0,\mathcal{T}]; L^{s_\gamma}(\Omega)), \\ s_\gamma = \min\{\gamma, 2\}, \end{gathered} \tag{2.43}$$

  $$\forall \mathcal{T} \in (0,\infty), \quad \mathbf{q}_\varepsilon \to \mathbf{q} \text{ weakly-}* \text{ in } L^\infty(0,\mathcal{T}; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^3)). \tag{2.44}$$

- *Vector field $\mathbf{U}$ is a weak solution with bounded energy (in the sense of Definition 1.3) to the Navier-Stokes equations (2.21), (2.7) with initial conditions*

  $$\mathbf{U}_0 = \mathbf{H}(\mathbf{u}_0), \tag{2.45}$$

  *where $\mathbf{H}$ is the Helmholtz projection defined in (2.33–2.35).*

- *If moreover*

$$\mathbf{H}(\mathbf{u}_0) \in W^{\frac{2}{5},\frac{5}{4}}(\Omega; R^3) \quad (\textit{Sobolev-Slobodeckii space}), \tag{2.46}$$

*then there exists a unique function*

$$\Pi \in L^{\frac{5}{4}}(0,T; W^{1,\frac{5}{4}}(\Omega)) \tag{2.47}$$

*such that the couple* $(\Pi, \mathbf{U})$ *satisfies the integral identity*

$$\begin{aligned} &\int_0^T \int_\Omega \overline{\varrho}\mathbf{U} \cdot \partial_t \varphi \, \mathrm{d}x \, \mathrm{d}t \\ &\quad + \int_0^T \int_\Omega \Big( \overline{\varrho}(\mathbf{U} \otimes \mathbf{U}) - \mu(\nabla_x \mathbf{U} + \nabla_x^T \mathbf{U}) \Big) : \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t \\ &\quad + \int_0^T \int_\Omega \Pi \mathrm{div} \varphi \, \mathrm{d}x \, \mathrm{d}t \\ &= - \int_0^T \int_\Omega \overline{\varrho}\mathbf{f} \cdot \varphi \, \mathrm{d}x \, \mathrm{d}t - \int_\Omega \overline{\varrho}\mathbf{U}_0 \cdot \varphi(0, \cdot) \, \mathrm{d}x \end{aligned} \tag{2.48}$$

*with test functions*

$$\varphi \in C_c^\infty([0,T) \times \overline{\Omega}; R^3), \ \ \varphi \cdot \mathbf{n}|_{(0,T)\times\partial\Omega} = 0. \tag{2.49}$$

- *The couple* $(R, \mathbf{Q})$*, where*

$$R = r^{(1)} - \Big(1/[p'(\overline{\varrho})]\Big)\Pi, \ \mathbf{Q} = \mathbf{q} - \overline{\varrho}\mathbf{U} \tag{2.50}$$

*satisfies equations* (2.1–2.2) *with*

$$\omega = p'(\overline{\varrho}), \ \mathbf{F} = \overline{\varrho}\mathbf{f}, \ \mathbb{T} = \overline{\varrho}\mathbf{U} \otimes \mathbf{U} - \mu(\nabla_x \mathbf{U} + \nabla_x^\perp \mathbf{U}), \ \Sigma = -\tfrac{1}{\omega}\Pi \tag{2.51}$$

*and with initial conditions*

$$\mathbf{Q}_0 = \overline{\varrho}\mathbf{u}_0 - \overline{\varrho}\mathbf{H}(\mathbf{u}_0), \quad R_0^{(1)} = \varrho_0^{(1)}. \tag{2.52}$$

Theorem 2.1 will be proved in several steps in Sections 3–5.

If the initial data are well prepared and close to a steady solution, the system in the limit is again the Lighthill equation with the more regular right-hand side which emanates from the steady Navier-Stokes equations. This result is the subject of the following theorem.

**Theorem 2.2.** *Let* $\Omega \subset R^3$ *be a bounded domain of class* $C^{2,\nu}$*,* $\nu \in (0,1)$*. Let* $\{(\varrho_\varepsilon, \mathbf{u}_\varepsilon)\}$ *be a family of the bounded energy weak solutions to the Navier-Stokes-Poisson system investigated in Theorem* 2.1*, which emanates from the same initial conditions and right-hand side, meaning that* $(\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon})$ *and* $\mathbf{f}$ *satisfy* (2.37–2.39)*. Suppose, in addition, that the right-hand side* $\mathbf{f}$ *is time independent, i.e.,*

$$\mathbf{f} \in L^2(\Omega; R^3), \tag{2.53}$$

*that the initial data are well prepared, i.e.,* (2.32) *holds, and close to the steady state corresponding to* $\mathbf{f}$*, meaning that*

$$\mathbf{u}_0 \ \textit{satisfies} \ (2.30\text{-}2.31). \tag{2.54}$$

*Then the sequences* $\varrho_\varepsilon$, $\mathbf{u}_\varepsilon$, $r_\varepsilon^{(1)}$, $\mathbf{q}_\varepsilon$, *where* $r_\varepsilon^{(1)}$, $\mathbf{q}_\varepsilon$ *are defined in* (2.40), *admit limits* $\overline{\varrho}$, $\mathbf{u}$, $r^{(1)}$, $\mathbf{q}$ *specified in* (2.41–2.44), *and these limits have the following properties:*

- $$\mathbf{U} = \mathbf{u}_0; \tag{2.55}$$
- *there exists a function* $\Pi \in L^2(\Omega)$ *such that*

$$\begin{aligned}&\int_\Omega \Big(\overline{\varrho}\mathbf{u}_0 \otimes \mathbf{u}_0 - \mu(\nabla_x \mathbf{u}_0 + \nabla_x^\perp \mathbf{u}_0)\Big) : \nabla_x \varphi \mathrm{d}x + \int_\Omega \Pi \mathrm{div}\varphi \mathrm{d}x \\ &= - \int_\Omega \overline{\varrho}\mathbf{f} \cdot \varphi \mathrm{d}x\end{aligned} \tag{2.56}$$

*for all*

$$\varphi \in C_c^\infty(\overline{\Omega}; R^3), \quad \varphi \cdot \mathbf{n}|_{\partial\Omega} = 0;$$

- *the couple*

$$(R, \mathbf{Q}) = \Big(r^{(1)} - (p(\overline{\varrho}))^{-1}\Pi\,,\, \mathbf{q}\Big) \tag{2.57}$$

*belongs to*

$$\Big(C_{\mathrm{weak}}([0,T]; L^{s_\gamma}(\Omega)) \cap L^\infty(0,T; L^{s_\gamma}(\Omega))\Big) \times L^\infty(0,T; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^3)) \tag{2.58}$$

*and satisfies equations* (2.5), (2.6) *with*

$$\omega = p'(\overline{\varrho}),\ \mathbf{F} = \overline{\varrho}\mathbf{f},\ \mathbb{T} = \overline{\varrho}\mathbf{u}_0 \otimes \mathbf{u}_0 - \mu(\nabla_x \mathbf{U} + \nabla_x^\perp \mathbf{U}), \quad \Sigma = 0 \tag{2.59}$$

*and with initial conditions*

$$\mathbf{Q}_0 = \overline{\varrho}\mathbf{u}_0, \quad R_0^{(1)} = \varrho_0^{(1)}. \tag{2.60}$$

Theorem 2.2 will be proved in Section 7.

From the point of view of mathematical modeling of acoustic waves, where the standard procedure consists in solving directly the non-homogenous wave equation (2.3) for unknowns $(R, \mathbf{Q})$ with zero initial data $R(0)$ and $\mathbf{Q}(0)$, Theorems 2.1 and 2.2 suggest an alternative approach: To construct the solution $(R, \mathbf{Q})$ of the Lighthill equation via formulas (2.50) resp. (2.57) by using the solution $(\Pi, \mathbf{U})$ of the incompressible Navier-Stokes equations and the solution $(r^{(1)}, \mathbf{q})$ of the homogenous wave equation with the initial data $r^{(1)}(0) = \frac{1}{\omega}\Pi(0)$, $\mathbf{q}(0) = 0$.

In what follows we briefly describe the organization of proofs.

Section 3 is devoted to the a priori estimates. Lemma 3.1 shows the energy inequality in the form which takes into account the way the initial data are bounded. It is then used in Lemma 3.2 to deduce estimates for the real-time variables $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$ and in Lemma 3.3 to derive estimates for the fast-time variables $(r_\varepsilon, \mathbf{v}_\varepsilon)$.

Section 4 concerns the passage to the limit in the Navier–Stokes–Poisson equations rescaled to the fast time; we show that the limiting density fluctuations and gradient part of the momentum satisfy a conveniently weakly formulated homogenous wave equation. These results are concisely announced in Lemmas 4.1.

Section 5 deals with the real-time limit of the bounded energy weak solutions of the Navier-Stokes-Poisson equations to a weak solution with bounded energy of the Navier-Stokes equations in the sense of Definition 1.3. The principal result is formulated in Lemma 5.1 and proved through Sections 6.1–6.5. In Section 6.1, we start to investigate basic limits which can be deduced from a priori estimates listed in Lemma 4.1 via classical compactness tools of functional analysis and we show, among others, the strong convergence of divergenceless parts of the sequence of velocity fields. The projection to the gradient part suffers from the lack of estimates of the time derivative; in fact, due to the presence of the singular term $(1/\varepsilon^2)\nabla p$ in the momentum equation, they may rapidly oscillate. Consequently, as usual in these type of problems, we will not be able to pass to the limit in the part $\mathrm{div}(\mathbf{H}^{\perp}(\varrho_\varepsilon \mathbf{u}_\varepsilon) \otimes \mathbf{H}^{\perp}(\mathbf{u}_\varepsilon))$ of the convective term. We shall rather prove that the latter expression tends to a gradient, and therefore is irrelevant from the point of view of definition of weak solutions. This property, discovered by Schochet [41] in the context of strong solutions (see also related papers by Kleinerman, Majda [24] da Veiga [9], Métivier, Schochet [34], Alazard [1] as well as the survey papers [42], [10], [11] plus references quoted there) , and by Lions, Masmoudi [30] (see also the the survey paper [32], [33]) in the context of weak solutions, can be proved nowadays by various methods see [12],[31], [6], [33], [15], [17], [16]. They are mostly based on the observation that $\mathbf{H}^{\perp}(\varrho_\varepsilon \mathbf{u}_\varepsilon)$ satisfy a non-homogenous wave equation with vanishing right-hand side and exploit in various ways its structure. For the sake of completeness, we show in Sections 6.2-6.3 a “short” proof based on the spectral analysis of the underlying wave operator following Lions, Masmoudi [30]. Then we show in Section 6.4 that the limit velocity field $\mathbf{U}$ satisfies the energy inequality.

The definition of weak solutions with bounded energy is apparently silent about the pressure field, whose knowledge is, however, necessary to discover the “low Mach number” Lighthill acoustic analogy. For the non-stationary Navier-Stokes equations this question is not an elementary problem (see, e.g., the survey paper of Galdi [21]). To complete the proof of Lemma 5.1, we investigate this problem in Section 6.5 following Ladyzhenskaya [27].

Finally, the weak solutions of the Lighthill acoustic analogy in the sense of Definition 1.1 are obtained combining the weak solutions of the homogenous wave equation in the fast time limit constructed in Lemma 4.1 with the weak solutions of the non-stationary Navier Stokes equation obtained in the real-time limit in Lemma 5.1. The time dependent pressure is responsible for the singular source term $\partial_t \Sigma$ which is equal to the distribution $-\frac{1}{\omega}\partial_t \Pi$.

In the steady case, when the initial data are well prepared in the sense of Theorem 2.2, the source term $\partial_t \Sigma = 0$ and we discover the weak formulation of the Lighthill equations with the right-hand side emanating from a weak solution of the steady incompressible Navier-Stokes problem (2.29), (2.7). The precise formulation of this result is subject of Theorem 2.2.

The proof is performed in Section 7. Its first part consists of the material of Section 4 and the main auxiliary result serving for the construction of the

Lighthill equation is formulated in Lemma 4.1. As far as the real-time limit, we start by Lemma 5.1, where we have showed existence of a weak limit in the case of ill-prepared data. We observe, that the weak solutions with bounded energy constructed in this lemma are Leray-Hopf solutions, provided the initial data are well prepared. If, in addition, the initial data are close to a steady state corresponding to the same specific external force **f**, the limit appears to be a weak solution of steady Navier-Stokes problem with the same external force **f**. Since this solution (as any steady weak solution) satisfies the Prodi-Serrin conditions, we can identify it with the Leray-Hopf weak solution constructed in Lemma 5.1. These observations are formulated in Lemmas 7.1–7.2. Theorem 2.2 is then obtained as a combination of the fast-time limit from Lemma 4.1 and real-time limit obtained in Lemma 7.2.

## 3. Estimates for real-time and fast-time variables

An immediate consequence of the energy inequality (2.20) is the following lemma:

**Lemma 3.1.** *Under assumptions of Theorem* 2.1 *the following estimate holds:*

$$\begin{aligned}\Big[\int_\Omega \Big(\frac{1}{2}\varrho_\varepsilon \mathbf{u}_\varepsilon^2 + \frac{1}{\varepsilon^2}\mathcal{H}(\varrho_\varepsilon)\Big)\mathrm{d}x\Big](\tau) + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla \mathbf{u}_\varepsilon \mathrm{d}x\mathrm{d}t \\ \leq \int_\Omega \Big(\frac{1}{2}\varrho_\varepsilon \mathbf{u}_{\varepsilon,0}^2 + \frac{1}{\varepsilon^2}\mathcal{H}(\varrho_{\varepsilon,0})\Big)\mathrm{d}x + \int_0^\tau \int_\Omega \varrho \mathbf{f}\cdot \mathbf{u}\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{3.1}$$

*where*

$$\mathcal{H}(\varrho) = H(\varrho) + \partial_\varrho H(\overline{\varrho})(\varrho - \overline{\varrho}) - H(\overline{\varrho}) = \varrho^\gamma - \gamma\overline{\varrho}^{\gamma-1}(\varrho - \overline{\varrho}) - \overline{\varrho}^\gamma \tag{3.2}$$

*is a strictly convex nonnegative function with minimum at* $\varrho = \overline{\varrho}$.

Lemma 3.1 implies several estimates; in order to write them in a concise way we shall introduce, inspired by [17], the essential and residual sets as follows:

$$\begin{aligned}\mathcal{M}_{\mathrm{ess}}(t) = \{x \in \Omega \,|\, \frac{\overline{\varrho}}{2} \leq \varrho_\varepsilon(t,x) \leq 2\overline{\varrho}\}, \quad \mathcal{M}_{\mathrm{res}}(t) = R^3 \setminus \mathcal{M}_{\mathrm{ess}}(t), \\ \tilde{\mathcal{M}}_{\mathrm{ess}}(t) = \mathcal{M}_{\mathrm{ess}}(\varepsilon t), \quad \tilde{\mathcal{M}}_{\mathrm{res}}(t) = \mathcal{M}_{\mathrm{res}}(\varepsilon t).\end{aligned} \tag{3.3}$$

For a function $h : \Omega \to R$, there holds

$$\begin{aligned} h = [h]_{\mathrm{ess}} + [h]_{\mathrm{res}}, \quad \text{where} \quad [h]_{\mathrm{ess}} = h1_{\mathcal{M}_{\mathrm{ess}}}, \quad [h]_{\mathrm{res}} = h1_{\mathcal{M}_{\mathrm{res}}}, \\ h = [h]_{\widetilde{\mathrm{ess}}} + [h]_{\widetilde{\mathrm{res}}}, \quad \text{where} \quad [h]_{\widetilde{\mathrm{ess}}} = h1_{\tilde{\mathcal{M}}_{\mathrm{ess}}}, \quad [h]_{\widetilde{\mathrm{res}}} = h1_{\tilde{\mathcal{M}}_{\mathrm{res}}}.\end{aligned} \tag{3.4}$$

We shall collect the estimates in the following two lemmas. Lemma 3.2 deals with the estimates of "real-time" quantities while Lemma 3.3 deals with their "fast-time" counterparts.

**Lemma 3.2.** *Under assumptions of Theorem* 2.1 *we have the following uniform estimates uniformly with respect to $\epsilon$:*

$$\operatorname{ess}\sup_{t\in(0,T)} |\mathcal{M}_{\mathrm{res}}| \le c\varepsilon^2, \tag{3.5}$$

$$\operatorname{ess}\sup_{t\in(0,T)} \left\| \Big[\varrho_\varepsilon(t)\Big]_{\mathrm{res}} \right\|_{L^\gamma(\Omega)} \le c\varepsilon^{2/\gamma}, \tag{3.6}$$

$$\operatorname{ess}\sup_{t\in(0,T)} \left\| \Big[\varrho_\varepsilon(t) - \overline{\varrho}\Big]_{\mathrm{res}} \right\|_{L^\gamma(\Omega)} \le c\varepsilon^{2/\gamma}, \tag{3.7}$$

$$\operatorname{ess}\sup_{t\in(0,T)} \left\| \Big[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}(t)\Big]_{\mathrm{ess}} \right\|_{L^2(\Omega)} \le c, \tag{3.8}$$

$$\operatorname{ess}\sup_{t\in(0,T)} \left\| \Big[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}(t)\Big]_{\mathrm{res}} \right\|_{L^p(\Omega)} \le c\varepsilon^{\frac{2}{p}-1},\ p\le\gamma, \tag{3.9}$$

$$\operatorname{ess}\sup_{t\in(0,T)} \left\| \varrho_\varepsilon(t) \right\|_{L^\gamma(\Omega)} \le c, \tag{3.10}$$

$$\operatorname{ess}\sup_{t\in(0,T)} \|\varrho_\varepsilon \mathbf{u}_\varepsilon^2(t)\|_{L^1(\Omega)} \le c, \tag{3.11}$$

$$\operatorname{ess}\sup_{t\in(0,T)} \|\varrho_\varepsilon \mathbf{u}_\varepsilon(t)\|_{L^{\frac{2\gamma}{2+\gamma}}(\Omega)} \le c, \tag{3.12}$$

$$\|\mathbf{u}_\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega))} \le c. \tag{3.13}$$

**Lemma 3.3.** *Under assumptions of Theorem* 2.1 *we have the following uniform estimates uniformly with respect to $\epsilon$:*

$$\operatorname{ess}\sup_{t\in(0,T/\varepsilon)} |\tilde{\mathcal{M}}_{\mathrm{res}}| \le c\varepsilon^2, \tag{3.14}$$

$$\operatorname{ess}\sup_{t\in(0,T/\varepsilon)} \left\| \Big[r_\varepsilon(t)\Big]_{\widetilde{\mathrm{res}}} \right\|_{L^\gamma(\Omega)} \le c\varepsilon^{2/\gamma}, \tag{3.15}$$

$$\operatorname{ess}\sup_{t\in(0,T/\varepsilon)} \left\| \Big[r_\varepsilon(t) - \overline{\varrho}\Big]_{\widetilde{\mathrm{res}}} \right\|_{L^\gamma(\Omega)} \le c\varepsilon^{2/\gamma}, \tag{3.16}$$

$$\operatorname{ess}\sup_{t\in(0,T/\varepsilon)} \left\| \Big[\frac{r_\varepsilon - \overline{\varrho}}{\varepsilon}(t)\Big]_{\widetilde{\mathrm{ess}}} \right\|_{L^2(\Omega)} \le c, \tag{3.17}$$

$$\operatorname{ess}\sup_{t\in(0,T/\varepsilon)} \left\| \Big[\frac{r_\varepsilon - \overline{\varrho}}{\varepsilon}(t)\Big]_{\widetilde{\mathrm{res}}} \right\|_{L^p(\Omega)} \le \varepsilon^{\frac{2}{p}-1},\ p\le\gamma, \tag{3.18}$$

$$\operatorname{ess}\sup_{t\in(0,T/\varepsilon)} \left\| r_\varepsilon(t) \right\|_{L^\gamma(\Omega)} \le c, \tag{3.19}$$

$$\operatorname{ess}\sup_{t\in(0,T/\varepsilon)} \|r_\varepsilon \mathbf{v}_\varepsilon^2(t)\|_{L^1(\Omega)} \le c, \tag{3.20}$$

$$\operatorname{ess}\sup_{t\in(0,T/\varepsilon)} \|\mathbf{q}_\varepsilon(t)\|_{L^{\frac{2\gamma}{2+\gamma}}(\Omega)} \le c, \tag{3.21}$$

$$\|\mathbf{v}_\varepsilon\|_{L^2(0,T/\varepsilon;W^{1,2}(\Omega))} \le \frac{c}{\sqrt{\varepsilon}}. \tag{3.22}$$

## 4. Limit passage in fast-time variables and homogenous wave equation

**Lemma 4.1.** *Under assumptions of Theorem* 2.1 *we have, at least for a chosen subsequence of* $\varepsilon \to 0+$,

$$\begin{aligned} &\forall \mathcal{T} > 0, \quad r_\varepsilon^{(1)} \to r^{(1)} \quad \textit{in } C_{\text{weak}}([0,\mathcal{T}]; L^{s_\gamma}(\Omega)),\ s_\gamma = \min\{\gamma, 2\}, \\ &\forall \mathcal{T} > 0, \quad \mathbf{q}_\varepsilon \to \mathbf{q} \quad \textit{weakly-}* \textit{ in } L^\infty(0,\mathcal{T}; L^{\frac{2\gamma}{\gamma+1}}(\Omega)), \end{aligned} \tag{4.1}$$

*where the fast-time variables* $r_\varepsilon^{(1)}$ *and* $\mathbf{q}_\varepsilon$ *have been defined in* (2.40). *Moreover, the weak limits* $r^{(1)}$, $\mathbf{q}$ *satisfy the homogenous wave equation*

$$\begin{gathered} \forall \varphi \in C_c^\infty([0,\infty) \times \overline{\Omega})), \\ \int_0^\infty \int_\Omega \Big( r^{(1)} \partial_t \varphi + \mathbf{q} \cdot \nabla_x \varphi \Big) \mathrm{d}x\mathrm{d}t = - \int_\Omega \varrho_0^{(1)} \varphi(0) \mathrm{d}x, \end{gathered} \tag{4.2}$$

$$\begin{gathered} \forall \varphi \in C_c^\infty([0,\infty) \times \overline{\Omega}); R^3), \quad \varphi \cdot \mathbf{n} = 0 \quad \textit{on } (0,\infty) \times \partial\Omega, \\ \int_0^\infty \int_\Omega \Big( \mathbf{q} \partial_t \varphi + p'(\overline{\varrho}) r^{(1)} \mathrm{div} \varphi \Big) \mathrm{d}x\mathrm{d}t = - \int_\Omega \varrho_0 \mathbf{u}_0 \cdot \varphi(0) \mathrm{d}x, \end{gathered} \tag{4.3}$$

*where* $\varrho_0^{(1)}$ *and* $\mathbf{u}_0$ *are defined in* (2.38), $\varrho_0 = \overline{\varrho} + \varrho_0^{(1)}$ *and*

$$r^{(1)}(0) = \varrho_0^{(1)}. \tag{4.4}$$

*Proof of Lemma* 4.1*:* Let $\varphi \in C_c^\infty([0,\infty) \times \Omega; R^3)$, $\varphi \cdot \mathbf{n}|_{[0,\infty)\times\partial\Omega} = 0$; then there exists $\varepsilon_0$ such that for all $0 < \varepsilon < \varepsilon_0$, $\varphi \in C_c^\infty([0, T/\varepsilon) \times \overline{\Omega})$. We rewrite (2.19) with the test function $\varphi_\varepsilon(t,x) = \varphi(t/\varepsilon, x)$, where $\varphi_\varepsilon$ is compactly supported in $[0,T) \times \overline{\Omega}$. After the change of variables to the fast time variable $\tau = \frac{t}{\varepsilon}$, we obtain

$$\begin{aligned} &\int_0^{T/\varepsilon} \int_\Omega r_\varepsilon \mathbf{v}_\varepsilon \cdot \partial_t \varphi \ \mathrm{d}x\mathrm{d}t \\ &\quad + \varepsilon \int_0^{T/\varepsilon} \int_\Omega (r_\varepsilon \mathbf{v}_\varepsilon \otimes \mathbf{v}_\varepsilon) : \nabla_x \varphi \ \mathrm{d}x\mathrm{d}t + \frac{1}{\varepsilon} \int_0^{T/\varepsilon} \int_\Omega (r_\varepsilon^\gamma - \overline{\varrho}^\gamma) \ \mathrm{div}\varphi \ \mathrm{d}x\mathrm{d}t \\ &= \varepsilon \int_0^{T/\varepsilon} \int_\Omega \Big( \mathbb{S}(\nabla_x \mathbf{v}_\varepsilon) : \nabla_x \varphi + r_\varepsilon \mathbf{f}_\varepsilon \cdot \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t - \int_\Omega \varrho_{\varepsilon,0} \mathbf{u}_{\varepsilon,0} \cdot \varphi(0,\cdot) \ \mathrm{d}x, \end{aligned} \tag{4.5}$$

where

$$\mathbf{f}_\varepsilon(t,x) = \mathbf{f}(\varepsilon t, x).$$

Similarly, continuity equation (2.17), where we take $b = 0$, rescaled to the fast time yields

$$\int_0^{T/\varepsilon} \int_\Omega \Big( r_\varepsilon^{(1)} \partial_t \varphi + r_\varepsilon \mathbf{v}_\varepsilon \cdot \nabla_x \varphi \Big) \mathrm{d}x\mathrm{d}t = - \int_\Omega \varrho_{\varepsilon,0}^{(1)} \varphi(0,\cdot) \mathrm{d}x \tag{4.6}$$

where $\varphi \in C_c^\infty([0,\infty) \times \overline{\Omega}))$ and $0 < \varepsilon < \varepsilon_0$, where $\varepsilon_0 > 0$ is so small that $\varphi$ is supported in $[0, T/\varepsilon) \times \overline{\Omega}$.

By virtue of (3.21) and thanks to (4.6), the sequence

$$\Big\{ \int_\Omega r_\varepsilon^{(1)} \varphi \mathrm{d}x \Big\}_{\varepsilon>0}, \quad \text{where } \varphi \in C_c^\infty(\overline{\Omega}),$$

is a bounded and equi-uniformly continuous subset of $C[0,\mathcal{T}]$ with any $\mathcal{T} \in (0,\infty)$. Arguing by the Arzelà-Ascoli theorem, the separability of $L^{s_\gamma}(\Omega)$ and the diagonalization, keeping in mind estimates (3.17), (3.18), we conclude that

$$\forall \mathcal{T} > 0, \quad r_\varepsilon^{(1)} \to r^{(1)} \quad \text{in } C_{\text{weak}}([0,\mathcal{T}]; L^{s_\gamma}(\Omega)), \tag{4.7}$$

at least for a chosen subsequence. Moreover, (4.4) holds.

By virtue of (3.21) we also have

$$\forall \mathcal{T} > 0, \quad \mathbf{q}_\varepsilon = r_\varepsilon \mathbf{v}_\varepsilon \to \mathbf{q} \quad \text{weakly-}* \text{ in } L^\infty(0,\mathcal{T}; L^{\frac{2\gamma}{\gamma+1}}(\Omega)). \tag{4.8}$$

This allows us to pass to the limit in (4.6) and to get equation (4.2). Writing

$$\begin{aligned} \frac{r^\gamma - \overline{\varrho}^\gamma}{\varepsilon} &= \Big[\frac{r^\gamma - \overline{\varrho}^\gamma}{\varepsilon}\Big]_{\widetilde{\text{ess}}} + \Big[\frac{r^\gamma - \overline{\varrho}^\gamma}{\varepsilon}\Big]_{\widetilde{\text{res}}} \\ &= \gamma \overline{\varrho}^{\gamma-1} \Big[\frac{r - \overline{\varrho}}{\varepsilon}\Big]_{\widetilde{\text{ess}}} + \varepsilon\gamma(\gamma-1) z^{\gamma-2} \Big[\Big(\frac{r - \overline{\varrho}}{\varepsilon}\Big)^2\Big]_{\widetilde{\text{ess}}} + \Big[\frac{r^\gamma - \overline{\varrho}^\gamma}{\varepsilon}\Big]_{\widetilde{\text{res}}}, \end{aligned}$$

where $\overline{\varrho}/2 \le z \le 2\overline{\varrho}$, and exploiting (3.14), (3.16), (3.17–3.18), (3.20), we obtain

$$\forall \mathcal{T} > 0, \quad \frac{r_\varepsilon^\gamma - \overline{\varrho}^\gamma}{\varepsilon} \to r^{(1)} \quad \text{weakly in } L^1(0,\mathcal{T}; L^1(\Omega)). \tag{4.9}$$

For a given test function $\varphi$ the upper bound of the time integrals in (4.5) and (4.6) are independent of $\epsilon \to 0+$. With this observation and with estimates (3.19), (3.20) and (3.22) in mind, we verify that

$$\varepsilon \int_0^{T/\varepsilon} \int_\Omega (r_\varepsilon \mathbf{v}_\varepsilon \otimes \mathbf{v}_\varepsilon) : \nabla_x \varphi \ \mathrm{d}x\mathrm{d}t \to 0,$$

$$\varepsilon \int_0^{T/\varepsilon} \int_\Omega \mathbb{S}(\nabla_x \mathbf{v}_\varepsilon) : \nabla_x \varphi \, \mathrm{d}x\mathrm{d}t \to 0,$$

$$\varepsilon \int_0^{T/\varepsilon} \int_\Omega r_\varepsilon \mathbf{f}_\varepsilon \cdot \varphi \ \mathrm{d}x \ \mathrm{d}t \to 0.$$

Now, we are ready to let $\varepsilon \to 0+$ in (4.5) to get (4.3). The proof of Lemma 4.1 is complete. □

## 5. Limit passage in the real-time variables and the Navier-Stokes equations

**Lemma 5.1.** *Under assumptions of Theorem* 2.1 *we have at least for a chosen subsequence of* $\varepsilon \to 0+$,

**(i)** $\mathbf{u}_\varepsilon \to \mathbf{U}$ *weakly in* $L^2(0,T; W^{1,2}(\Omega; R^3))$. (5.1)

*Moreover,*

$$\mathbf{U} \in L^\infty(0,T;L^2(\Omega;R^3)) \cap C_{\mathrm{weak}}([0,T];L^2(\Omega;R^3)), \tag{5.2}$$

$$\mathrm{div}\mathbf{U} = 0 \ \textit{a.a. on} \ (0,T)\times\Omega, \ \mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0 \tag{5.3}$$

*and the integral identity*

$$\begin{aligned} &\int_0^T \int_\Omega \Big(\overline{\varrho}\mathbf{U}\cdot\partial_t\varphi + \overline{\varrho}(\mathbf{U}\otimes\mathbf{U}) : \nabla_x\varphi\Big)\, \mathrm{d}x\mathrm{d}t = -\int_0^T\int_\Omega \overline{\varrho}\mathbf{f}\cdot\varphi \mathrm{d}x\mathrm{d}t \\ &\quad + \int_0^T \int_\Omega \mu(\nabla_x\mathbf{U} + \nabla_x^\perp\mathbf{U}) : \nabla_x\varphi \ \mathrm{d}x \ \mathrm{d}t - \int_\Omega \overline{\varrho}\mathbf{H}(\mathbf{u}_0)\cdot\varphi(0,\cdot)\ \mathrm{d}x \end{aligned} \tag{5.4}$$

*holds for any test function*

$$\varphi \in C_c^\infty([0,T)\times\overline{\Omega};R^3), \ \mathrm{div}\ \varphi = 0, \ \varphi\cdot\mathbf{n}|_{[0,T)\times\partial\Omega} = 0. \tag{5.5}$$

*In addition,* $\mathbf{u}$ *satisfies the energy inequality*

$$\begin{aligned} &\Big[\frac{1}{2}\overline{\varrho}\int_\Omega |\mathbf{u}|^2\mathrm{d}x\Big](\tau) + \int_0^\tau\int_\Omega \mathbb{S}(\nabla\mathbf{u}) : \nabla_x\mathbf{u}\mathrm{d}x\mathrm{d}t \\ &\quad \le \liminf_{\varepsilon\to 0+}\Big[\int_\Omega \Big(\frac{1}{2}\varrho_\varepsilon\mathbf{u}_{\varepsilon,0}^2 + \frac{1}{\varepsilon^2}\mathcal{H}(\varrho_{\varepsilon,0})\Big)\mathrm{d}x\Big] + \int_0^\tau\int_\Omega \varrho\mathbf{f}\cdot\mathbf{U}\mathrm{d}x\mathrm{d}t \end{aligned} \tag{5.6}$$

*for a.a.* $\tau \in (0,T)$.

*In other words,* $\mathbf{U}$ *is a weak solution with bounded energy (in the sense of Definition* 1.3*) to the Navier-Stokes equations* (2.21) *with slip boundary conditions* (2.7) *and initial conditions* $\mathbf{U}(0) = \mathbf{H}(\mathbf{u}_0)$.

**(ii)** *Moreover, if* $\mathbf{H}(u_0)$ *satisfies conditions* (2.46)*, then*

$$\mathbf{U} \in W^{1,\frac{5}{4}}(0,T;L^{\frac{5}{4}}(\Omega;R^3)) \cap L^{\frac{5}{4}}(0,T;W^{2,\frac{5}{4}}(\Omega))$$

*and there exists a function* $\Pi \in L^{\frac{5}{4}}(0,T;W^{1,\frac{5}{4}}(\Omega))$ *such that the couple* $(\Pi,\mathbf{U})$ *verifies*

$$\overline{\varrho}\partial_t\mathbf{U} + \overline{\varrho}\mathrm{div}(\mathbf{U}\otimes\mathbf{U}) + \nabla\Pi = \overline{\mu}\mathrm{div}(\nabla_x\mathbf{U} + \nabla_x^T\mathbf{U}) + \overline{\varrho}\mathbf{f} \tag{5.7}$$

*almost everywhere in* $(0,T)\times\Omega$.

**Lemma 5.2.** *Let assumptions of Theorem* 5.1 *be satisfied. Suppose that the initial data satisfy in addition conditions* (2.30–2.32)*. Then we have:*

**(i)** *The energy inequality* (5.6) *is replaced by*

$$\begin{aligned} &\frac{1}{2}\overline{\varrho}\int_\Omega |\mathbf{U}|^2(\tau)\,\mathrm{d}x + \int_0^\tau\int_\Omega \frac{\overline{\mu}}{2}(\nabla_x\mathbf{U} + \nabla_x^T\mathbf{U}) : (\nabla_x\mathbf{U} + \nabla_x^T\mathbf{U})\,\mathrm{d}x\mathrm{d}t \\ &\quad \le \overline{\varrho}\int_0^\tau\int_\Omega \mathbf{f}\cdot\mathbf{U}\mathrm{d}x\mathrm{d}t + \frac{1}{2}\overline{\varrho}\int_\Omega |\mathbf{u}_0|^2\,\mathrm{d}x \ \textit{for a.a.}\ \tau\in(0,T). \end{aligned} \tag{5.8}$$

*In the other words,* $\mathbf{U}$ *is a Leray-Hopf weak solution (Definition* 1.3*) to the Navier-Stokes equations* (2.21) *with slip boundary conditions* (2.7) *and initial conditions* $\mathbf{U}_0 = \mathbf{u}_0$.

**(ii)** *Moreover* $\mathbf{U} = \mathbf{U}_0 \in W^{2,\frac{3}{2}}(\Omega)$ *and there exists a unique function* $\Pi \in W^{1,\frac{3}{2}}(\Omega)$, $\int_\Omega \Pi \, \mathrm{d}x = 0$ *such that*

$$\overline{\varrho}\mathrm{div}(\mathbf{U} \otimes \mathbf{U}) - \overline{\mu}\mathrm{div}(\nabla_x \mathbf{U} + \nabla_x^T \mathbf{U}) + \nabla_x \Pi = \overline{\varrho}\mathbf{f}, \tag{5.9}$$

*almost everywhere in* $\Omega$.

Lemma 5.1 will be proved throughout Sections 6.1–6.5. Lemma 5.2 will be proved in Section 7.

## 6. Proof of Lemma 5.1 and Theorem 2.1

### 6.1. Limits in the density, velocity and momentum

Since $\varrho_\varepsilon$ satisfies (2.17) with $\mathbf{u}_\varepsilon$ on place of $\mathbf{u}$ and (3.6–3.10) it is a routine matter to establish that $\varrho_\varepsilon \in C([0,T]; L^r(\Omega)$, $1 \le r < \gamma$ and that

$$\varrho_\varepsilon \to \overline{\varrho} \quad \text{in } L^\infty(0,T; L^\gamma(\Omega) \text{ and } C([0,T]; L^r(\Omega)),\ r \in [1,\gamma), \tag{6.1}$$

where the second convergence is deduced from the fact that $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$ satisfies the renormalized continuity equation (2.17) with $\mathbf{u}_\varepsilon$ obeying the bound (3.13). In accordance with this bound,

$$\mathbf{u}_\varepsilon \to \mathbf{U} \quad \text{weakly in } L^2(0,T; W^{1,2}(\Omega; R^3)),\ \mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{6.2}$$

By virtue of (6.1), (6.2) and (3.12),

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \overline{\varrho}\mathbf{U} \quad \text{weakly-}* \text{ in } L^\infty(0,T; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^3)). \tag{6.3}$$

Thus the limit $\varepsilon \to 0+$ in the continuity equation (2.17) yields

$$\mathrm{div}\mathbf{U} = 0. \tag{6.4}$$

Due to the continuity properties of the projections $\mathbf{H}$, $\mathbf{H}^\perp$ (see (2.33–2.35)), we conclude from (6.2) and (6.4) that

$$\begin{aligned} \mathbf{H}(\mathbf{u}_\varepsilon) \to \mathbf{U} &\quad \text{weakly in } L^2(0,T; W^{1,2}(\Omega; R^3)), \\ \mathbf{H}^\perp(\mathbf{u}_\varepsilon) \to 0 &\quad \text{weakly in } L^2(0,T; W^{1,2}(\Omega; R^3)). \end{aligned} \tag{6.5}$$

We deduce from (2.19) that the sequence of functions

$$t \to \Big[ \int_\Omega \mathbf{H}(\varrho_\varepsilon \mathbf{u}_\varepsilon) \cdot \varphi \mathrm{d}x \Big](t), \quad \varphi \in C^\infty(\overline{\Omega}),\ \varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$$

is bounded and equi-uniformly continuous in $C[0,T]$. Then, using the Arzelà-Ascoli theorem, separability of $L^{[\frac{2\gamma}{2\gamma+1}]'}(\Omega; R^3)$ and density plus diagonalization argument, we obtain

$$\mathbf{H}(\varrho_\varepsilon \mathbf{u}_\varepsilon) \to \overline{\varrho}\mathbf{H}(\mathbf{U}) = \overline{\varrho}\mathbf{U} \quad \text{in } C_{\mathrm{weak}}([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^3)). \tag{6.6}$$

For $\gamma > \frac{3}{2}$, $W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^{\frac{2\gamma}{\gamma+1}}(\Omega)$; standard compactness argument then yields

$$\mathbf{H}(\varrho_\varepsilon \mathbf{u}_\varepsilon) \to \overline{\varrho}\mathbf{U} \quad \text{in } L^2(0,T; [W^{1,2}(\Omega; R^3)]^*). \tag{6.7}$$

We also observe that

$$\begin{gathered}(\varrho_\varepsilon - \overline{\varrho})\mathbf{u}_\varepsilon \to 0, \quad \mathbf{H}\Big((\varrho_\varepsilon - \overline{\varrho})\mathbf{u}_\varepsilon\Big) \to 0,\\ \mathbf{H}^\perp\Big((\varrho_\varepsilon - \overline{\varrho})\mathbf{u}_\varepsilon\Big) \to 0 \quad \text{in } L^2(0,T;L^{\frac{6}{5}}(\Omega;R^3)),\end{gathered} \tag{6.8}$$

where we have used (6.1–6.2) and continuity of $\mathbf{H}, \mathbf{H}^\perp$. Writing

$$\overline{\varrho}\Big(\mathbf{H}(\mathbf{u}_\varepsilon)\Big)^2 = \mathbf{H}\Big((\varrho_\varepsilon - \overline{\varrho})\mathbf{u}_\varepsilon\Big)\cdot\mathbf{H}(\mathbf{u}_\varepsilon) + \mathbf{H}(\varrho_\varepsilon\mathbf{u}_\varepsilon)\cdot\mathbf{H}(\mathbf{u}_\varepsilon)$$

we infer due to (6.5), (6.7) and (6.8) that

$$\mathbf{H}(\mathbf{u})_\varepsilon \to \mathbf{H}(\mathbf{U}) = \mathbf{U} \quad \text{in } L^2(0,T;L^2(\Omega;R^3)). \tag{6.9}$$

The projection of the convective term in (2.19) on the divergenceless vector fields can be written as

$$\begin{aligned}&\int_0^T\int_\Omega \varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon : \nabla_x\varphi\mathrm{d}x\mathrm{d}t\\ &= \int_0^T\int_\Omega \mathbf{H}^\perp\Big(\varrho_\varepsilon\mathbf{u}_\varepsilon\Big)\otimes\mathbf{H}\Big(\mathbf{u}_\varepsilon\Big) : \nabla_x\varphi\mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega \mathbf{H}\Big(\varrho_\varepsilon\mathbf{u}_\varepsilon\Big)\otimes\mathbf{u}_\varepsilon : \nabla_x\varphi\mathrm{d}x\mathrm{d}t\\ &\quad + \int_0^T\int_\Omega \mathbf{H}^\perp\Big(\varrho_\varepsilon\mathbf{u}_\varepsilon\Big)\otimes\mathbf{H}^\perp\Big(\mathbf{u}_\varepsilon\Big) : \nabla_x\varphi\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{6.10}$$

where

$$\varphi\in C_c^\infty([0,T)\times\overline{\Omega};R^3),\ \operatorname{div}\varphi = 0,\ \varphi\cdot\mathbf{n}|_{[0,T)\times\partial\Omega} = 0. \tag{6.11}$$

By virtue of (6.2) and (6.7),

$$\int_0^T\int_\Omega \mathbf{H}\Big(\varrho_\varepsilon\mathbf{u}_\varepsilon\Big)\otimes\mathbf{u}_\varepsilon : \nabla_x\varphi\mathrm{d}x\mathrm{d}t \to \int_0^T\int_\Omega \overline{\varrho}\mathbf{U}\otimes\mathbf{U} : \nabla_x\varphi\mathrm{d}x\mathrm{d}t.$$

Further

$$\begin{aligned}&\int_0^T\int_\Omega \mathbf{H}^\perp\Big(\varrho_\varepsilon\mathbf{u}_\varepsilon\Big)\otimes\mathbf{H}\Big(\mathbf{u}_\varepsilon\Big) : \nabla_x\varphi\mathrm{d}x\mathrm{d}t\\ &= \int_0^T\int_\Omega \mathbf{H}^\perp\Big((\varrho_\varepsilon - \overline{\varrho})\mathbf{u}_\varepsilon\Big)\otimes\mathbf{H}\Big(\mathbf{u}_\varepsilon\Big) : \nabla_x\varphi\mathrm{d}x\mathrm{d}t\\ &\quad + \overline{\varrho}\int_0^T\int_\Omega \mathbf{H}^\perp(\mathbf{u}_\varepsilon)\otimes\mathbf{H}(\mathbf{u}_\varepsilon) : \nabla_x\varphi\mathrm{d}x\mathrm{d}t,\end{aligned}$$

where the first term tends to 0 due to (6.5) and (6.8), while the second one converges to 0 by virtue of (6.5) and (6.9).

Thus, we have

$$\int_0^T\int_\Omega \varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon : \nabla_x\varphi\mathrm{d}x\mathrm{d}t \to \int_0^T\int_\Omega \overline{\varrho}\mathbf{U}\otimes\mathbf{U} : \nabla_x\varphi\mathrm{d}x\mathrm{d}t \tag{6.12}$$

for all $\varphi$ belonging to (6.11) provided we show

$$\int_0^T\int_\Omega \mathbf{H}^\perp\Big(\varrho_\varepsilon\mathbf{u}_\varepsilon\Big)\otimes\mathbf{H}^\perp\Big(\mathbf{u}_\varepsilon\Big) : \nabla_x\varphi\mathrm{d}x\mathrm{d}t \to 0 \tag{6.13}$$

with any $\varphi$ in (6.11). In classical interpretation the last identity means that

$$\operatorname{div}\Big[\mathbf{H}^{\perp}\Big(\varrho_\varepsilon\mathbf{u}_\varepsilon\Big)\otimes\mathbf{H}^{\perp}\Big(\mathbf{u}_\varepsilon\Big)\Big]$$

as $\varepsilon\to 0$ becomes a gradient.

We shall devote the next two paragraphs to the proof of (6.13).

**6.2. Wave equation in real time and its spectral analysis**

Following Schochet [41] and Lions-Masmoudi [30] we rewrite equations (2.17) (with $b=0$) and (2.19) as a wave equation

$$\begin{gathered}\forall\varphi\in C_c^\infty((0,T)\times\overline{\Omega})),\\ \int_0^T\int_\Omega\Big(\varepsilon\varrho_\varepsilon^{(1)}\partial_t\varphi+\mathbf{z}_\varepsilon\cdot\nabla_x\varphi\Big)\mathrm{d}x\mathrm{d}t=0,\end{gathered}\tag{6.14}$$

$$\begin{gathered}\forall\varphi\in C_c^\infty((0,T)\times\overline{\Omega};R^3),\quad \varphi\cdot\mathbf{n}=0\quad\text{on }(0,T)\times\partial\Omega,\\ \int_0^\infty\int_\Omega\Big(\varepsilon\mathbf{z}_\varepsilon\partial_t\varphi+p'(\overline{\varrho})\varrho_\varepsilon^{(1)}\operatorname{div}\varphi\Big)\mathrm{d}x\mathrm{d}t\\ =\varepsilon\int_0^T\int_\Omega\Big(-\mathbb{T}_\varepsilon:\nabla\varphi-\mathbf{F}_\varepsilon\cdot\varphi+g_\varepsilon\operatorname{div}\varphi\Big)\mathrm{d}x\mathrm{d}t,\end{gathered}\tag{6.15}$$

where we have set

$$\varrho_\varepsilon^{(1)}=\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon},\quad \mathbf{z}_\varepsilon=\varrho_\varepsilon\mathbf{u}_\varepsilon\tag{6.16}$$

and

$$\begin{aligned}&\mathbb{T}_\varepsilon=\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon-\mathbb{S}(\nabla\mathbf{u}_\varepsilon),\quad \mathbf{F}_\varepsilon=\varrho_\varepsilon\mathbf{f},\\ &g_\varepsilon=\Big(\gamma\overline{\varrho}^{\gamma-1}\frac{1}{\varepsilon}\Big[\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}\Big]_{\rm res}-\gamma(\gamma-1)z^{\gamma-2}\Big[\Big(\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}\Big)^2\Big]_{\rm ess}\\ &\qquad-\frac{1}{\varepsilon}\Big[\frac{\varrho_\varepsilon^\gamma-\overline{\varrho}^\gamma}{\varepsilon}\Big]_{\rm res}\Big),\quad z\in(\overline{\varrho}/2,2\overline{\varrho}).\end{aligned}\tag{6.17}$$

To identify the basic modes of (6.14), (6.15), we are naturally led to the eigenvalue problem

$$\nabla_x\omega=\lambda\mathbf{V},\quad \operatorname{div}\mathbf{V}=\lambda\omega\quad\text{in }\Omega,\quad \mathbf{V}\cdot\mathbf{n}|_{\partial\Omega}=0,\tag{6.18}$$

which is equivalent to the eigenvalue problem for the Laplace operator

$$\Delta\omega=-\Lambda\omega,\quad\text{in }\Omega,\quad \nabla_x\omega\cdot\mathbf{n}|_{\partial\Omega}=0,\quad \lambda^2=-\Lambda.\tag{6.19}$$

The latter problem admits in $L^2(\Omega)$ a system of real eigenfunctions $\{\omega_{j,m}\}_{j=0,m=1}^{\infty,m_j}$, which forms an orthonormal basis in $L^2(\Omega)$, corresponding to real eigenvalues

$$\begin{gathered}(\Lambda_0=)\Lambda_{0,1}=0,\quad \omega_0=\omega_{0,1}=1/|\Omega|\text{ and }m_0=1,\\ 0<\Lambda_{1,1}=\cdots=\Lambda_{1,m_1}=(\Lambda_1)<\Lambda_{2,1}=\cdots=\Lambda_{2,m_2}(=\Lambda_2)<\cdots,\end{gathered}\tag{6.20}$$

where $m_j$ denotes the multiplicity of the eigenvalue $\Lambda_j$.

Accordingly, the original system (6.18) admits solutions

$$\begin{gathered}\mathbf{V}_{j,m}=-\mathrm{i}(\sqrt{\Lambda_j})^{-1}\nabla\omega_{j,m},\quad \lambda_j=\mathrm{i}\sqrt{\Lambda_j},\quad\text{where } j=1,2,\ldots,\\ \lambda_0=0,\quad\text{with eigenspace }\dot{\mathbf{L}}(\Omega)=\mathbf{H}(L^2(\Omega;R^3)),\end{gathered}\tag{6.21}$$

meaning that

$$L^2(\Omega;R^3) = \mathbf{H}(L^2(\Omega;R^3)) \oplus \mathbf{H}^{\perp}(L^2(\Omega;R^3)),$$
$$\text{where } \mathbf{H}^{\perp}(L^2(\Omega;R^3)) = \overline{\{\mathrm{span}\{\mathrm{i}\mathbf{V}_{j,m}\}_{j=1,m=1}^{\infty,m_j}\}}^{L^2(\Omega;R^3)}. \tag{6.22}$$

In the sequel we denote, for $a \in L^1(\Omega)$, $\mathbf{z} \in L^1(\Omega;R^3)$,

$$[a]_{j,m} = \int_\Omega a\omega_{j,m}\mathrm{d}x, \quad [\mathbf{z}]_{j,m} = \int_\Omega \mathbf{z}\cdot\mathbf{V}_{j,m}\mathrm{d}x \tag{6.23}$$

and

$$\{a\}_M = \sum_{\{j>0|\Lambda_j\le M\}}\sum_{m=1}^{m_j}[a]_{j,m}\omega_{j,m}, \quad \{\mathbf{z}\}_M = \sum_{\{j>0|\Lambda_j\le M\}}\sum_{m=1}^{m_j}[\mathbf{z}]_{j,m}\mathbf{V}_{j,m}, \tag{6.24}$$

where $M$ is a fixed integer.

Since $\omega_{j,m}$, $\mathbf{V}_{j,m}$ are smooth functions on $\overline{\Omega}$ and $\mathbf{V}_{j,m}\cdot\mathbf{n}|_{\partial\Omega} = 0$, we can use them as test functions in (6.14–6.15) to obtain

$$\begin{aligned} \varepsilon\partial_t[\varrho_\varepsilon^{(1)}]_{j,m} + \mathrm{i}\sqrt{\Lambda_j}[\mathbf{z}_\varepsilon]_{j,m} &= 0,\\ \varepsilon\partial_t[\mathbf{z}_\varepsilon]_{j,m} + \mathrm{i}\sqrt{\Lambda_j}p'(\overline{\varrho})[\varrho_\varepsilon^{(1)}]_{j,m} &= \varepsilon A_\varepsilon^{j,m} \end{aligned} \tag{6.25}$$

where

$$A_\varepsilon^{j,m} = \int_\Omega \Big( - \mathbb{T}_\varepsilon : \nabla\mathbf{V}_{j,m} - \varrho_\varepsilon\mathbf{f}\cdot\mathbf{V}_{j,m} + g_\varepsilon\mathrm{div}\mathbf{V}_{j,m}\Big)\mathrm{d}x$$

is bounded in $L^2(0,T)$.

Finally multiplying the first equation in (6.25) by $\omega_{j,m}$ and the second one by $\mathbf{V}_{j,m}$, we get, after some calculus,

$$\left\{\begin{aligned} \varepsilon\partial_t\{\varrho_\varepsilon^{(1)}\}_M + \mathrm{div}\{\mathbf{z}_\varepsilon\}_M &= 0,\\ \varepsilon\partial_t\{\mathbf{z}_\varepsilon\}_M + p'(\overline{\varrho})\nabla_x\{\varrho_\varepsilon^{(1)}\}_M &= \varepsilon\mathbf{a}_{\varepsilon,M}, \end{aligned}\right\} \quad \text{a.e. in } (0,T)\times\Omega, \tag{6.26}$$

where

$$\begin{gathered}\mathbf{a}_{\varepsilon,M} = \sum_{\{j>0|\Lambda_j\le M\}}\sum_{m=1}^{m_j} A_\varepsilon^{j,m}\mathbf{V}_{j,m}\\ \text{is bounded in } L^2(0,T;C^k(\overline{\Omega};R^3)),\ k\in N\end{gathered} \tag{6.27}$$

for any fixed natural number $M$, uniformly with respect to $\varepsilon \to 0$.

### 6.3. Treatment of the convective term

As far as term (6.13) is concerned, we can write

$$\begin{aligned}\mathbf{H}^{\perp}(\mathbf{z}_\varepsilon)\otimes\mathbf{H}^{\perp}(\mathbf{u}_\varepsilon) = &\Big[\Big\{\mathbf{H}^{\perp}(\mathbf{z}_\varepsilon)\Big\}_M + \Big[\mathbf{H}^{\perp}(\mathbf{z}_\varepsilon) - \Big\{\mathbf{H}^{\perp}(\mathbf{z}_\varepsilon)\Big\}_M\Big]\Big]\\ &\otimes\Big[\Big\{\mathbf{H}^{\perp}(\mathbf{u}_\varepsilon)\Big\}_M + \Big[\mathbf{H}^{\perp}(\mathbf{u}_\varepsilon) - \Big\{\mathbf{H}^{\perp}(\mathbf{u}_\varepsilon)\Big\}_M\Big]\Big].\end{aligned} \tag{6.28}$$

We have

$$\mathbf{H}^{\perp}(\mathbf{z}_\varepsilon) - \Big\{\mathbf{H}^{\perp}(\mathbf{z}_\varepsilon)\Big\}_M = \Big[\mathbf{H}^{\perp}\Big((\varrho_\varepsilon - \overline{\varrho})\mathbf{u}_\varepsilon\Big) - \Big\{\mathbf{H}^{\perp}\Big((\varrho_\varepsilon - \overline{\varrho})\mathbf{u}_\varepsilon\Big)\Big\}_M\Big] + \overline{\varrho}\mathbf{H}^{\perp}(\mathbf{u}_\varepsilon) - \overline{\varrho}\Big\{\mathbf{H}^{\perp}(\mathbf{u}_\varepsilon)\Big\}_M,$$

where, according to (6.8),

$$\Big[\mathbf{H}^{\perp}\Big((\varrho_\varepsilon - \overline{\varrho})\mathbf{u}_\varepsilon\Big) - \Big\{\mathbf{H}^{\perp}\Big((\varrho_\varepsilon - \overline{\varrho})\mathbf{u}_\varepsilon\Big)\Big\}_M\Big] \to 0 \quad \text{in } L^2(0,T;L^{\frac{6}{5}}(\Omega)).$$

Furthermore, in agreement with (6.18–6.22),

$$\begin{aligned}\|\mathrm{div}\mathbf{u}_\varepsilon\|^2_{L^2(\Omega)} &= \Big\|\sum_{j=1}^{\infty}\sum_{m=1}^{m_j}[\mathbf{u}_\varepsilon]_{j,m}\mathrm{div}\mathbf{V}_{j,m}\Big\|^2_{L^2(\Omega)} \\ &= \Big\|\sum_{j=1}^{\infty}\sum_{m=1}^{m_j}[\mathbf{u}_\varepsilon]_{j,m}\sqrt{\Lambda_j}\omega_{j,m}\Big\|^2_{L^2(\Omega)} = \sum_{j=1}^{\infty}\sum_{m=1}^{m_j}\Lambda_j[\mathbf{u}_\varepsilon]^2_{j,m};\end{aligned}$$

whence

$$\begin{aligned}\|\mathbf{H}^{\perp}(\mathbf{u}_\varepsilon) - \{\mathbf{H}^{\perp}(\mathbf{u}_\varepsilon)\}_M\|_{L^2(\Omega)} &= \sum_{\{j>0|\Lambda_j>M\}}\sum_{m=1}^{m_j}[\mathbf{u}_\varepsilon]^2_{j,m} \\ &\leq \frac{1}{M}\|\mathrm{div}\mathbf{u}_\varepsilon\|^2_{L^2(\Omega)}.\end{aligned} \tag{6.29}$$

In view of these observations, the proof of (6.13) reduces to showing

$$\int_0^T\int_\Omega \Big\{\mathbf{H}^{\perp}\Big(\mathbf{z}_\varepsilon\Big)\Big\}_M \otimes \Big\{\mathbf{H}^{\perp}\Big(\mathbf{u}_\varepsilon\Big)\Big\}_M : \nabla_x\varphi \mathrm{d}x\mathrm{d}t \to 0$$

or equivalently, due to (6.1), (6.8),

$$\int_0^T\int_\Omega \Big\{\mathbf{H}^{\perp}\Big(\mathbf{z}_\varepsilon\Big)\Big\}_M \otimes \Big\{\mathbf{H}^{\perp}\Big(\mathbf{z}_\varepsilon\Big)\Big\}_M : \nabla_x\varphi \mathrm{d}x\mathrm{d}t \to 0, \tag{6.30}$$

for any divergenceless $\varphi$ as in (6.11).

By virtue of (6.21–6.22), $\{\mathbf{H}^{\perp}(\mathbf{z}_\varepsilon)\}_M = \nabla_x\Psi_\varepsilon$ and identity (6.26) can be rewritten

$$\begin{aligned}\varepsilon\partial_t d_\varepsilon + \Delta\Psi_\varepsilon &= 0,\\ \varepsilon\partial_t\nabla\Psi_\varepsilon + p'(\overline{\varrho})\nabla d_\varepsilon &= \varepsilon\mathbf{a}_{\varepsilon,M},\end{aligned} \tag{6.31}$$

where

$$\begin{aligned}\Psi_\varepsilon &= \mathrm{i}\sum_{\{j>0|\Lambda_j\leq M\}}\sum_{m=1}^{m_j}[\mathbf{z}_\varepsilon]_{j,m}\sqrt{\Lambda_j}\omega_{j,m},\\ d_\varepsilon &= \sum_{\{j>0|\Lambda_j\leq M\}}\sum_{m=1}^{m_j}[\varrho_\varepsilon^{(1)}]_{j,m}\omega_{j,m}.\end{aligned} \tag{6.32}$$

With this notation, and recalling that $d_\varepsilon$ is bounded in $W^{1,\infty}(0,T;C^k(\overline{\Omega}))$ and $\Psi_\varepsilon$ in $L^\infty(0,T;C^k(\overline{\Omega}))\cap W^{1,2}(0,T;C^k(\overline{\Omega}))$, $k\in N$, as a consequence of (6.23), (6.25), we can rewrite (6.30) as the following chain of identities:

$$
\begin{aligned}
&\int_0^T\int_\Omega \Big\{\mathbf{H}^\perp\Big(\mathbf{z}_\varepsilon\Big)\Big\}_M \otimes \Big\{\mathbf{H}^\perp\Big(\mathbf{z}_\varepsilon\Big)\Big\}_M : \nabla_x\varphi \mathrm{d}x\mathrm{d}t\\
&= \sum_{j,k=1}^3 \int_0^T\int_\Omega \partial_k\Psi_\varepsilon\partial_j\Psi_\varepsilon\partial_j\varphi_k\mathrm{d}x\mathrm{d}t\\
&= \frac12\sum_{k=1}^3\int_0^T\int_\Omega \partial_k|\nabla\Psi_\varepsilon|^2\varphi_k\mathrm{d}x\mathrm{d}t + \sum_{k=1}^3\int_0^T\int_\Omega \partial_k\Psi_\varepsilon\Delta\Psi_\varepsilon\varphi_k\mathrm{d}x\mathrm{d}t\\
&= \frac12\sum_{k=1}^3\int_0^T\int_\Omega \partial_k|\nabla\Psi_\varepsilon|^2\varphi_k\mathrm{d}x\mathrm{d}t - \varepsilon\sum_{k=1}^3\int_0^T\int_\Omega \partial_k\Psi_\varepsilon\partial_t d_\varepsilon\varphi_k\mathrm{d}x\mathrm{d}t\\
&= \frac12\sum_{k=1}^3\int_0^T\int_\Omega \partial_k|\nabla\Psi_\varepsilon|^2\varphi_k\mathrm{d}x\mathrm{d}t - \varepsilon\sum_{k=1}^3\int_0^T\int_\Omega \partial_t\Big(\partial_k\Psi_\varepsilon d_\varepsilon\Big)\varphi_k\mathrm{d}x\mathrm{d}t \qquad (6.33)\\
&\quad + \varepsilon\sum_{k=1}^3\int_0^T\int_\Omega \partial_t(\partial_k\Psi_\varepsilon)d_\varepsilon\varphi_k\mathrm{d}x\mathrm{d}t\\
&= \frac12\sum_{k=1}^3\int_0^T\int_\Omega \partial_k|\nabla\Psi_\varepsilon|^2\varphi_k\mathrm{d}x\mathrm{d}t + \varepsilon\sum_{k=1}^3\int_0^T\int_\Omega \partial_k\Psi_\varepsilon d_\varepsilon\partial_t\varphi_k\mathrm{d}x\mathrm{d}t\\
&\quad - \frac12 p'(\overline{\varrho})\sum_{k=1}^3\int_0^T\int_\Omega \partial_k|d_\varepsilon|^2\varphi_k\mathrm{d}x\mathrm{d}t + \varepsilon\int_0^T\int_\Omega d_\varepsilon\mathbf{a}_{\varepsilon,M}\cdot\varphi\mathrm{d}x\mathrm{d}t\\
&= \varepsilon\sum_{k=1}^3\int_0^T\int_\Omega \partial_k\Psi_\varepsilon d_\varepsilon\partial_t\varphi_k + \varepsilon\int_0^T\int_\Omega d_\varepsilon\mathbf{a}_{\varepsilon,M}\cdot\varphi\mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

where we have used several times equations (6.31) and the properties (6.11) of $\varphi$. By virtue of (6.27) the right-hand side of this chain of identities tends to 0 as $\varepsilon\to 0$. Proof of the limit (6.13) and of the part (i) of Lemma 5.1 is complete.

### 6.4. Incompressible energy inequality

We can rewrite (3.1) in the form

$$
\begin{aligned}
&\int_0^T\eta'(s)\int_\Omega \frac12\varrho_\varepsilon\mathbf{u}_\varepsilon^2\mathrm{d}x\mathrm{d}s + \int_0^T\eta'(s)\int_0^s\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}_\varepsilon):\nabla\mathbf{u}_\varepsilon\mathrm{d}x\mathrm{d}t\mathrm{d}s \qquad (6.34)\\
&\quad\le \int_0^T\eta'(s)\int_\Omega\Big(\frac12\varrho_\varepsilon\mathbf{u}_{\varepsilon,0}^2 + \frac{1}{\varepsilon^2}\mathcal{H}(\varrho_{\varepsilon,0})\Big)\mathrm{d}x\mathrm{d}s + \int_0^T\eta'(s)\int_0^s\int_\Omega \varrho_\varepsilon\mathbf{f}\cdot\mathbf{u}_\varepsilon\mathrm{d}x\mathrm{d}t\mathrm{d}s,
\end{aligned}
$$

where $\eta\in C_c^\infty([0,T))$, $\eta\le 0$, $\eta'\ge 0$. Due to (6.1–6.2) and (3.11),

$$\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon \to \sqrt{\overline{\varrho}}\mathbf{U} \quad \text{weakly in } L^\infty(0,T;L^2(\Omega)). \qquad (6.35)$$

Letting $\varepsilon \to 0+$ in (6.34), taking advantage of (6.2), (6.35), the lower weak continuity at the left-hand side and (6.3) at the right-hand side, we get

$$\begin{aligned} &\int_0^T \eta' \int_\Omega \frac{1}{2}\overline{\varrho}\mathbf{U}^2 \mathrm{d}x\mathrm{d}s + \int_0^T \eta'(s) \int_0^s \int_\Omega \mathbb{S}(\nabla_x \mathbf{U}) : \nabla \mathbf{U} \mathrm{d}x\mathrm{d}t\mathrm{d}s \\ &\leq \liminf_{\varepsilon\to 0+} \Big[ \int_0^\infty \eta' \int_\Omega \Big( \frac{1}{2}\varrho_\varepsilon \mathbf{u}_{\varepsilon,0}^2 + \frac{1}{\varepsilon^2}\mathcal{H}(\varrho_{\varepsilon,0})\Big) \mathrm{d}x\mathrm{d}s \Big] \\ &\quad + \int_0^T \eta' \int_0^s \int_\Omega \varrho \mathbf{f}\cdot \mathbf{U} \mathrm{d}x\mathrm{d}t\mathrm{d}s. \end{aligned} \tag{6.36}$$

By properly choosing $\eta$'s we deduce by standard arguments (5.6).

### 6.5. Reconstruction of pressure in the non-steady case

In this Section we complete the proof of Lemma 5.1 by reconstructing the pressure. The reconstruction of pressure in the non-steady case will be based on the *maximal $L^p - L^p$ regularity to the non-stationary Stokes system*

$$\begin{gathered} \partial_t \mathbf{U} + \nabla \Pi = \mu \mathrm{div}(\nabla_x \mathbf{U} + \nabla_x^\perp \mathbf{U}) + \mathbf{F} \text{ a.e. in } (0,T)\times\Omega, \\ \mathrm{div}\mathbf{U} = 0, \text{ a. e. in } (0,T)\times\Omega \end{gathered} \tag{6.37}$$

endowed with the initial conditions

$$\mathbf{U}(0,x) = \mathbf{U}_0(x), \quad x\in\Omega \tag{6.38}$$

and boundary conditions

$$\mathbf{U}\cdot\mathbf{n}|_{\partial\Omega} = 0, \ (\nabla_x \mathbf{U} + \nabla_x^\perp \mathbf{U})\mathbf{n}\times\mathbf{n}|_{\partial\Omega} = 0 \text{ a.e. in } (0,T) \tag{6.39}$$

in the sense of traces on $\partial\Omega$. The theorem reads:

**Lemma 6.1.** *Let $\Omega \subset R^3$ be a bounded domain of class $C^{2,\nu}$, $\nu\in(0,1)$, $1<p<\infty$, $\mu>0$. Suppose that*

$$\begin{gathered} \mathbf{F}\in L^p((0,T)\times(\Omega;R^3)), \ \mathbf{U}_0 \in W^{2-\frac{2}{p},p}(\Omega;R^3), \ \mathrm{div}\mathbf{U}_0 = 0, \\ \mathbf{U}_0\cdot\mathbf{n}|_{\partial\Omega} = 0 \textit{ if } 1-\tfrac{3}{p} < 0, \end{gathered}$$

*where $W^{2-\frac{2}{p},p}(\Omega;R^3)$ denotes the Sobolev-Slobodeckii space.*

*Then the problem (6.37–6.39) admits a solution $(\Pi,\mathbf{U})$, unique in the class*

$$\begin{gathered} \mathbf{U}\in L^p(0,T;W^{2,p}(\Omega;R^3)), \ \partial_t\mathbf{U}\in L^p(0,T;L^p(\Omega;R^3)), \\ \mathbf{U}\in C([0,T];W^{2-\frac{2}{p},p}(\Omega;R^3)), \ \Pi\in L^p(0,T;W^{1,p}(\Omega)), \ \int_\Omega \Pi \mathrm{d}x = 0. \end{gathered}$$

*Moreover, there exists a positive constant $c = c(p,q,\Omega,T,\overline{\mu})$ such that*

$$\begin{aligned} &\mathbf{U}(t)\|_{W^{2-\frac{2}{p},p}(\Omega;R^3)} + \|\partial_t U\|_{L^p(0,T;L^p(\Omega;R^3))} \\ &\quad + \|\mathrm{div}(\nabla_x\mathbf{U}+\nabla_x^\perp\mathbf{U})\|_{L^p(0,T;L^p(\Omega;R^3))} + \|\nabla\Pi\|_{L^p((0,T)\times\Omega;R^3)} \\ &\leq c(\|\mathbf{F}\|_{L^p((0,T)\times\Omega;R^3)} + \|\mathbf{U}_0\|_{W^{2-\frac{2}{p},p}(\Omega;R^3)}), \end{aligned} \tag{6.40}$$

*for any $t\in[0,T]$.*

The original formulation and proof of this result is due to Solonnikov [45]. For the more general $L^p - L^q$-versions see, e.g., Shibata, Shimizu [44] or Saal, [40].

**Remark.** *The reader shall notice that the bounds for* $\mathrm{div}(\nabla_x \mathbf{U} + \nabla_x^\perp \mathbf{U})$ *and for* $\mathbf{U}$ *in* (6.40) *imply in addition*

$$\|\mathbf{U}\|_{L^p(0,T;W^{2,p}(\Omega;R^3))} \le c(\|\mathbf{F}\|_{L^p((0,T)\times\Omega;R^3)} + \|\mathbf{U}_0\|_{W^{2-\frac{2}{p},p}(\Omega;R^3)}) \tag{6.41}$$

*via a standard uniqueness Agmon, Douglis, Nirenberg argument and the* $L^p$ *version of the Korn inequality.*

Coming back to the proof, we rewrite identity (5.4) in the form

$$\begin{aligned}\int_0^T \int_\Omega \Big(\overline{\varrho}\mathbf{U}\cdot\partial_t\varphi + \mu\mathbf{U}\cdot\mathrm{div}(\nabla_x\varphi + \nabla_x^\perp\varphi)\Big)\,\mathrm{d}x\mathrm{d}t \\ = \int_0^T\int_\Omega \mathbf{G}\cdot\varphi\,\mathrm{d}x\mathrm{d}t - \int_\Omega \overline{\varrho}\mathbf{H}(\mathbf{U}_0)\cdot\varphi(0,x)\,\mathrm{d}x,\end{aligned} \tag{6.42}$$

where

$$\mathbf{G} = \overline{\varrho}\mathbf{U}\cdot\nabla_x\mathbf{U} - \overline{\varrho}\mathbf{f}$$

and

$$\begin{aligned}\varphi \in C_c^\infty([0,T)\times\overline{\Omega};R^3),\ \varphi\cdot\mathbf{n}|_{[0,T)\times\partial\Omega} = 0, \\ (\nabla_x\varphi + \nabla_x^\perp\varphi)\mathbf{n}\times\mathbf{n}|_{[0,T)\times\partial\Omega} = 0,\ \mathrm{div}\varphi = 0.\end{aligned} \tag{6.43}$$

We compare $\mathbf{U}$ with $\mathbf{V}$, the unique strong solution of the Stokes problem,

$$\begin{aligned}\overline{\varrho}\partial_t\mathbf{V} - \mu\mathrm{div}(\nabla_x\mathbf{V} + \nabla_x^\perp\mathbf{V}) + \nabla_x\Pi = -\mathbf{G}, \\ \mathrm{div}\mathbf{V} = 0,\ \mathbf{V}(0,x) = [(\mathbf{H}(\mathbf{u}_0)](x), \\ \mathbf{V}\cdot\mathbf{n}|_{\partial\Omega} = 0\ \big((\nabla_x\mathbf{V} + \nabla_x^T\mathbf{V})\mathbf{n}\big)\times\mathbf{n}|_{\partial\Omega} = 0 \text{ a.e. in } (0,T).\end{aligned} \tag{6.44}$$

Due to (5.1–5.2), $\mathbf{G}\in L^{\frac{5}{4}}((0,T)\times\Omega))$; whence Theorem 6.1 affirms, in particular, that

$$\begin{aligned}\mathbf{V}\in C([0,T];L^{\frac{5}{4}}(\Omega;R^3))\cap L^{\frac{5}{4}}(0,T;W^{2,\frac{5}{4}}(\Omega;R^3)), \\ \partial_t\mathbf{V}\in L^{\frac{5}{4}}((0,T)\times\Omega;R^3)),\ \Pi\in L^{\frac{5}{4}}(0,T;L^{\frac{5}{4}}(\Omega)),\ \int_\Omega \Pi\mathrm{d}x = 0.\end{aligned} \tag{6.45}$$

Subtracting (6.42) and (6.44) we obtain

$$\int_0^T\int_\Omega \Big(\overline{\varrho}(\mathbf{U}-\mathbf{V})\cdot\partial_t\varphi + \mu(\mathbf{U}-\mathbf{V})\mathrm{div}(\nabla_x\varphi + \nabla_x^T\varphi)\Big)\,\mathrm{d}x\mathrm{d}t = 0 \tag{6.46}$$

with any $\varphi$ such that

$$\begin{aligned}\mathrm{div}\varphi = 0, \\ \varphi\in\cap_{r\in(1,\infty)}L^r(0,T;W^{2,r}(\Omega;R^3)),\ \partial_t\varphi\in\cap_{r\in(1,\infty)}L^r((0,T)\times\Omega;R^3), \\ \varphi(T,x) = 0,\ x\in\Omega, \\ \varphi\cdot\mathbf{n}|_{\partial\Omega} = 0,\ (\nabla_x\varphi + \nabla_x^T\varphi)\mathbf{n}\times\mathbf{n}|_{\partial\Omega} = 0 \text{ a.e.in } (0,T),\end{aligned} \tag{6.47}$$

where the last two identities are satisfied in the sense of traces. In order to enlarge the set of test functions from (6.43) to (6.47), we have employed (5.1–5.3), (6.45) and density argument.

In view of Theorem 6.1 we can use in (6.46) any test function $\varphi$, $\varphi(t,x) = \phi(T-t,x)$, where $(p,\phi)$ solves the Stokes problem

$$\overline{\varrho}\partial_t\phi - \overline{\mu}\mathrm{div}(\nabla_x\phi + \nabla_x^T\phi) + \nabla_x p = \mathbf{F},\ \mathrm{div}\phi = 0,\ \text{a.e. in } (0,T)\times\Omega,$$
$$\phi(0,x) = 0,\ x\in\Omega,$$
$$\phi\cdot\mathbf{n}|_{\partial\Omega} = 0,\ (\nabla_x\phi + \nabla_x^T\phi)\mathbf{n}\times\mathbf{n}|_{\partial\Omega} = 0 \text{ a.e.in } (0,T),$$

where $\mathbf{F}\in C_c^\infty((0,T)\times\Omega;R^3)$. We thus get

$$\int_0^T\int_\Omega(\mathbf{U}-\mathbf{V})\cdot\mathbf{F}\,\mathrm{d}x\mathrm{d}t = 0 \text{ for all } \mathbf{F}\in C_c^\infty((0,T)\times\Omega;R^3),$$

where we have used the fact that $\int_0^T\int_\Omega \mathbf{U}\cdot\nabla_x p\,\mathrm{d}x\mathrm{d}t = \int_0^T\int_\Omega \mathbf{V}\cdot\nabla_x p\,\mathrm{d}x\mathrm{d}t = 0$. Therefore $\mathbf{U}=\mathbf{V}$. This completes the proof of identity (5.7) as well as of the whole Lemma 5.1.

### 6.6. Conclusion

Combining the "fast" time solutions constructed in Lemma 4.1 with the "real" time solutions of Lemma 5.1 finishes the proof of Theorem 2.1.

## 7. Proof of Lemma 5.2 and Theorem 2.2

### 7.1. Prodi-Serrin conditions

Sequences $\varrho_\varepsilon$, $\mathbf{u}_\varepsilon$, $r_\varepsilon^{(1)}$, $\mathbf{q}_\varepsilon$ satisfy assumptions of Theorem 2.1. They therefore admit limits $\overline{\varrho}$, $\mathbf{U}$, $r^{(1)}$, $\mathbf{q}$ in the sense (2.41–2.44); $r^{(1)}$, $\mathbf{q}$ satisfy equations (4.2) and (4.3) of Lemma 4.1, while $\mathbf{U}$ verifies equation (5.4) and inequality (5.6), where

$$\liminf_{\varepsilon\to 0+}\Big[\int_\Omega\Big(\frac{1}{2}\varrho_\varepsilon\mathbf{u}_{\varepsilon,0}^2 + \frac{1}{\varepsilon^2}\mathcal{H}(\varrho_{\varepsilon,0})\Big)\mathrm{d}x\Big] + \int_0^\tau\int_\Omega\varrho\mathbf{f}\cdot\mathbf{U}\mathrm{d}x\mathrm{d}t = \int\frac{1}{2}\overline{\varrho}|\mathbf{u}_0|^2\,\mathrm{d}x.$$

Therefore, $\mathbf{U}$ is a Leray-Hopf weak solution of problem (2.21), (2.7) with initial data $\mathbf{U}_0 = \mathbf{u}_0$.

At this point, we recall the celebrated Prodi-Serrin uniqueness conditions for the Navier-Stokes equations (cf. Serrin [43]).

**Lemma 7.1.** *Let* $\mathbf{v}$ *and* $\mathbf{w}$ *be two weak solutions of Navier-Stokes equations* (2.21) *with boundary conditions* (2.7) *and the same initial conditions. Let* $\mathbf{v}$ *be a Leray-Hopf weak solution and*

$$\mathbf{w}\in L^r(0,T;L^s(\Omega;R^3)),\ \textit{for some } r,s \textit{ such that } \tfrac{3}{s}+\tfrac{2}{r}=1,\ s\in(3,\infty).$$

*Then* $\mathbf{v}=\mathbf{w}$.

We apply this lemma with $\mathbf{v} = \mathbf{U}$ and $\mathbf{w} = \mathbf{u}_0$. We already know that $\mathbf{U}$ is a Leray-Hopf weak solution with initial conditions $\mathbf{u}_0$. Recall that $\mathbf{u}_0$ verifies (2.30–2.31) meaning that $\mathbf{w}$ is a weak solution of the non-stationary problem with initial data $\mathbf{w}(0) = \mathbf{u}_0$. Moreover, being time independent, $\mathbf{w}$ belongs to $L^\infty(0,T;L^6(\Omega;R^3))$ and verifies therefore the Prodi-Serrin conditions. Lemma 7.1 yields $\mathbf{v} = \mathbf{w}$ or equivalently $\mathbf{U} = \mathbf{u}_0$.

### 7.2. Reconstruction of the pressure in the steady case

At this point we make a pause to introduce natural spaces useful for the investigation of pressure in the Navier-Stokes equations (2.21) with boundary conditions (2.7), and to list some of their properties needed in the sequel.

For $1 \le p < \infty$, we set

$$W^{1,p}_{\mathbf{n}}(\Omega;R^3) = \{\mathbf{z} \in W^{1,p}(\Omega;R^3)\,|\,\mathbf{z}\cdot\mathbf{n}|_{\partial\Omega} = 0\}$$

a closed subspace of $W^{1,p}(\Omega;R^3)$ and by

$$\dot{W}^{1,p}_{\mathbf{n}}(\Omega;R^3) = \{\mathbf{z} \in W^{1,p}(\Omega;R^3)\,|\,\mathbf{z}\cdot\mathbf{n}|_{\partial\Omega} = 0,\ \mathrm{div}\mathbf{z} = 0\}.$$

The set

$$C^{2,\nu}_{c,\mathbf{n}} = \{\mathbf{z} \in C^{2,\nu}(\overline{\Omega};R^3)\,|\,\mathbf{z}\cdot\mathbf{n}|_{\partial\Omega} = 0\}$$

is dense in $W^{1,p}_{\mathbf{n}}(\Omega;R^3)$ and the set

$$\dot{C}^{2,\nu}_{c,\mathbf{n}} = \{\mathbf{z} \in C^{2,\nu}(\overline{\Omega};R^3)\,|\,\mathbf{z}\cdot\mathbf{n}|_{\partial\Omega} = 0,\ \mathrm{div}\mathbf{z} = 0\}$$

is dense in $\dot{W}^{1,p}_{\mathbf{n}}(\Omega;R^3)$, see, e.g., [18, Section 10.7].

The functional

$$\mathcal{F} : W^{1,2}_{\mathbf{n}}(\Omega) \to R \quad \text{defined by}$$

$$\begin{aligned}\mathcal{F}_\tau(\varphi) = &\int_\Omega \overline{\varrho}(\mathbf{u}(\tau) - \mathbf{u}_0)\cdot\varphi \mathrm{d}x \\ &+ \int_0^\tau\int_\Omega \Big[\Big(\mu(\nabla_x\mathbf{u} + \nabla_x^\perp\mathbf{u}) - \overline{\varrho}(\mathbf{u}\otimes\mathbf{u})\Big) : \nabla_x\varphi - \overline{\varrho}\mathbf{f}\cdot\varphi\Big]\,\mathrm{d}x\mathrm{d}t\end{aligned} \tag{7.1}$$

is a continuous linear functional on $W^{1,2}_{\mathbf{n}}(\Omega;R^3)$ vanishing on $\dot{W}^{1,2}_{\mathbf{n}}(\Omega;R^3)$.

Now, we shall recall several facts from functional analysis of (unbounded) linear operators. Let $A : X \to Y$, where $X$ and $Y$ are Banach spaces, be a linear (unbounded) closed and densely defined operator. Denote by $A^* : Y^* \to X^*$ its adjoint, where $X^*$ and $Y^*$ are duals to $X$ and $Y$, respectively. It is well known that $\ker(A) = (\mathcal{R}(A^*))^\perp$, $\ker(A^*) = (\mathcal{R}(A))^\perp$, and also $(\mathcal{R}(A)) = (\ker(A*))^\perp$ provided $\mathcal{R}(A)$ is closed (in $Y$), $(\mathcal{R}(A^*)) = (\ker(A))^\perp$ provided $\mathcal{R}(A^*)$ is closed (in $X^*$), see, e.g., Brezis [7]. Thus, if a continuous linear functional $\mathcal{F} : X \to R$ vanishes on $\ker(A)$ then $\mathcal{F} \in (\ker(A))^\perp = \mathcal{R}(A^*)$, provided the last subspace is closed (in $X^*$).

Next, we mention some properties of the Bogovskii solution operator to the problem $\mathrm{div}\mathbf{z} = g$, [5]. We shall need the following facts. For any bounded Lipschitz domain, there exists a linear operator $\mathcal{B}$ with the following properties:

- $$\mathcal{B}:\overline{L^p}(\Omega):=\left\{g\in L^p(\Omega)\,|\,\int_\Omega z\mathrm{d}x=0\right\}\to W_0^{1,p}(\Omega,R^3),\quad \forall 1<p<\infty; \tag{7.2}$$
- $$\mathrm{div}\mathcal{B}(g)=g,\quad \forall g\in\overline{L^p}(\Omega); \tag{7.3}$$
- $$\|\mathcal{B}(g)\|_{W^{1,p}(\Omega)}\le c(p,\Omega)\|g\|_{L^p(\Omega)},\quad \forall g\in\overline{L^p}(\Omega); \tag{7.4}$$
- $$\text{if } g\in\overline{C_c^\infty}(\Omega):=\{g\in C_c^\infty(\Omega)\,|\,\textstyle\int_\Omega g dx=0\}\text{ then }\mathcal{B}(g)\in C_c^\infty(\Omega). \tag{7.5}$$

Such an operator can be constructed explicitly; we refer to Galdi [20] or to [38], Section 3.3 for more details and further properties.

The operator $A=\mathrm{div}$ from $X=W^{1,p}_{\mathbf{n}}(\Omega;R^3)$ to $Y=L^p(\Omega)$ with $\mathcal{D}(A)=W^{1,p}_{\mathbf{n}}(\Omega;R^3)$, $\ker(A)=\dot W^{1,p}_{\mathbf{n}}(\Omega;R^3)$, $\mathcal{R}(A)=\overline{L^p}(\Omega)$ is a continuous operator (hence $\mathcal{R}(A)$ is closed). (The surjectivity of $A$ onto $\overline{L^p}(\Omega)$ follows from the property (7.3) of the Bogovskii operator.) Therefore, for all $\mathbf{v}\in W^{1,p}_{\mathbf{n}}(\Omega;R^3)$, $\psi\in C_c^\infty(\Omega)$ a dense subset of $Y^*=L^{p'}(\Omega)$, we have

$$\langle\psi\,,\,A\mathbf{v}\rangle_{Y^*,Y}=\int_\Omega\psi\mathrm{div}\mathbf{v}\mathrm{d}x=\int_\Omega\mathbf{v}\cdot\nabla_x\psi\mathrm{d}x=\langle A^*\psi\,,\,\mathbf{v}\rangle_{X^*,X}. \tag{7.6}$$

The last identity implies that $A^*:L^{p'}(\Omega)\to(W^{1,p}_{\mathbf{n}}(\Omega;R^3))^*$ is continuous with $\mathcal{D}(A^*)=L^{p'}(\Omega)$ and a range $\mathcal{R}(A^*)$ closed in $(W^{1,p}_{\mathbf{n}}(\Omega;R^3))^*$.

We apply this result to the linear functional

$$\mathcal{F}:W^{1,2}_{\mathbf{n}}(\Omega)\to R,$$
$$\mathcal{F}(\varphi)=\int_\Omega\Big[\Big(\mu(\nabla_x\mathbf{u}+\nabla_x^\perp\mathbf{u})-\overline{\varrho}\mathbf{u}\otimes\mathbf{u}\Big):\nabla_x\varphi-\overline{\varrho}\mathbf{f}\cdot\varphi\Big]\mathrm{d}x$$

and obtain existence of $\Pi\in L^2(\Omega)$, $\int_\Omega\Pi\mathrm{d}x=0$ such that $\mathcal{F}(\varphi)=\int_\Omega\Pi\mathrm{div}\varphi\mathrm{d}x$. We thus have

**Lemma 7.2.** *Let $\mathbf{u}_0$ satisfy* (2.30), (2.31). *Then there exists a unique*

$$\Pi\in L^2(\Omega),\quad \int_\Omega\Pi\mathrm{d}x=0$$

*such that*

$$\begin{gathered}\forall\varphi\in C_c^\infty((0,T)\times\overline{\Omega}),\quad \varphi\cdot\mathbf{n}|_{(0,T)\times\partial\Omega},\\ \int_0^T\int_\Omega\Big(\overline{\varrho}\mathbf{u}_0\otimes\mathbf{u}_0-\mu(\nabla_x\mathbf{u}_0+\nabla_x^\perp\mathbf{u}_0)\Big):\nabla_x\varphi\mathrm{d}x\mathrm{d}t\\ +\int_0^T\int_\Omega\Pi\mathrm{div}\varphi\mathrm{d}x\mathrm{d}t=-\int_0^T\int_\Omega\overline{\varrho}\mathbf{f}\cdot\varphi\mathrm{d}x\mathrm{d}t.\end{gathered} \tag{7.7}$$

Now, regarding (7.7) as a (steady) Stokes problem with the right-hand side $-\overline{\varrho}\mathbf{u}_0\cdot\nabla_x\mathbf{u}_0\in L^{\frac{3}{2}}(\Omega;R^3)$, we may use the standard regularity results for the Stokes problem combined with the uniqueness (cf., e.g., Galdi [20]) to conclude that $\mathbf{u}_0$ must necessarily belong to $W^{2,\frac{3}{2}}(\Omega;R^3)$ and $\Pi$ to $W^{1,\frac{3}{2}}(\Omega)$.

### 7.3. Conclusion

Now, we conclude the proof of Theorem 2.2 by replacing in (4.2) $r^{(1)}$ by $r^{(1)} - (p(\overline{\varrho}))^{-1}\Pi$ ($\Pi$ being time independent, $\int_0^T \int_\Omega \Pi \partial_t \varphi \mathrm{d}x\mathrm{d}t = 0$) and by subtracting (7.7) from equation (4.3).

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# On the Uniqueness of Solutions to Boundary Value Problems for Non-stationary Euler Equations

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**Abstract.** We seek for extremely wide classes of generalized solutions to boundary value problems for nonsteady Euler equations where uniqueness still holds. Such classes appear to be formulated rather compactly in terms of the Orlicz spaces. This result is obtained with extrapolatory techniques in the scale of symmetric spaces which have been developed by the author based on integral representations and transforms of $N$-functions generating Orlicz spaces.

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## 1. Introduction

The subject of the paper is the Euler system which describes nonsteady flows of an ideal incompressible fluid:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f}, \qquad \mathrm{div}\boldsymbol{u} = 0. \tag{1.1}$$

Here, as usual, $\boldsymbol{u}$ stands for the velocity vector, $p$ is the pressure, and $\boldsymbol{f}$ is the specified vector of external mass forces; all these quantities are functions of the time $t$ and the space variables $\boldsymbol{x} \in \mathbb{R}^n$, $n \in \mathbb{N}$; the operators div and $\nabla$ act in $\boldsymbol{x}$.

The model (1.1) is much simplified in the sense of mechanics, and nevertheless appears to be rather substantial, which attracts mathematicians. The history of

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study of its mathematical well-posedness is extremely extensive and it cannot be reported in such a small paper as the present one. A more complete historical review can be found in [79], [23], [9], [16] and [52]. A number of important problems were solved as a result of this study, but, on the other hand, many mathematical questions for (1.1) are still waiting for their answers. We mention here the results, milestones and problems which are quite close to the subject of the paper.

### 1.1. Basic existence results

The study of solvability of the boundary value problems for the Euler equations was started in [26] and [41], where the local existence and uniqueness of the classical solutions (as $n = 3$) were proved. These results concern two boundary value problems: the Cauchy problem, i.e., the problem in $\mathbb{R}^n \times (0, T)$ with the initial data

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \tag{1.2}$$

and the "nonpenetration problem" in $Q_T = \Omega \times (0, T)$ which describes flows in a bounded domain $\Omega \subset \mathbb{R}^n$, with the initial data (1.2) and the boundary condition

$$\boldsymbol{u} \cdot \boldsymbol{n}|_{\partial\Omega\times(0,T)} = 0 \tag{1.3}$$

(which means the impermeable boundary) where $T > 0$ is a sufficiently small number, and $\boldsymbol{n}$ is the outward normal to $\partial\Omega$. These problems do not seem to be very important in applications, but they are the traditional starting point in the study of well-posedness of hydrodynamical models, since this eliminates additional difficulties in boundary conditions. In the framework of these two boundary value problems the first group of global results for two-dimensional flows was obtained, cf. [73] (classical solutions without external forces with some redundant smoothness of input data), the final exposition and generalization of this result in [28] (see also [27]); and finally the famous result [76] concerning the existence of solutions with vorticity in $L_p$ and their uniqueness in the class of bounded vorticity. The paper [76] also deals with the external problem with the decay condition at $\infty$, and this restriction (i.e., the decay) was later eliminated in [13] and [16]. In [68] the global existence was proved for three-dimensional axisymmetric flows, and the paper [21] contains the same result for similar but more general flows (helical etc.); further development of these ideas is described in [16].

### 1.2. Irregular flows

Nevertheless, there are no global theorems for general three-dimensional flows yet, even in "bad" classes or for small data. It may be caused, in particular, by fundamental differences between the cases $n = 2$ and $n = 3$ in the transport equation for the vorticity $\boldsymbol{\omega} = \mathrm{rot}\boldsymbol{u}$. Thus, the $L_\infty$-norm of the vorticity may not be conserved in time as $n = 3$ [59]. Moreover, the famous result [10] showed that, specifically, the infinite growth of this norm causes the appearance of singularities in a finite time period, observed in numerical experiments and special classes of solutions (such examples are mentioned, e.g., in [14], [15], [53] and [24]). Further development of the result [10] is shown in [24] and [9].

Singularities that inevitably appear in solutions of three-dimensional Euler equations stimulate appropriate definitions of irregular solutions (and the proof of their existence), as well as the proof of uniqueness in the widest possible classes. As far as the former problem is concerned, we should mention first the result [54] of global solvability of the two-dimensional non-penetration problem

$$(\mathrm{NP})=\{(1.1),(1.2),(1.3)\}$$

with vorticity slightly better than $L_1$ (see also the similar result [50] with more convenient formulations), and the papers [18], [19], [20] and [17] (see also [9], [42], [43] and [22]), where the similar result (three-dimensional axisymmetric case inclusively) is obtained for measure-valued vorticity (but definite-signed, with locally finite kinetic energy). Thus, the problem of a vortex sheet is partly solved. The problem of smoothness of the boundary of a vortex patch was investigated in [12] and [11]. The problem of point vortex (singularities of vorticity take the form of a $\delta$-function, the energy being locally infinite) is the subject of a series of papers; here the local results were obtained in [71] and [72], and global ones were initially obtained in [51] and [65] in particular cases, and then generally in [62] and [63] (as $n = 2$).

### 1.3. Uniqueness problem

As against the global existence, the dimension of the flow is unessential in the uniqueness problem. There exist nontrivial solutions of (1.1) with compact support in space-time provided that no restrictions for the vorticity are accepted (cf. [60] and [58]); this fact, in particular, stimulates the formulation of the uniqueness classes in terms of the vorticity. The first advancement in the uniqueness problem after [76] (where the uniqueness is proved for bounded vorticity) was the paper [79], where the uniqueness for (NP) is proved in the class of vorticity in specially normed scale of the spaces $L_p$ (see also the paper [70] with formulations in Besov spaces). A similar result is presented in [50] in terms of Orlicz spaces. The aim of the present paper is to complete this result, in particular, to consider the "through flow problems"[1] (i.e., the problems of the flow of a fluid through a domain), which are of greater interest in applications but are more difficult and less investigated mathematically.

### 1.4. Through flow problems

In through flow problems the boundary $\partial\Omega$ of the flow domain $\Omega$ splits into three non-intersecting fragments: $\Gamma_1$, $\Gamma_2$ and $\Gamma_0$ (which are the inflow section, outflow section and impermeable section correspondingly), where:

$$\boldsymbol{u}\cdot\boldsymbol{n}|_{\Gamma_0\times(0,T)} = 0, \tag{1.4}$$

$$\boldsymbol{u}\cdot\boldsymbol{n}|_{\Gamma_1\times(0,T)} = g_1 < 0, \tag{1.5}$$

$$\boldsymbol{u}\cdot\boldsymbol{n}|_{\Gamma_2\times(0,T)} > 0 \tag*{$(1.6)_1$}$$

[1]When we write the term "through flow" we however do not imply, as in [56], that there are no stagnant basins in the flow domain.

(only the sign is specified) or

$$\boldsymbol{u} \cdot \boldsymbol{n}|_{\Gamma_2 \times (0,T)} = g_2 > 0 \tag{1.6$_2$}$$

(the exact value is specified). It is clear that these conditions (as against (1.3)) are not sufficient now, and the choice of additional boundary conditions leads to several versions of the problem.

The first experience of the statement and solving the through flow problem was the paper [37], where the natural idea to specify the vorticity on the entrance (since the trajectories, on which the transport equation for the vorticity holds, start namely at $\Gamma_1 \times (0,T)$) was proposed, and the local existence and uniqueness theorem for the three-dimensional problem was announced. However, such a form of this idea is faulty since the value $\mathrm{div}\boldsymbol{\omega}$, which must be zero in $Q_T$, is transported along the mentioned trajectories, and consequently the compatibility condition for the vorticity at the entrance arises: $\mathrm{div}\boldsymbol{\omega}|_{\Gamma_1 \times (0,T)} = 0$. Nevertheless, the idea of [37] was completed (after necessary corrections) in further papers devoted to correct statements of three-dimensional through flow problem of the version under consideration (i.e., with the vorticity specified at the entrance). Namely, zero normal component and arbitrary tangent components of the vorticity were specified at the entrance (with the corresponding compatibility conditions) as the additional condition in [67], [80], and local existence and uniqueness of the strong solution were proved for arbitrary $n$'s (with the use of preceding results [40] and [25]). To specify two tangent components of the vorticity

$$\boldsymbol{\omega}_\sigma|_{\Gamma_1 \times (0,T)} = \boldsymbol{h} \tag{1.7}$$

(here $\boldsymbol{h}$ is a tangent vector field, and the index $\sigma$ stands for tangent component of a vector) as the additional condition at the entrance in the three-dimensional problem was the idea of the papers [36] and [30], where the local existence and uniqueness of classical solutions to the resulting problem

$$\text{(TF.I)}=\{(1.1),(1.2),(1.4),(1.5),(1.6)_2,(1.7)\}$$

(we call it "through flow problem of type I")[2] were proved. A.V.Kazhikhov (cf. [30], [36] and [6]) also considered other versions of boundary conditions for the vorticity (instead of (1.7)) and the case of a nonhomogeneous fluid, and he formulated the compatibility conditions in the case of the plain vortex though specified at the entrance, as in [37].

In the two-dimensional case, the problem (TF.I), as well as (NP), is deeply investigated globally. When $n = 2$, the problem may be considered as a three-dimensional one with $\Omega$ being the cylinder of infinite height, and the vorticity taking the form $\boldsymbol{\omega} = (0,0,\omega)$, so that the condition (1.7) means just that the scalar vorticity $\omega$ is specified at the entrance. Namely such a formulation of two-dimensional problem (TF.I) was studied in the classical paper [77], where the global existence and uniqueness of rather smooth (up to classical) solutions were

[2]Further we also meet through flow problems of types II and III; such numeration follows the paper [36].

proved. This result was generalized to the three-dimensional axisymmetric case in the paper [66]. Some additional constraints on $\Gamma_k$ and input data made in [77] were further released in [4], where (based on [8]) the global solvability of the two-dimensional problem (TF.I) was shown in the class of bounded vorticity, and further smoothness (and consequently the uniqueness)[3] of the solutions (up to the classical solution) was proved for more smooth data; in other words, the result of [28] was generalized to the problem (TF.I). Finally, the class of global existence for the two-dimensional problem (TF.I) was extended in [69] almost up to $\omega \in L_1$ (similarly to the result of [54] for the problem (NP)).

The steady version of the problem (TF.I) is also studied well. In the two-dimensional (and three-dimensional axisymmetric) case the global existence was proved in [2], and the uniqueness and smoothness of solutions were shown in [3], where stagnant basins were also studied. The local existence of the generalized solution to the three-dimensional steady problem is proved in [55]. Mention should be also made of the remarkable result concerning the conditions of "washout of vorticity" in a finite time period obtained in [5]. Additional information on the study of the problem (TF.I) may be found in [55].

Thus, the through flow problem in the form (TF.I) has been investigated for a long time and rather completely, however it does not seem to be very appropriate in applications, since prescription of the velocity at the entrance is more natural than the vorticity. The problems of the latter type were studied in detail in the papers of A.V.Kazhikhov. In the first version of such problems (let us call it "the through flow problem of type II") the full velocity vector is specified at the entrance, i.e., the tangent components of the velocity are specified in addition to (1.5):

$$\boldsymbol{u}_\sigma|_{\Gamma_1\times(0,T)} = \boldsymbol{r}, \tag{1.8}$$

and the pressure (instead of $(1.6)_2$) is specified at the exit:

$$p|_{\Gamma_2\times(0,T)} = p_*. \tag{1.9}$$

As a result, we come to the problem

$$\text{(TF.II)}=\{(1.1),(1.2),(1.4),(1.5),(1.6)_1,(1.8),(1.9)\}.$$

The papers [36] and [35] contain the proof of classical local unique solvability of two-dimensional (and three-dimensional axisymmetric) problem (TF.II) under the condition $\boldsymbol{u}_0\cdot\boldsymbol{n}|_{\Gamma_2} \geqslant C > 0$, nonhomogeneous fluid inclusively.

Finally, one more version of the through flow problem with the condition (1.8) at the entrance (let us call it "the through flow problem of type III") differs from the preceding one in the condition at the exit which is $(1.6)_2$ instead of (1.9), and the resulting problem is

$$\text{(TF.III)}=\{(1.1),(1.2),(1.4),(1.5),(1.6)_2,(1.8)\}.$$

[3]The specificity of the problem (TF.I) is that the uniqueness of its solution requires smoothness of the vorticity one order higher than in the problem (NP) or in the problems (TF.II) and (TF.III) stated below; see the details in Subsection 3.2.

This problem was studied in the steady version in [1]. The existence and uniqueness of a local classical solution to the nonsteady problem (TF.III) (as a matter of fact, only in two-dimensional and three-dimensional axisymmetric cases) was proved in [36] and [29] based on the ideas of [64] under the condition $g_1 \leqslant -C < 0$ (nonhomogeneous fluid inclusively). Further investigation in [32] and [31] gave a global result (in the class of bounded vorticity) for the two-dimensional problem (TF.III) in a rectangular domain and under the condition that the initial data are sufficiently close to the uniform (straight) flow[4].

It is appropriate to mention here that most of the cited results of A.V. Kazhikhov are included in the book of his selected works [33] published recently.

Thus, we have listed the main statements of boundary value problems for (1.1), and we proceed to our main objective which is to describe the widest classes of uniqueness for these problems.

## 2. Classes of uniqueness

Suppose we are given a number $T > 0$ and a bounded domain $\Omega \subset \mathbb{R}^n$ with the boundary such that the Gauss theorem holds (a Lipschitz-continuous boundary is sufficient). The topological conditions will be mentioned as they will be necessary. Let us consider, following [79], solutions to (1.1) of the following class:

$$\nabla \otimes \boldsymbol{u},\ \boldsymbol{u}_t \in L_\infty(0,T,L_r(\Omega)), \qquad \forall r < +\infty. \tag{2.1}$$

This provides the corresponding regularity for $p$ (provided that $\boldsymbol{f}$ is sufficiently regular), so that the equations (1.1) may be treated almost everywhere, and initial and boundary values in the problems (NP), (TF.II) and (TF.III) are accepted in the sense of continuity. This way of defining a solution is more convenient in boundary value problems than integral identities.

### 2.1. Gronwall-type lemma

Thus, if $(\boldsymbol{u}_1, p_1)$ and $(\boldsymbol{u}_2, p_2)$ are the solutions to (1.1) of the mentioned class, then

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}_1 \cdot \nabla)\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}_2 + \nabla p = 0, \qquad \mathrm{div}\boldsymbol{u} = 0,$$

where $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, $p = p_1 - p_2$; and we come immediately to the relation

$$\frac{\partial |\boldsymbol{u}|^2}{\partial t} + \mathrm{div}(|\boldsymbol{u}|^2 \boldsymbol{u}_1) + 2(\boldsymbol{u} \otimes \boldsymbol{u}) : \mathbb{D}(\boldsymbol{u}_2) + 2\mathrm{div}(p\boldsymbol{u}) = 0, \tag{2.2}$$

almost everywhere in $Q_T$, where $\mathbb{D}(\boldsymbol{v})$ stands for the rate of deformations tensor (for the vector field $\boldsymbol{v}$) which has the components $D_{ij}(\boldsymbol{v}) = \dfrac{1}{2}\left(\dfrac{\partial v_i}{\partial x_j} + \dfrac{\partial v_j}{\partial x_i}\right)$. After

[4]This result is of particular interest because, as it was mentioned in Subsection 1.2, global results for the Euler equations are difficult to obtain even for small data.

integration of (2.2) over $Q_t$ we derive the equality

$$\begin{aligned}\int\limits_{\Omega}|\boldsymbol{u}|^2 d\boldsymbol{x} + \int\limits_0^t d\tau \int\limits_{\partial\Omega}|\boldsymbol{u}|^2(\boldsymbol{u}_1\cdot\boldsymbol{n})dS + 2\int\limits_0^t d\tau\int\limits_{\partial\Omega} p(\boldsymbol{u}\cdot\boldsymbol{n})dS \\ + 2\int\limits_0^t d\tau\int\limits_{\Omega}(\boldsymbol{u}\otimes\boldsymbol{u}):\mathbb{D}(\boldsymbol{u}_2)d\boldsymbol{x} = 0\end{aligned} \tag{2.3}$$

for almost all $t\in(0,T)$. It is clear that in the problems (NP), (TF.II) and (TF.III) the third integral in (2.3) equals 0, and the second one is nonnegative, which leads finally to the relation

$$\int\limits_{\Omega}|\boldsymbol{u}|^2 d\boldsymbol{x} \leqslant 2\int\limits_0^t d\tau\int\limits_{\Omega}|\boldsymbol{u}|^2|\mathbb{D}(\boldsymbol{u}_2)|d\boldsymbol{x}. \tag{2.4}$$

In this way, during the proof of uniqueness of solutions in the class (2.1) we come to the Gronwall-type inequality (Osgood's inequality)

$$\int\limits_{\Omega}\psi(t,\boldsymbol{x})d\boldsymbol{x} \leqslant \int\limits_0^t\int\limits_{\Omega} g(s,\boldsymbol{x})\psi(s,\boldsymbol{x})d\boldsymbol{x}ds \tag{2.5}$$

with nonnegative functions $\psi$ and $g$, in which it is necessary to prove that $\psi = 0$ (under minimal restrictions for $g$). This is trivial in the case of bounded $g$'s since it reduces to classical Gronwall's lemma, but the similar result for (2.5) remains true for some classes of unbounded $g$'s as well. This fact was discovered in [76], [79], and it was formulated in terms of specially normed scales of the spaces $L_p$. Namely, in these papers the classes of the form

$$\|g\|_{L_\infty(0,T,L_r(\Omega))} \leqslant \theta(r), \qquad r \gg 1 \tag{2.6}$$

were considered with the functions $\theta$ growing sufficiently slowly (in [76] it was $\theta(r) = Cr$, and in [79] the growth rate was slightly diminished). However, the result looks rather bulky in these terms[5] and, in particular, it makes the formulation of uniqueness classes for the Euler equations difficult.

Our present aim is to demonstrate how the final (and unimprovable) result for the inequality (2.5) may be formulated in terms of Orlicz spaces and how it affects the formulation of the uniqueness classes in boundary value problems for the Euler equations.

[5] The formulation in terms of the whole family of estimates (2.6) seems to be bulky, and especially the requirements for admissible $\theta$'s found in [79] are complicated.

### 2.2. Orlicz spaces

Let us recall some notions of the theory of Orlicz spaces (detailed exposition of this theory may be found, e.g., in [38] and [39]). A function $M$ of one real variable is called an $N$-function if it is convex and even (hence, one can examine its behavior only on the right half-axis), and if, on the right half-axis, it increases strictly and satisfies the relations

$$\frac{M(s)}{s} \to 0 \quad \text{as} \quad s \to 0, \qquad \frac{M(s)}{s} \to +\infty \quad \text{as} \quad s \to +\infty.$$

For any $N$-function $M$, it is possible to determine its complementary $N$-function $\overline{M}$ which is its Legendre transform. Since any $N$-function is differentiable almost everywhere, $\overline{M}$ can be defined as $\overline{M}' = (M')^{-1}$.

If the set $\Omega \subset \mathbb{R}^n$ has a finite measure (for example, is a bounded domain which will further be considered)[6], the Orlicz class $K_M(\Omega)$ can be introduced as a set of measurable functions $u$ such that $\int\limits_\Omega M(u(\boldsymbol{x}))d\boldsymbol{x} < \infty$. The Orlicz space $L_M(\Omega)$ is the linear span of the class $K_M(\Omega)$; therefore, it is natural to introduce the Luxemburg norm in it:

$$\|u\|_{L_M(\Omega)} = \inf\left\{ k \;\middle|\; \int\limits_\Omega M\left(\frac{u(\boldsymbol{x})}{k}\right) d\boldsymbol{x} \leqslant 1 \right\}.$$

Since the measure of $\Omega$ is finite, only the behavior of $N$-functions at $+\infty$ is important; therefore, it is assumed below that the formulae for them are written for large arguments.

**Example 2.1.** $M(s) = s^p/p$, $p > 1$, $\overline{M}(s) = s^q/q$, $q = p/(p-1)$, $L_M(\Omega) = L_p(\Omega)$, $L_{\overline{M}}(\Omega) = L_q(\Omega)$.

**Example 2.2.** $M(s) = e^s - s - 1$, $\overline{M}(s) = (s+1)\ln(s+1) - s$. The space $L_M(\Omega)$ consists of functions which belong to all $L_r(\Omega)$, $r < \infty$ (but are nevertheless unbounded, generally speaking), and the space $L_{\overline{M}}(\Omega)$ consists of functions which are integrable (and even possess somewhat better properties) but do not belong to any $L_{1+\varepsilon}(\Omega)$. □

An increase of $N$-functions at $\infty$ can be compared by means of the relations $\prec$ and $\prec\!\!\!\prec$ defined as follows:

$$M_1 \prec M_2, \quad \text{if} \quad M_1(u) \leqslant M_2(Cu), \qquad u \gg 1,$$
$$M_1 \prec\!\!\!\prec M_2, \quad \text{if} \quad \frac{M_2(u)}{M_1(Cu)} \to \infty, \qquad u \to \infty, \quad \forall C > 0.$$

In the first case, there is the continuous embedding $L_{M_2}(\Omega) \hookrightarrow L_{M_1}(\Omega)$, and, in the second case, this embedding is strict (in the theoretical-set sense) and, in some sense, compact (for example, $L_{M_2}(\Omega) \subset K_{M_1}(\Omega)$). Accordingly, the relation

[6] Generally speaking, the measure of $\Omega$ does not have to be finite, but this assumption makes the considerations much simpler. This case is sufficient for the aims of the paper. In a general case $\Omega$ does not have to be a subset of $\mathbb{R}^n$.

$M_1 \sim M_2$ (which means the relations $M_1 \prec M_2$ and $M_2 \prec M_1$ simultaneously) is a criterion of the coincidence $L_{M_1}(\Omega) = L_{M_2}(\Omega)$.

For the functions with growth rate faster than polynomial growth, the so-called $\Delta^2$-condition can be considered, which implies that $M^2 \sim M$, i.e., $M^2(u) \leqslant M(Cu)$ as $u \gg 1$. Ignoring "pathological" cases (which do not arise in the article), this condition is satisfied by all $N$-functions $M$ that increase not more slowly than the function $F(s) = \exp(s^{\varepsilon})$.

The Lebesgue spaces $L_r(\Omega)$ are a particular case of the Orlicz spaces, for which the theory is partly similar to the theory of the spaces $L_r$ (especially in the case where $M$ and $\overline{M}$ simultaneously possess growth rate not faster than polynomial growth) but differs from it in many respects and it allows a description of the subtle properties of functions in $\Omega$.

### 2.3. Orlicz spaces and scales of spaces $L_p$

There is also the following relationship between the Lebesgue and Orlicz spaces. Let us consider the set (we denote it by $L_{\omega,\beta}$) of measurable functions $u$ which belong to all $L_p(\Omega)$ as $p \in [\alpha, \beta)$, and such that $\|u\|_{L_p(\Omega)} \leqslant C\omega(p)$, $p \in [\alpha, \beta)$, with the specified function $\omega$. As has been noted by various researchers in various situations, the sets of the type $L_{\omega,\beta}$ are contained in the Orlicz spaces $L_M(\Omega)$ with appropriate $M$'s. For example, we may mention the papers [74], [57] and [61][7]. However, a fairly complete investigation of the relation between $L_{\omega,\beta}$ and other symmetric spaces (the relation with $L_p$ is most important) was made only recently, see, e.g., [48], [49] and [7]. It is convenient to use here the terms and some results of our paper [48]. Below, we will need only the case $\beta = +\infty$. The set $L_{\omega,\infty}$ becomes a Banach space if it is endowed with the norm

$$\|u\|_{L_{\omega,\infty}} = \sup_{p\in[\alpha,+\infty)} \frac{\|u\|_{L_p(\Omega)}}{\omega(p)}.$$

This space does not depend on the choice of $\alpha$ (i.e., the corresponding norms are equivalent) and does not vary if $\omega$ varies to within equivalence of the special form:

$$\omega_1 \overset{\mathscr{L}}{\sim} \omega_2 \qquad \overset{\text{def}}{\Longleftrightarrow} \qquad C_1\omega_1(p) \leqslant \omega_2(p) \leqslant C_2\omega_1(p).$$

The operators

$$\mathbf{In}_\infty[\omega](v) = \int\limits_{\alpha}^{+\infty} \frac{v^p\,dp}{\omega^p(p)} \qquad \text{and} \qquad \mathbf{Sc}_\infty[\Phi](p) = \max_{v\geqslant 1} \frac{v}{\Phi^{1/p}(v)}$$

generate the correspondence between the functions $\omega = \omega(p)$ and $N$-functions $\Phi = \Phi(v)$. In particular, the relations $\overset{\mathscr{L}}{\sim}$ and $\sim$ correspond to each other under these mappings. We do not need to formulate here all the details (cf. [48]) since

[7]More detailed review can be found in [48] and [49].

it will suffice to use the fact that in the case of rapidly increasing $\Phi$, i.e., slowly increasing $\omega$ (which are used below) the equality

$$L_{\omega,\infty} = L_\Phi \qquad \text{as} \qquad \Phi = \mathbf{In}_\infty[\omega], \quad \text{i.e.,} \quad \omega = \mathbf{Sc}_\infty[\Phi]$$

is valid. And, conversely, any space $L_\Phi$ with a rapidly increasing $\Phi$ can be represented as $L_{\omega,\infty}$ with the corresponding function $\omega$.

Thus, in the case under consideration, the family of estimates in $L_p$ of the type (2.6) is equivalent to the estimate in an appropriate Orlicz space. Using this fact, we could translate the results of the paper [79] into "the language of the Orlicz spaces" (and this was done in [50]). However, such translation does not seem to be very reasonable for the inequality (2.5), since the investigation of (2.5) is already made in [34] in terms of Orlicz spaces, and the results were proved to be exact (unimprovable). Below we formulate a part of this result.

### 2.4. Gronwall-type lemma in the Orlicz spaces

**Definition 2.3.** The class $\mathcal{K}$ is the set of $N$-functions that satisfy one of the following equivalent conditions:

$$\int^{+\infty} \frac{\ln M(s)}{s^2} ds = +\infty, \qquad \int^{+\infty} \frac{ds}{\overline{M}(s)} = +\infty, \qquad \int^{+\infty} \frac{ds}{sM^{-1}(s)} = +\infty. \tag{2.7}$$

**Remark 2.4.** All elements of $\mathcal{K}$ satisfy the $\Delta^2$-condition except few for a"pathological" examples.

**Statement 2.5.** Let the functions $g$ and $\psi$ be specified in $Q_T$ and nonnegative, $\psi \in L_{1+\varepsilon}(Q_T)$, $g \in K_M(Q_T)$, $M \in \mathcal{K}$. Then the relation (2.5) implies that $\psi = 0$. If $M \notin \mathcal{K}$ then there exist nonnegative functions $\psi \in L_\infty(Q_T)$ and $g \in L_\infty(0, T, L_M(\Omega))$ such that (2.5) holds but $\psi \not\equiv 0$. □

It is obvious from (2.7) that the class $\mathcal{K}$ consists of functions growing faster than all polynomials. It is clear that, given any $M \in \mathcal{K}$, one can find $M_1 \in \mathcal{K}$ such that $M_1 \prec\!\!\!\prec M$, so that the conditions of belonging to $L_M$ or $K_M$ generated by *some* $M \in \mathcal{K}$ are the same.

**Example 2.6.** $M_\alpha(s) = \exp(s/\ln^\alpha s)$, $M_\alpha \in \mathcal{K}$ as $\alpha \leqslant 1$. The corresponding $\omega(p) = \mathbf{Sc}_\infty[M_\alpha](p) \overset{\mathcal{L}}{\sim} p\ln^\alpha p$. Thus, if we formulate the conditions for $g$ in the form $\|g\|_{L_p(Q_T)} \leqslant Cp\ln^\alpha p$, then the Gronwall lemma for (2.5) is valid as $\alpha \leqslant 1$. It is possible to refine the conditions for $M$ by choosing, for example, $M(s) = \exp\left(\dfrac{s}{\ln s \ln^\alpha \ln s}\right)$, which will produce new logarithms in $\omega$ (cf. [79]), etc.

### 2.5. Uniqueness in terms of $\nabla \otimes \boldsymbol{u}$

Coming back to the problem (2.4), in view of Statement 2.5 we obtain the uniqueness for the problems (NP), (TF.II) and (TF.III) in the class

$$\nabla \otimes \boldsymbol{u} \in K_\Phi(Q_T), \qquad \Phi \in \mathcal{K}. \tag{2.8}$$

However, it is usual (in particular, for the reasons mentioned in Subsection 1.3) to formulate the properties of solutions and classes of uniqueness for the Euler equations in terms of the vorticity. In order to do that, we need the properties of the mapping $\boldsymbol{\omega} \mapsto \nabla \otimes \boldsymbol{u}$ in the Orlicz spaces. Since these properties are well-known in $L_p$, it is convenient to make an abstraction from the nature of an operator and use extrapolatory methods. If we used the representation of the Orlicz spaces in the form of the spaces $L_{\omega,\infty}$ then these methods would become a tautology, but this representation is cumbersome and, hence, inconvenient. For the purposes of the article, it is convenient to use the constructive extrapolatory method developed in [46] and [47], which consists in the following.

### 2.6. Extrapolation of operators in Orlicz spaces

Let $A$ be a bounded linear operator in $L_p$ for all $p \gg 1$, and its norm $\|A\|_{\mathcal{L}(L_p)} \leqslant C\varphi(p)$. Then, one should calculate the inverse Mellin transform of the function $\varphi^p(p)$:

$$\varphi^p(p) = \int\limits_{\sigma}^{+\infty} \psi(s) s^p ds \tag{2.9}$$

(here $\sigma \geqslant 0$ may be chosen arbitrarily, and the formula (2.9) is extended in some sense to the case of non-analytical functions $\varphi$ [47]) and use the obtained function $\psi$ as the kernel of the convolution type integral transform:

$$\mathbf{F}_{\psi,\sigma}[\Phi](v) = \int\limits_{\sigma}^{+\infty} \psi(s)\Phi(vs) ds. \tag{2.10}$$

As a result, it can be argued that $A$ acts boundedly from $L_M$ into $L_\Phi$ for any $N$-functions $\Phi$ and $M$ linked by the relation $M = \mathbf{F}_{\psi,\sigma}[\Phi]$. In particular, if $\varphi(p) = p$ then the relation (2.9) (to within $\stackrel{\mathcal{L}}{\sim}$ which is insignificant as shown in [47]) gives $\psi(s) = e^{-s}$, and the operator (2.10) becomes the operator $\mathbf{S}$ which was studied in [44] and [45]:

$$\mathbf{S}[\Phi](v) = \int\limits_{0}^{+\infty} e^{-s}\Phi(vs) ds. \tag{2.11}$$

Thus, we obtain the following

**Statement 2.7.** If the linear operator $A$ has the property $\|A\|_{\mathcal{L}(L_p)} \leqslant Cp$ for all $p \gg 1$ then $A \in \mathcal{L}(L_M, L_\Phi)$ for all $N$-functions $M$ and $\Phi$ linked by the relation $M = \mathbf{S}[\Phi]$, where the operator $\mathbf{S}$ is defined in (2.11).

### 2.7. Uniqueness in terms of vorticity

In the problem under consideration the operator $A$ is the mapping $\boldsymbol{\omega} \mapsto \nabla \otimes \boldsymbol{u}$. The properties of this mapping in $L_p$ are easy to formulate at least in the problems (NP) and (TF.III) since this is the operator of the problem

$$\mathrm{rot}\boldsymbol{u} = \boldsymbol{\omega}, \qquad \mathrm{div}\boldsymbol{u} = 0; \qquad \boldsymbol{u} \cdot \boldsymbol{n}|_{\partial\Omega} = g_0, \tag{2.12}$$

where $g_0 = 0$ in the problem (NP), and

$$g_0 = \begin{cases} 0 & \text{at } \Gamma_0, \\ g_1 & \text{at } \Gamma_1, \\ g_2 & \text{at } \Gamma_2 \end{cases} \tag{2.13}$$

in the problem (TF.III). Let us, for the sake of simplicity, consider below only singly connected domains $\Omega$. This constraint may be released with the use of a special technique (see, for example, [55]).

The problem (2.12) in the case $g_0 = 0$ possesses the following properties in $L_p$ (cf. [75], [76] and [78]):

$$\|\nabla \otimes \boldsymbol{u}\|_{L_r(\Omega)} \leqslant C_1 r \|\boldsymbol{\omega}\|_{L_r(\Omega)}, \qquad r \gg 1.$$

In view of Statement 2.7, this leads to the estimate $\|\nabla \otimes \boldsymbol{u}\|_{L_\Phi(\Omega)} \leqslant C_2 \|\boldsymbol{\omega}\|_{L_M(\Omega)}$ for all pairs $\Phi$, $M$ such that $M = \mathbf{S}[\Phi]$. The properties of $g_{1,2}$ are insignificant for us, and we may suppose that $g_0 \in L_\infty(0, T, W^1_\infty(\partial\Omega))$. As a result for (2.12), we can finally write down the estimate

$$\|\nabla \otimes \boldsymbol{u}\|_{L_\Phi(\Omega)} \leqslant C_2 \|\boldsymbol{\omega}\|_{L_M(\Omega)} + C_3 \|g_0\|_{W^1_\infty(\partial\Omega)}, \qquad M = \mathbf{S}[\Phi]. \tag{2.14}$$

It remains for us to understand which $M$'s provide $\Phi \in \mathcal{K}$ if $M = \mathbf{S}[\Phi]$. This problem was solved in [50]:

**Statement 2.8.** If $M = \mathbf{S}[\Phi]$, and $M$ satisfies the $\Delta^2$-condition, then $\Phi \in \mathcal{K}$ is equivalent to $M \in \mathcal{K}_1$, where $\mathcal{K}_1$ consists of functions $M$ that satisfy the condition

$$\int^{+\infty} \frac{\ln \ln M(s)}{s^2} ds = +\infty. \tag{2.15}$$

**Remark 2.9.** As follows from Remark 2.4 and the fact that $\mathbf{S}$ increases the rate of increase of functions to which it is applied, the $\Delta^2$-condition in Statement 2.8 is not an additional essential constraint but it only filters the pathological functions insignificant in applications.

### 2.8. Final result

Let us sum up Subsections 2.5 and 2.7:

**Theorem 2.10.** The problems (NP), (TF.II) and (TF.III) cannot have more than one solution in the class (2.8). In particular, for singly connected domains $\Omega$ the problems (NP) and (TF.III)[8] cannot have more than one solution in the class

$$\operatorname{rot} \boldsymbol{u} \in L_\infty(0, T, L_M(\Omega)), \qquad M \in \mathcal{K}_1 \tag{2.16}$$

(see (2.15)). As usual, the uniqueness of the pressure holds to within an additive constant.

[8]The condition $g_0 \in L_\infty(0, T, W^1_\infty(\partial\Omega))$ is sufficient in the problem (TF.III), see (2.13).

## 3. Additional remarks

### 3.1. Theorem 2.10 and global existence

Generally speaking, Theorem 2.10 is a conditional result since we cannot guarantee global existence of solutions in the corresponding classes in the framework of known results. As far as we know, only two situations are exceptional:

1. The two-dimensional problem (NP): global existence theorem in the class (2.16) can be easily obtained, as shown in [50], based on the technique of [76]. This idea is also mentioned in [79]; as a matter of fact, global solvability of the two-dimensional problem (NP) in the class (2.1) is proved in [76]. Moreover, as it was mentioned in Subsection 1.2, there are global existence theorems for this problem in wider classes as well [54], [50].
2. As it was mentioned in Subsection 1.4, the two-dimensional problem (TF.III) was studied in [31] and [32], where the global result was obtained in the simplest case (however, locally in the initial data), in the class $\mathrm{rot}\boldsymbol{u} \in L_\infty(Q_T)$ inclusively. The similar result in the class (2.16) does not offer any difficulties: it suffices to use the estimate for $\omega = \mathrm{rot}\boldsymbol{u}$ in the class (2.16) from the equation
$$\frac{\partial \omega}{\partial t} + \boldsymbol{u} \cdot \nabla \omega = \mathrm{rot}\boldsymbol{f}$$
(with the use of the technique developed in [34]) instead of the similar estimate of $\omega$ in $L_\infty(Q_T)$ used in [31].

It is curious to note that specifically these two types of boundary value problems (certainly, for any $n$) admit formulation of the result of Theorem 2.10 in the final terms of the vorticity.

In other cases we can only argue the following: if the problems (NP), (TF.II) and (TF.III) have solutions in $Q_T$ of the classes mentioned in Theorem 2.10, then these solutions are unique.

### 3.2. Problem (TF.I)

The problem (TF.I) essentially differs from the others in the fact that the smoothness should be one order higher to provide the uniqueness. This assumption (i.e., the higher smoothness) was accepted, e.g., in [77] and [4]. However, global solvability of the two-dimensional problem (TF.I) was proved in [4] in the class $\mathrm{rot}\boldsymbol{u} \in L_\infty(Q_T)$, but one order higher smoothness was essential in the proof of uniqueness. It is not clear how to avoid this difficulty. However, the problem (TF.I) is the least interesting (among other through flow problems) from the physical point of view, despite the fact that it has historical priority and is the most studied globally (certainly, only in the two-dimensional case) among other through flow problems, in the "bad" classes inclusively [77], [4], [69].

### 3.3. Optimality and comparison of the results

In connection with the Gronwall-type lemma for (2.5) and Statement 2.5 we should mention that the class $\mathcal{K}$ is an extremely wide class where the condition $\nabla \otimes \boldsymbol{u} \in$

$L_M$ with $M \in \mathcal{K}$ still provides uniqueness of trajectories of fluid particles, i.e., uniqueness of solutions to the Cauchy problem for ODE

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{u}(t, \boldsymbol{x}), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0.$$

This fact was proved in [34]. A similar result in other terms (namely, in terms of scales of the spaces $L_p$, which correspond to the spaces $L_{\omega,\infty}$ in our notation) was obtained in [79] (for solenoidal fields). More details can be found in [50], where the comparison between the results of [79] and the result of Theorem 2.10 for the problem (NP) is also made. These results appeared to be equivalent, which, however, is discovered only a posteriori, with the use of extrapolatory representations of the Orlicz spaces (as the scales $L_{\omega,\infty}$), developed in [48] (see Subsection 2.3). Moreover, the terms of Theorem 2.10 seem to be more convenient since they need the verification of only one clear condition (2.15), (2.16). However, the mentioned equivalence is not surprising since both papers [79] and [34] contain unimprovable results for the Gronwall-type lemma (which is the basis in the uniqueness result for the Euler equations), obtained in different terms (the scales of the spaces $L_p$, and the Orlicz spaces correspondingly). Thus, the result of Theorem 2.10 is unimprovable in the framework of the method used here to obtain uniqueness (i.e., with the use of (2.4)), but there is no other method yet, as noted in [79].

More details about applications of the Orlicz spaces and extrapolatory techniques (used in the paper) to other hydrodynamical problems can be found in our paper [49].

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# On Nonlinear Stability of MHD Equilibrium Figures

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**Abstract.** We study the problem of equilibrium figures of electro-conducting fluids. Our first goal is to set correctly the initial boundary value problem for the equations governing both incompressible and compressible flows of electrically conducting fluids with unknown free surface. Notice that when the exterior is a dielectric or a vacuum, attention cannot be confined merely to the region of the conducting fluid; this represents a crucial point for the well-posedness problem. The second goal is to study a new criterion of nonlinear stability of the rest state of a heavy electro-conducting, incompressible or compressible fluid in a section of horizontal layer with rigid plane bottom, and upper unknown free boundary in a vacuum. The new criterion proposes an alternative definition of perturbation, and is deeply related to the unknown motion of the boundary. The third goal is to prove nonlinear stability, in the class of global regular solutions, if the system has non-significant magnetic susceptibility, in absence of surface currents, for large initial data. Kinematic viscosity, magnetic diffusivity, surface tension are only non-negative.

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## 1. Introduction

During the second half of the twentieth century, a great number of papers and books appeared to study mathematical properties of plasmas through the two models: kinetic [20] and continuum [3], [6], [21], [13]. Also we quote the very recent paper [12], where a non-relativistic theory for studying thermo-mechanical-electromagnetic processes in deformable media has been presented, see also [1], [2], [8], [22] [15] for further bibliography.

One of the most mathematically challenging, and physically crucial points is the stability problem of equilibrium figures, such as the Tokamak figure plasma confinement, or electro-magnetic casting problems, where one meets a matching problem between solutions of Maxwell's equations in the region occupied by plasma, and in the region exterior to the plasma. In these cases the fluid doesn't tend to conform to external influences, rather it modifies itself as if it had a mind of its own.

This problem constitutes one of the main research subject for mathematicians and physicists, and remains at the present time still open. We note in such literature a proliferation of variational stability criteria, that arise from the introduction of special variations. A condition which is sufficient for stability with regard to a class of variations is not sufficient to drawn any conclusion at all with regard to stability with respect to another class of variations, cf. [19], [20], therefore a different criterion should be used. Moreover, in the study of stability several different linearizations of the unsteady problem around equilibrium configurations have been proposed, hence even the *definition of a linearized initial value problem is still unclear*.

Let us begin by recalling some papers related to the equilibrium configurations [1], [2], [8], [13], [18], [19], [20], [21], [27], [35]. As a matter of fact, in the literature of well-posedness of magnetohydrodynamic fluid flows with free surface there is much confusion, see [15] where the initial value problem is not clearly formulated, see equations (1.38), (1.39) of Chapter I. Indeed the mathematical problem of confinement of MHD fluid flows in external insulating media has several different formulations, and besides the case in which the surface is assumed fixed, cf., e.g., [6], [11],[16], [23], [25], [34], [36], [37], [39], [32], [26], the complete problem with unknown free surface is studied mostly in the linear case cf. [35], [4], [12], [5], [33], [17], [24]. In most papers linear stability of the rest state is proved when the magnetic permeability $\mu$ of the fluid coincides with that $\mu_0$ of vacuum, i.e., the fluid is a plasma, and in absence of surface currents, see [35].

The analysis here developed is motivated by a criticism of previous approaches to the problem, see [15], and it brings us to a reformulation of the well-posedness problem. A physical discussion of the loss of a balance equation for the total energy arises naturally, cf., Subsection 3.6. Furthermore, we prove nonlinear stability of an equilibrium figure with respect to large initial perturbations of domain, velocity, electric and magnetic fields when the magnetic permeability $\mu$ of the fluid coincides with that $\mu_0$ of vacuum, i.e., the fluid is a plasma cf. [35], and has no surface currents. To our knowledge, the problem of existence of global solutions for the problem as stated here remains at the present time an open problem, and will be the subject of future work.

On this note, we consider the equations governing electrically conducting, incompressible and compressible fluid flows in a horizontal layer bounded below by a rigid dielectric, and above by a free surface in the presence of an isolating external medium. To simplify the problem we neglect thermal effects, and assume periodicity on the horizontal variables $x_*$ and denote by $\Sigma$ the periodicity cell. We consider a simple geometry in $R^3$, where we may use cartesian coordinates $\mathcal{R} =: \{O,\, x_* \in R^2,\, z \in R\}$, where $O$ belongs to the bottom plane $\pi$ of the layer,

$z$ is the vertical axis upward oriented, and with $x_* = x\mathbf{i} + y\mathbf{j}$ on $\pi$, and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ the corresponding ortho-normal basis. The entire problem develops in an infinite vertical strip $\mathcal{P}$, directed along $\mathbf{k}$, and with constant cross section $\Sigma$. The fluid is confined in the bounded section of $\mathcal{P}$ between the plane section

$$\pi = \{(x_*, z) \in R^3 : \quad x_* \in \Sigma,\ z = 0\}$$

and the surface

$$\Gamma_t = \{(x_*, z) \in R^3 : \quad x_* \in \Sigma,\ \ z = \zeta(x_*, t)\},$$

where $\zeta$ is an unknown scalar function; finally

$$\Omega_t = \{(x_*, z) \in R^3 : \quad x_* \in \Sigma,\ 0\ <\ z < \zeta(x_*, t)\}. \tag{1.1}$$

Below $\Omega_t$, in the half-strip

$$\Omega^- = \{x_*, z :\ x_* \in \Sigma; \quad 0 > z > -\infty\}, \tag{1.2}$$

there is a perfect insulating rigid bar (dielectric). We add a superscript $^-$ to the functions defined in $\Omega^-$. Above $\Gamma_t$, in the half strip

$$\widehat{\Omega}_t = \{x_*, z :\ x_* \in \Sigma; \quad \zeta(x_*, t) < z < \infty\} \tag{1.3}$$

there is a vacuum. We add a superscript $\widehat{\cdot}$ to functions defined in $\widehat{\Omega}_t$. The basic rest state is a function of external data. In a vacuum we consider extensions to $\mathcal{P}$ of pressure, density, and velocity defined in $\Omega_t$ as constants, generally, except for the density in $\Omega^-$, we take the constant to be zero. Since these variables are discontinuous through $\Gamma_t$ and $\Sigma$, the motion equations are intended to be written in the open domains $\widehat{\Omega}_t$, $\Omega_t$, $\Omega^-$. If there is a vacuum in $\widehat{\Omega}_t$, the only solenoidal field satisfying the Maxwell equation is a constant field $H_b\mathbf{k}$. In this note we have studied the stability of two basic states, say

$$\mathbf{H}_b = H_b\mathbf{k}, \qquad \widehat{\mathbf{H}}_b = H_b\frac{\mu}{\widehat{\mu}}\frac{L}{z}\mathbf{k}, \quad \mathbf{H}_b^- = H_b\frac{\mu}{\mu^-}\mathbf{k}, \qquad \text{model rest state,}$$

$$\mathbf{H}_b = H_b\mathbf{k}, \qquad \widehat{\mathbf{H}}_b = \frac{\mu}{\widehat{\mu}}H_b\mathbf{k}, \qquad \mathbf{H}_b^- = H_b\frac{\mu}{\mu^-}\mathbf{k}, \qquad \text{true rest state,}$$

$$H_b = \text{const.}$$

The model rest state helps in pointing out the differences between our approach and the previous ones. The model rest state satisfies the Maxwell's equations with the magnetic field non solenoidal at initial time.

For incompressible fluids the unknowns are the velocity $\mathbf{u}$, the pressure $p$, the magnetic field $\mathbf{H}$, the height $\zeta$. If in correspondence of zero external forces, a vertical uniform magnetic field is maintained, there exists the static solution $S_b = \{(p_b, \mathbf{u}_b, \mathbf{H}_b, \zeta_b)\}$, with the pressure $p_b = \overline{p}$, the velocity $\mathbf{u}_b = 0$, the magnetic field $\mathbf{H}_b = H_b\mathbf{k}$, the height $\zeta_b = L$, and with $\overline{p}$, $H_b$, $L$ constants. For compressible fluids the unknown are the velocity $\mathbf{u}$, the density $\rho$, the magnetic field $\mathbf{H}$, the height $\zeta$. If in correspondence of potential external forces, a vertical uniform magnetic field is maintained, there exists the static solution $S_b = \{(\rho_b, \mathbf{u}_b, \mathbf{H}_b, \zeta_b)\}$, with the density $\rho_b = \overline{\rho}$, the velocity $\mathbf{u}_b = 0$, the magnetic field $\mathbf{H}_b = H_b\mathbf{k}$, the height $\zeta_b = L$, and with $\overline{\rho}$, $H_b$, $L$ constants. As concrete applications of the stability problem in this geometry, we refer to electromagnetic casting processes, cf. [5], [10], [32], [33], [25], [26][1]. More general geometries are by no means restrictions, and will be considered in forthcoming papers.

The plan of the work is the following. In Section 2 we write the equations and the initial and boundary conditions both for incompressible and compressible fluids. In Section 3 we derive the energy balance equation in case of large magnetic susceptibility, and observe that in case of a moving boundary, the contribution to the time derivative of the total energy is given by mechanical surface tension. In Section 4 we give an alternative definition of perturbation to a basic flow, and compare our definition with the previous one given in [4], [24], [26]. Moreover, in Section 5 we derive a differential equation for the $L^2$ norm of our perturbations. Let $E$ be the total energy of perturbations, it is worth distinguishing between the classical and the new definition of perturbation to the magnetic field. Precisely, assuming zero magnetic susceptibility, and in absence of surface currents, with standard procedures, using the old perturbations, in the model rest state, it is not known how to obtain a global estimate in the time for the total energy of perturbation $\mathcal{E}(t) := \mathcal{E}(p, \mathbf{u}, \widetilde{\mathbf{H}}, \zeta) - E(p_b, 0, \widetilde{\mathbf{H}}_b, \zeta_b)$, while using the new perturbations it is possible to deduce an a priori estimate in time of $\mathcal{E}(t)$ in terms of initial data; in particular nonlinear stability follows. We end the paper with some concluding remarks.

## 2. Initial boundary value problem

In this section we set the indefinite equation of magnetofluiddynamics both for incompressible and compressible fluids.

We begin by recalling the Maxwell equations in the whole space $R^3$:

$$\begin{aligned} \mu\partial_t \mathbf{H} &= -\nabla \times \mathbf{E}, \\ \mathbf{j} &= \nabla \times \mathbf{H}, \end{aligned} \tag{2.1}$$

where $\mathbf{j}$ is the current density, $\mathbf{H}$ the magnetic field in the fluid, $\mathbf{E}$ the electric field. Notice that $\mathbf{H}$ is related to $\mathbf{E}$ by

$$\mathbf{E} = \frac{1}{\sigma}\nabla \times \mathbf{H} + \mu\mathbf{H} \times \mathbf{u}, \tag{2.2}$$

the constants $\mu > 0$ and $\sigma > 0$ are the magnetic permeability and the electric conductivity. If the fluid is highly conducting ($\sigma \to \infty$), the magnetic viscosity (dissipation) disappears. This minor role to which $\mathbf{E}$ is reduced, is peculiar to hydromagnetics where displacements currents are neglected. Moreover in a vacuum it is $\mathbf{u} = 0$, $\nabla \times \mathbf{H} = 0$, $\sigma = 0$, and relation (2.2) fails, hence *in a vacuum the electrical field is an independent field.*

[1]It has been shown in [14] that the surface tension must be a function of temperature, thus more elaborate computations arise. This is not considered in this paper.

### 2.1. Incompressible fluids in $\Omega_t$

Let us consider a layer of fluid $\Omega_t$ as defined in the introduction. We add the subscript $_*$ to denote quantities calculated on $\pi$. The unit normal $\mathbf{n}$ has components $(-\nabla_*\zeta, 1)/\sqrt{g}$, where $g = 1 + |\nabla_*\zeta|^2$ is the metric element.

The equations governing magnetohydrodynamical, incompressible fluid flows are:

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} &= \nabla\cdot\mathbf{T}(\mathbf{u},p) + \nabla\cdot\mathbf{T}(\mathbf{H}) + \nabla U, \\ \mu\partial_t\mathbf{H} &= -\frac{1}{\sigma}\nabla\times(\nabla\times\mathbf{H}) - \mu\nabla\times(\mathbf{H}\times\mathbf{u}), \\ \nabla\cdot\mathbf{u} &= 0, \qquad\qquad x\in\Omega_t,\ t\in(0,\infty), \end{aligned} \tag{2.3}$$

where $\Omega_t$, $\mathbf{u}$, $p$, $\mathbf{H}$ are the unknown domain, velocity, pressure, and magnetic field, while $\nabla U$ is the potential force. To (2.3) we add the state equations

$$\begin{aligned} &\mathbf{T}(\mathbf{u},p) = -p\mathbf{I} + \nu S(\mathbf{u}), \qquad p = p(x,t), \\ &\mathbf{T}(\mathbf{H}) = \mu\Big(-\frac{H^2}{2}\mathbf{I} + \mathbf{H}\otimes\mathbf{H}\Big), \qquad\qquad x\in\Omega_t,\ t\in(0,\infty), \end{aligned} \tag{2.4}$$

where $\nu > 0$ is the kinematic viscosity, and $S(\mathbf{u}) = \nabla\mathbf{u} + \nabla\mathbf{u}^T$.

In (2.3) $\nabla\cdot\mathbf{T}(\mathbf{H})$ denotes the Lorentz force

$$\nabla\cdot\mathbf{T}(\mathbf{H}) \;=\; \mu\mathbf{j}\times\mathbf{H} \;=\; \mu(\nabla\times\mathbf{H})\times\mathbf{H}.$$

### 2.2. Compressible fluids in $\Omega_t$

The equations governing magnetohydrodynamical, compressible fluid flows are:

$$\begin{aligned} \partial_t\rho + \mathbf{u}\cdot\nabla\rho &= -\rho\nabla\cdot\mathbf{u}, \\ \rho\partial_t\mathbf{u} + \rho\mathbf{u}\cdot\nabla\mathbf{u} &= \nabla\cdot\mathbf{T}(\mathbf{u},p) + \nabla\cdot\mathbf{T}(\mathbf{H}) + \rho\nabla U, \\ \mu\partial_t\mathbf{H} &= -\frac{1}{\sigma}\nabla\times(\nabla\times\mathbf{H}) - \mu\nabla\times(\mathbf{H}\times\mathbf{u}), \\ \nabla\cdot\mathbf{H} &= 0, \end{aligned} \tag{2.5}$$

where $\Omega_t$, $\mathbf{u}$, $\rho$, $\mathbf{H}$ are the unknown velocity, density, and magnetic fields, while $\nabla U$ is the potential force.

In (2.5) $\mathbf{T}(\mathbf{H})$ is given by $(2.4)_2$, while $\mathbf{T}(\mathbf{u},p)$ has the state equations

$$\mathbf{T}(\mathbf{u},p) = -p\mathbf{I} + 2\nu S(\mathbf{u}), \qquad p = p(\rho), \tag{2.6}$$

where

$$\nu S(\mathbf{u}) = \lambda_1(\nabla\mathbf{u} + \nabla\mathbf{u}^T) + \lambda_2\nabla\cdot\mathbf{u}\mathbf{I},$$

and $\lambda_1$, and $\lambda_2$ are the constant shear and bulk viscosity coefficients.

### 2.3. Equations in the exterior to $\Omega_t$

We notice that systems (2.3), (2.5) are not sufficient to solve the problem when the fluid is embedded in an insulating medium, such as a vacuum. Actually, since the vacuum and the bottom are perfect insulators, we must consider as unknown the magnetic field $\mathbf{H}$ both inside and outside the strip containing $\Omega_t$. In (1.1), (1.2), (1.3) we have defined the fluid domain $\Omega_t$, the rigid dielectric lower strip $\Omega^-$, and the empty upper strip $\widehat{\Omega}_t$. Also we put a $\widetilde{\cdot}$ to denote functions defined on $\widetilde{\Omega}_t = \Omega_t \cup \Omega^- \cup \widehat{\Omega}_t$. We recall that outside $\Omega_t$ the region is insulating, either vacuum or a rigid dielectric. In the vacuum there is zero material density, and we assume $\widehat{\mathbf{u}} = 0$, $-\widehat{p} = 0$, in the dielectric we set $\mathbf{u}^- = 0$, $p^- = 0$. We assume the equation of motion to be satisfied. Then for both compressible and incompressible fluids the vector $\widehat{\mathbf{H}}$ must satisfy the system

$$\begin{aligned}
&(\nabla \times \mathbf{H}^-) \times \mathbf{H}^- = 0, \qquad \nabla \cdot \mathbf{H}^- = 0,\\
&\mathbf{E}^- \;=\; \frac{\nabla \times \mathbf{H}^-}{\sigma^-},\\
&\mu^- \frac{\partial}{\partial t}\mathbf{H}^- + \nabla \times \mathbf{E}^- = 0, \qquad x \in \Omega^-,\ t \in (0,\infty),\\
&(\nabla \times \widehat{\mathbf{H}}) \times \widehat{\mathbf{H}} = 0, \qquad \nabla \cdot \widehat{\mathbf{H}} = 0,\\
&\widehat{\mu}\frac{\partial}{\partial t}\widehat{\mathbf{H}} + \nabla \times \widehat{\mathbf{E}} = 0, \qquad x \in \widehat{\Omega}_t,\ t \in (0,\infty).
\end{aligned} \tag{2.7}$$

Equations (2.7) contain all equations of motion for the fluid and for the magnetic field as well as in the exterior region $\Omega^- \cup \widehat{\Omega}_t$.

### 2.4. Boundary conditions

To study the boundary conditions we introduce some preliminary geometrical formulas on the surface $\Gamma_t$. Let us consider a layer of fluid over a fixed plane $\pi$. On $\pi$ we choose the coordinates $x_* = (x_1, x_2)$ and denote by $z$ the axis orthogonal to $\pi$. We assume that the deformable surface may be represented during all times in cartesian coordinates by the equation

$$\zeta = \zeta(x_*, t) = L + \eta(x_*, t), \qquad L > 0.$$

We add the subscript $_*$ to denote quantities calculated on $\pi$. In the sequel, we shall denote by $\Sigma$ the periodicity cell given by $\widetilde{\Omega}_t \cap \pi$. The unit normal $\mathbf{n}$ has components $(-\nabla_*\zeta, 1)/\sqrt{g}$, where $g = 1 + |\nabla_*\zeta|^2$ is the metric element. We recall that on $\Gamma_t$ the two vectors

$$(\mathbf{t}_1, \mathbf{t}_2) := \nabla_*(x\mathbf{i} + y\mathbf{j} + \zeta\mathbf{k}) \equiv (\mathbf{i} + \partial_x\eta\mathbf{k}, \mathbf{j} + \partial_y\eta\mathbf{k})$$

represent two linearly independent, non-orthogonal, non-unitary vectors tangent to $\Gamma_t$, with components $\mathbf{t}_1 = (1, 0, \partial_x\eta)$, $\mathbf{t}_2 = (0, 1, \partial_y\eta)$, $i = 1, 2$. Furthermore, the unit normal vector $\mathbf{n}$ directed outward to $\Omega_t$, is given by

$$\mathbf{n} = -\frac{1}{\sqrt{g}}\mathbf{t}_1 \times \mathbf{t}_2, \qquad g = |\mathbf{t}_1 \times \mathbf{t}_2|^2 = 1 + |\nabla_*\eta|^2,$$

and the doubled mean curvature $\mathcal{K}(\zeta)$ is given by

$$\mathcal{K}(\zeta) = \mathcal{K}(\eta) = -\nabla \cdot \mathbf{n} = \nabla_* \cdot \Big(\frac{1}{\sqrt{g}} \nabla_* \eta\Big).$$

For the velocity of fluid on the material surface we get

$$\begin{aligned} &\frac{\partial \eta}{\partial t} = \mathbf{u} \cdot \mathbf{n}', \qquad \mathbf{n}' = \sqrt{g}\,\mathbf{n}, \qquad \text{on} \quad \Gamma_t, \\ &\mathbf{u} = 0 \qquad\qquad \text{on} \quad \Sigma. \end{aligned} \tag{2.8}$$

The boundary conditions on $\mathbf{H}$ depend on electrical properties of the exterior domains $\widehat{\Omega}_t$, $\Omega^-$, and of the material boundary $\Gamma_t$. It is customary to distinguish between the magnetic field $\mathbf{H}$, and the induced magnetic field $\mathbf{B} = \mu \mathbf{H}$, where $\mu$ is the magnetic permeability of the medium. In the sequel we assume $\mu$ is constant, and we use only the magnetic field $\mathbf{H}$. We suppose $\widehat{\Omega}_t$, $\Omega^-$ non-electrically conducting, and denote by $-\mathbf{n}$ its normal unit vector. For a vector field $\mathbf{v}$ we set

$$v_n := \mathbf{v} \cdot \mathbf{n}, \qquad \mathbf{v}_\tau = \mathbf{v} - v_n \mathbf{n}. \tag{2.9}$$

The normal and tangential components of $\mathbf{H}$, $\mathbf{E}$ satisfy the following boundary conditions:

$$\begin{aligned} &\mu H_n = \widehat{\mu} \widehat{H}_n, \qquad \mathbf{H}_\tau = \widehat{\mathbf{H}}_\tau, \qquad \text{on}\, \Gamma_t, \\ &\Big(\mu \mathbf{H} \times \mathbf{u} + \frac{1}{\sigma} \nabla \times \mathbf{H}\Big)_\tau = \widehat{\mathbf{E}}_\tau + \mathbf{j}_S, \qquad \text{on}\, \Gamma_t \\ &\mu H_n = \mu^- H_{-,n}, \qquad \mathbf{H}_\tau = \mathbf{H}^-_\tau, \qquad \text{on}\, \pi, \\ &\Big(\mu \mathbf{H} \times \mathbf{u} + \frac{1}{\sigma} \nabla \times \mathbf{H}\Big)_\tau = \mathbf{E}^-_\tau + \mathbf{j}_S, \qquad \text{on}\, \pi, \end{aligned} \tag{2.10}$$

where $\mathbf{j}_S$ denote surface currents tangent to $\Gamma_t$, cf. [22, Section 53]. Furthermore we have

$$\mathbf{u} = 0 \qquad \text{on } \pi.$$

$$\mathbf{T}(\mathbf{u}, p) \cdot \mathbf{n} + \mathbf{T}(\mathbf{H}) \cdot \mathbf{n} = k \mathcal{K}(\zeta) \mathbf{n} + \mathbf{T}(\widehat{\mathbf{H}}) \cdot \mathbf{n}, \qquad \text{on}\, \Gamma_t, \tag{2.11}$$

where $k$ denotes the mechanical surface tension. Notice that in the vacuum region $\widehat{\Omega}_t$ the exterior velocity $\widehat{\mathbf{u}}$ and the pressure $\widehat{p}$ are zero. Furthermore, $(2.11)_2$ can equivalently be written as

$$\mathbf{T}(\mathbf{u}, p) \cdot \mathbf{n} = k \mathcal{K}(\zeta) \mathbf{n} - \Big[\mathbf{T}(\widetilde{\mathbf{H}})\Big] \cdot \mathbf{n}, \qquad \text{on}\, \Gamma_t, \tag{2.12}$$

where

$$\Big[\mathbf{T}(\widetilde{\mathbf{H}})\Big](\zeta) = \lim_{x,y \to \zeta} \Big(\mathbf{T}(\mathbf{H})(x) - \mathbf{T}(\widehat{\mathbf{H}}(y))\Big), \qquad x \in \Omega_t,\, y \in \widehat{\Omega}_t,\, \zeta \in \Gamma_t.$$

If no confusion arises, using the jump symbol $[\,\cdot\,]$ we omit the term $(\zeta)$.

Finally, to have a uniquely determined exterior magnetic field $\widehat{\mathbf{H}}$, cf., [35], we set

$$\lim_{|x| \to \infty} \widehat{\mathbf{H}}(x) = \widehat{\mathbf{H}}_\infty, \qquad \lim_{|x| \to \infty} \mathbf{H}^-(x) = \mathbf{H}^-_\infty.$$

In the sequel $\widehat{\mathbf{H}}_\infty$ will be zero for $\widehat{\mathbf{H}}$ variable, will be $\mu/\widehat{\mu}$ for $\widehat{\mathbf{H}}$ constant, and $\mathbf{H}^-_\infty$ will be $\mu/\mu^-$. Also we shall continue to assume $\mathbf{j}_S \neq 0$, $\mu \neq \widehat{\mu}$ for the sake of generality.

### 2.5. Initial conditions

To deal with unsteady incompressible motions, to (2.3), (2.7) we add the initial conditions:

$$\zeta(x_*, 0) = \zeta_0(x), \qquad x \in \Sigma, \tag{2.13}$$

$$\begin{aligned}
\Omega_0 &:= \{x_*, z : \quad x_* \in \Sigma, \quad 0 < z < \zeta(x_*, 0) = \zeta_0(x_*)\}, \\
\widehat{\Omega}_0 &:= \{x_*, z : \quad x_* \in \Sigma, \quad \zeta(x_*, 0) = \zeta_0(x_*) < z < \infty\},
\end{aligned}$$

$$\begin{aligned}
\mathbf{u}(x, 0) &= \mathbf{u}_0(x), && x \in \Omega_0, \\
\mathbf{H}(x, 0) &= \mathbf{H}_0(x), && x \in \Omega_0, \\
\widehat{\mathbf{H}}(x, 0) &= \widehat{\mathbf{H}}_0(x) && x \in \widehat{\Omega}_0, \\
\mathbf{H}^-(x, 0) &= \mathbf{H}^-_0(x), && x \in \Omega^-.
\end{aligned} \tag{2.14}$$

To deal with unsteady compressible motions, to (2.5), (2.7) we add the initial conditions:

$$\begin{aligned}
\zeta(x_*, 0) &= \zeta_0, && x_* \in \Sigma, \\
\rho(x, 0) &= \rho_0(x), && x \in \Omega_0, \\
\mathbf{u}(x, 0) &= \mathbf{u}_0(x), && x \in \Omega_0, \\
\mathbf{H}(x, 0) &= \mathbf{H}_0(x), && x \in \Omega_0, \\
\widehat{\mathbf{H}}(x, 0) &= \widehat{\mathbf{H}}_0(x), && x \in \widehat{\Omega}_0, \\
\mathbf{H}^-(x, 0) &= \mathbf{H}^-_0(x), && x \in \Omega^-.
\end{aligned} \tag{2.15}$$

Conditions at infinity are

$$\mathbf{H}_\infty = \lim_{|z| \to \infty} \widetilde{\mathbf{H}}, \qquad \mathbf{E}_\infty = \lim_{|z| \to \infty} \widetilde{\mathbf{E}}.$$

Since $\mathbf{H}_\infty$ is parallel to $\mathbf{k}$, we can state

$$\mathbf{k} \times \mathbf{E} \cdot \mathbf{H} \to 0, \qquad \text{as } |z| \to \infty.$$

## 3. Energy identity

The aim of this section is to furnish an energy estimate both for incompressible and compressible fluids.

### 3.1. Magnetic field

We first obtain separately an energy estimate for the magnetic field $\widetilde{\mathbf{H}}$.

Let us notice that since the domain is unbounded, in general the energy is not finite, and we need to introduce some cutoff energies, as follows:

$$\begin{aligned}\mathcal{E}_z(\widehat{\mathbf{H}}) &:= \int_\Sigma\int_\zeta^z \frac{\widehat{\mu}\widehat{H}^2}{2} dz dx_* \;=: \int_{\widehat{\Omega}_{tz}} \frac{\widehat{\mu}\widehat{H}^2}{2} dz dx_*, \\ \mathcal{E}_z(\mathbf{H}^-) &:= \int_\Sigma\int_z^0 \frac{\mu^-(H^-)^2}{2} dz dx_* := \int_{\Omega_z^-} \frac{\mu^-(H^-)^2}{2} dz dx_* = \mathcal{E}_z(\mathbf{H}_0^-).\end{aligned} \tag{3.1}$$

Notice that in (3.1) the first symbol := defines the cutoff energies, while the second =: defines the cutoff domains.

To derive energy estimates for the magnetic field, we first multiply by $\mathbf{H}$ $(2.5)_3$ and integrate over $\Omega_t$, by the Reynolds transport theorem, and recalling that the normal to $\partial\Omega_t$ is $\mathbf{n}$, we get

$$\begin{aligned}&\frac{d}{dt}\int_{\Omega_t} \frac{\mu\mathbf{H}^2}{2} dx - \frac{\mu}{2}\int_{\Gamma_t} H^2\, u_n dS = \int_{\Gamma_t} \mathbf{n}\times\left(\mathbf{u}\times\mathbf{H} - \frac{1}{\sigma}\nabla\times\mathbf{H}\right)\cdot\mathbf{H} dS \\ &+\frac{1}{\sigma}\int_\pi \mathbf{k}\times(\nabla\times\mathbf{H})\cdot\mathbf{H}\, dx_* - \frac{1}{\sigma}\int_{\Omega_t}(\nabla\times\mathbf{H})^2 dx - \mu\int_{\Omega_t}(\nabla\times\mathbf{H})\cdot(\mathbf{H}\times\mathbf{u})\, dx.\end{aligned} \tag{3.2}$$

Thus we multiply $(2.7)_3$ by $\mathbf{H}^-$ and integrate over $\Omega_z^-$, by the Reynolds transport theorem, and recalling that the exterior normal to $\partial\Omega_z^-$ on $\pi$ is $\mathbf{k}$, taking into account periodicity on the lateral surface, we obtain

$$\frac{d}{dt}\int_{\Omega_z^-} \frac{\mu^-(\mathbf{H}^-)^2}{2} dx = \int_\pi \mathbf{k}\times\mathbf{E}^-\cdot\mathbf{H}^- dx_*\Big|_0 + \int_\Sigma \mathbf{k}\times\mathbf{E}^-\cdot\mathbf{H}^- dx_*\Big|_z. \tag{3.3}$$

Next we multiply $(2.7)_5$ by $\widehat{\mathbf{H}}$ and integrate over $\widehat{\Omega}_{tz}$, by the Reynolds transport theorem, and recalling that the normal to $\partial\widehat{\Omega}_{tz}$ is $-\mathbf{n}$, taking into account periodicity on the lateral surface, we obtain

$$\frac{d}{dt}\int_{\widehat{\Omega}_{tz}} \frac{\widehat{\mu}\widehat{\mathbf{H}}^2}{2} dx + \frac{\widehat{\mu}}{2}\int_{\Gamma_t} \widehat{H}^2\, u_n dS = \int_{\Gamma_t} \mathbf{n}\times\widehat{\mathbf{E}}\cdot\widehat{\mathbf{H}} dS - \int_\Sigma \mathbf{k}\times\mathbf{E}^-\cdot\mathbf{H}^- dx_*\Big|_z, \tag{3.4}$$

where $\sqrt{g}\, u_n = \partial_t\zeta = \dfrac{\partial\eta}{\partial t}$ denotes the velocity of points at boundary $\Gamma_t$.

Let us now observe that the boundary condition $(2.10)_3$ yields

$$\begin{aligned}&\int_{\Gamma_t} \mathbf{n}\times\left(\mu\mathbf{u}\times\mathbf{H} - \frac{1}{\sigma}\nabla\times\mathbf{H}\right)\cdot\mathbf{H} dS + \int_{\Gamma_t} \mathbf{n}\times\widehat{\mathbf{E}}\cdot\widehat{\mathbf{H}} dS \\ &= -\int_{\Gamma_t} \mathbf{j}_S\cdot\mathbf{H}_\tau\, dS = -\int_{\Gamma_t} \mathbf{j}_S\cdot\widehat{\mathbf{H}}_\tau\, dS =: \mathcal{B}_S.\end{aligned} \tag{3.5}$$

**Remark 3.1** *It is important to notice that, since $\mathbf{j}_S$ is given, $\mathcal{B}_S(t)$ involves only tangential components of* $\mathbf{H}$.

**Remark 3.2** *Concerning the time derivative of magnetic energy in* $\Omega_z^-$, *and* $\widehat{\Omega}_{tz}$ *we assume that it is summable in the limit* $z \to \pm\infty$.

Adding (3.2), (3.3), (3.4), using the boundary conditions $(2.10)_{4,5}$ and (3.5) we get, in the limits $z \to \pm\infty$, the wanted energy equation for a magnetic field:

$$\begin{aligned}
&\frac{d}{dt}\frac{\widetilde{\mu}}{2}\int_{\widetilde{\Omega}_t} \widetilde{\mathbf{H}}^2 dx - \frac{1}{2}\int_{\Gamma_t}\left[\widetilde{\mu}\widetilde{H}^2\right] u_n dS \\
&= -\frac{1}{\sigma}\int_{\Omega_t}(\nabla\times\mathbf{H})^2 dx + \mathcal{B}_S + \mu\int_{\Omega_t}(\nabla\times\mathbf{H})\cdot(\mathbf{H}\times\mathbf{u})\,dx\,.
\end{aligned} \tag{3.6}$$

We set $\widetilde{\Omega}_z = \widehat{\Omega}_{tz}\cup\Omega_z^-$, then

$$\begin{aligned}
\mathcal{E}(\widetilde{\mathbf{H}}) &= \lim_{|z|\to\infty}\frac{\widetilde{\mu}}{2}\int_{\widetilde{\Omega}_z}\widetilde{\mathbf{H}}^2\,dx - \frac{\widetilde{\mu}}{2}\int_{\widetilde{\Omega}_{bz}}\widetilde{\mathbf{H}}_b^2\,dx\,, \\
D(\widetilde{\mathbf{H}}) &:= \mathcal{D}(\mathbf{H}) := \frac{1}{\sigma}\int_{\Omega_t}(\nabla\times\mathbf{H})^2 dx\,.
\end{aligned} \tag{3.7}$$

It holds that

$$\frac{d}{dt}\mathcal{E}(\widetilde{\mathbf{H}}) = \frac{d}{dt}\mathcal{E}_z(\widetilde{\mathbf{H}})\,, \tag{3.8}$$

$$\frac{d}{dt}\mathcal{E}(\widetilde{\mathbf{H}}) + D(\widetilde{\mathbf{H}}) = \frac{1}{2}\int_{\Gamma_t}\left[\widetilde{\mu}\widetilde{H}^2\right] u_n\,dS + \mathcal{B}_S - \mu\int_{\Omega_t}(\nabla\times\mathbf{H})\cdot(\mathbf{H}\times\mathbf{u})\,dx\,. \tag{3.9}$$

### 3.2. Incompressible fluids

In this subsection we derive a balance equation for the total energy of the system governing incompressible fluid flows. Let us multiply $(2.3)_1$ times $\mathbf{u}$, and integrate over $\Omega_t$; using Reynold's transport theorem, and recalling that $\mathbf{u}$ is solenoidal we obtain

$$\begin{aligned}
\frac{d}{dt}\int_{\Omega_t}\Big(\frac{\mathbf{u}^2}{2} - U\Big)dx &= -\frac{\nu}{2}\int_{\Omega_t}|S(\mathbf{u})|^2 dx + \mu\int_{\Omega_t}\mathbf{u}\cdot(\nabla\times\mathbf{H})\times\mathbf{H}\,dx \\
&\quad + \int_{\Gamma_t}\mathbf{u}\cdot\mathbf{T}(\mathbf{u},p)\cdot\mathbf{n}\,dS.
\end{aligned} \tag{3.10}$$

Furthermore we have

$$\begin{aligned}
&\int_{\Gamma_t}\mathbf{u}\cdot\mathbf{T}(\mathbf{u},p)\cdot\mathbf{n}\,dS = \int_{\Gamma_t}\Big(k\mathcal{K}(\eta)\mathbf{n} + \mathbf{u}\cdot[\mathbf{T}(\widehat{\mathbf{H}}) - \mathbf{T}(\mathbf{H})]\cdot\mathbf{n}\Big)\,dS \\
&= \int_{\Gamma_t}\Big(k\mathcal{K}(\eta)\mathbf{n} - \Big[\mathbf{T}(\widetilde{\mathbf{H}})\Big]\cdot\mathbf{n}\Big)\,dS.
\end{aligned} \tag{3.11}$$

Moreover since the external pressure is constant, and $\mathbf{u}$ is solenoidal we get

$$\int_{\Gamma_t}\widehat{p}\mathbf{n}\cdot\mathbf{u}\,dS = 0. \tag{3.12}$$

Also from (2.8), and the definition of $\mathcal{K}$ we get

$$\int_{\Gamma_t} \mathcal{K}(\eta)\mathbf{u}\cdot\mathbf{n}dS = -\frac{d}{dt}\int_{\Sigma}\sqrt{1+|\nabla_*\eta|^2}\,dx_*. \tag{3.13}$$

Notice that

$$\frac{1}{2}\int_{\Gamma_t}\left[\widetilde{\mu}\widetilde{H}^2\right]u_n\,dS - \int_{\Gamma_t}\left[\mathbf{T}(\widetilde{\mathbf{H}})\right]\cdot\mathbf{n}\,dS = \left[\widetilde{\mu}\right]\int_{\Gamma_t}\widehat{\mathbf{H}}_\tau^2\,u_n\,dS. \tag{3.14}$$

Adding (3.10), (3.9), and using boundary condition $(2.10)_3$, and identities (3.14), (3.13), we get

$$\begin{aligned}&\frac{d}{dt}\left[\int_{\Omega_t}\Big(\frac{\mathbf{u}^2}{2}-U\Big)dx + \mathcal{E}(\widetilde{\mathbf{H}}) + k\int_{\Sigma}\sqrt{1+|\nabla_*\eta|^2}\,dx_*\right]\\ &\quad= -\frac{\nu}{2}\int_{\Omega_t}|S(\mathbf{u})|^2dx - \mathcal{D}(\widetilde{\mathbf{H}}) + \mathcal{B}_S + \frac{\widehat{\mu}}{\mu}\left[\widetilde{\mu}\right]\int_{\Gamma_t}\widehat{\mathbf{H}}_n^2\,u_n\,dS,\end{aligned} \tag{3.15}$$

which can be rewritten as

$$\begin{aligned}&\frac{d}{dt}\left[\int_{\Omega_t}\Big(\frac{\mathbf{u}^2}{2}-U\Big)dx + \frac{\widetilde{\mu}}{2}\int_{\widetilde{\Omega}_t}\widetilde{\mathbf{H}}^2\,dx + \ k\int_{\Sigma}\ \sqrt{1+|\nabla_*\zeta|^2}dx_*\right]\\ &\quad+\frac{\nu}{2}\int_{\Omega_t}|S(\mathbf{u})|^2dx + \frac{1}{\sigma}\int_{\Omega_t}(\nabla\times\mathbf{H})^2dx = \mathcal{B}_S(t) + \left[\widetilde{\mu}\right]\mathtt{M}\end{aligned} \tag{3.16}$$

where

$$\mathtt{M} = \frac{\widehat{\mu}}{\mu}\int_{\Gamma_t}\widehat{\mathbf{H}}_n^2\,u_n\,dS.$$

We set

$$\begin{aligned}\mathcal{E}(t) &= \int_{\Omega_t}\Big(\frac{\mathbf{u}^2}{2}-U\Big)dx + \mathcal{E}(\widetilde{\mathbf{H}}) + k\int_{\Sigma}\sqrt{1+|\nabla_*\zeta|^2}dx_*\\ &= \mathcal{E}(\mathbf{u}) + \Pi(\zeta) + \mathcal{E}(\widetilde{\mathbf{H}}) + \mathcal{E}(\zeta),\end{aligned} \tag{3.17}$$

$$\mathcal{D}(t) = \frac{\nu}{2}\int_{\Omega_t}|S(\mathbf{u})|^2dx + \frac{1}{\sigma}\int_{\Omega_t}(\nabla\times\mathbf{H})^2dx = \mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{H}).$$

Here we have introduced the potential energy

$$\Pi(\zeta) = -\int_{\Sigma}\int_0^{\zeta}U\,dz\,dx_*.$$

With notations (3.17), equation (3.16) is written

$$\frac{d}{dt}\mathcal{E}(t) + D(t) = \mathcal{B}_S(t) + \left[\widetilde{\mu}\right]\mathtt{M}. \tag{3.18}$$

**Remark 3.3** *Equation* (3.18) *furnishes the wanted balance equation for the total energy. It states that the variations of the total energy arise from surface currents, and from a combination due to the variation of free surface with the tangential component of* $\mathbf{H}$ *through the magnetic susceptibilities.*

**3.3. Model rest state $S_b$**

We write the model rest state, still called rest state, for a horizontal layer of a heavy incompressible fluid. The model rest state is not divergence free. The basic domain $\Omega_b$ occupied by the fluid is a rectangular parallelepiped having as basis below a horizontal rigid basis $\Sigma$ and above the rectangle

$$\Sigma_L := \{(x_*, z) : \quad x_* \in \Sigma, \quad z = L\}.$$

The two basic domains exterior to $\Omega_b$ are two vertical half channels $\widehat{\Omega}_b$, $\Omega^-$ with rectangular section $\Sigma$. Let $\nabla U = -f\mathbf{k}$, with $f$ the positive constant gravity acceleration, and $\mathbf{j}_S$ be given. We know that there exists a basic equilibrium configuration $S_b = \{\mathbf{u}_b, p_b, \widetilde{\mathbf{H}}_b, \zeta_b\}$, in $\widetilde{\Omega}_b = \Omega_b \cup \widehat{\Omega}_b \cup \Omega^-$ with

$$\begin{array}{lll}
\widetilde{\mathbf{u}}_b = 0, & & \text{in}\, \widetilde{\Omega}_b \\
\zeta_b = L, & & \text{in}\Sigma \\
p_b = -\, f\, z, & & \text{in}\, \Omega_b, \\
\widehat{p}_b = const., & & \text{in}\, \widehat{\Omega}_b \cup \Omega^- \\
\mathbf{H}_b = H_b\mathbf{k}, & \mathbf{E}_b = 0, & \text{in } \Omega_b \\
\widehat{\mathbf{H}}_b = H_b \dfrac{\mu L}{\widehat{\mu}\, z}\mathbf{k}, & \widehat{\mathbf{E}}_b = -\mathbf{j}_S = 0, & \text{in } \widehat{\Omega}_b \\
\mathbf{H}_b^- = H_b \dfrac{\mu L}{\mu^-\, z}\mathbf{k}, & \mathbf{E}_b^- = 0, & \text{in } \Omega^-,
\end{array} \tag{3.19}$$

and with $H_b$ constant, as basic flow.

Also for the rest state we observe that the energy of the magnetic field is not finite in $\widehat{\Omega}_b$, and $\Omega^-$, thus we use the cutoff energies introduced by (3.1), and we shall work with

$$\widehat{\mathcal{E}}_{bz}(t) = \int_{\widehat{\Omega}_{bz}} \frac{\widehat{\mu}\widehat{H}_b^2}{2} dz dx_* = \widehat{\mathcal{E}}_{bz}(0),$$

$$\mathcal{E}_{bz}^-(t) = \int_{\mathcal{P}_{-z}} \frac{\mu^-(H^-)_b^2}{2} dz dx_* = \mathcal{E}_{bz}^-(0),$$

also the dissipation is given by $(3.17)_2$. In this way, the energy, and the dissipation in the basic state are given by

$$\widetilde{\mathcal{E}}_{bz}(t) = \int_{\widetilde{\Omega}_{bz}} \frac{\widetilde{\mu}\widetilde{\mathbf{H}}_b^2}{2} dx + \Big(\frac{f\, L^2}{2} + k\Big)|\Sigma| = \widetilde{\mathcal{E}}_{bz}(0),$$

$$\mathcal{D}_b = \frac{1}{\sigma} \int_{\Omega_b} (\nabla \times \mathbf{H}_b)^2 dx = 0.$$

Also, $\mathbf{j}_S$ is tangential to the boundary, in particular, since $\Gamma_b$ is parallel to $\Sigma$, $\mathbf{j}_S$ is orthogonal to $\mathbf{k}$, and it holds the trivial relation

$$\int_{\Gamma_b} \mathbf{j}_S \cdot \widehat{\mathbf{H}}_b \, dS = \int_\Sigma \mathbf{j}_S \cdot \mathbf{k} H_b dx_* \Big|_0 = 0.$$

We remark that, under assumptions $\mathbf{j}_S = 0$, $\mu - \widehat{\mu} = 0$, (3.18) reduces to

$$\frac{d}{dt}\mathcal{E}_b(t) + D_b(t) = 0. \tag{3.20}$$

### 3.4. True rest state $S_b$

All calculations given in the previous subsection hold true except the fact that $\widehat{\mathbf{H}}_b = \mathbf{H}_b = \text{const}$. Here we write the differences.

$$\mathbf{H}_b = H_b\mathbf{k}, \qquad \mathbf{E}_b = 0, \qquad \widehat{\mathbf{H}}_b = H_b\frac{\mu}{\widehat{\mu}}\mathbf{k}, \qquad \widehat{\mathbf{E}}_b = \widehat{\mathbf{E}}_b(\mathbf{j}_S) = 0, \tag{3.21}$$

$H_b$ constant, as basic flow. Again the magnetic energy is not finite in $\widehat{\Omega}_t$, and $\Omega^-$, therefore we shall adopt all definitions previously introduced. Also, for plane horizontal surface $\pi$ it holds the trivial relation

$$\int_{\Gamma_t} \mathbf{j}_S \cdot \widehat{\mathbf{H}}_b\, dS = \int_{\Sigma} \mathbf{j}_S \cdot \mathbf{k}\, H_b dx_*\Big|_0 = 0.$$

We remark that, assumptions $\mathbf{j}_S = 0$, $\mu - \widehat{\mu} = 0$, and the property (3.18) up to a constant,

$$\frac{d}{dt}\mathcal{E}_{bz}(t) = \frac{d}{dt}\mathcal{E}_b(t) = 0,$$

yields (3.20).

### 3.5. Compressible fluids

To obtain the energy identity we multiply $(2.5)_2$ by $\mathbf{u}$, and integrate over $\Omega_t$, using the Reynolds transport theorem, utilizing the continuity equation, cf. [29] we get

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_t} \rho\frac{\mathbf{u}^2}{2}dx &= -\frac{\nu}{2}\int_{\Omega_t} |S(\mathbf{u})|^2 dx + \int_{\Omega_t} \rho\nabla U \cdot \mathbf{u} dx\\ &+ \int_{\Omega_t} p(\rho)\nabla\cdot\mathbf{u} dx + \mu\int_{\Omega_t} \mathbf{u}\cdot(\nabla\times\mathbf{H})\times\mathbf{H}\, dx + \int_{\Gamma_t} \mathbf{u}\cdot\mathbf{T}(\mathbf{u},p)\cdot\mathbf{n}\, dS.\end{aligned} \tag{3.22}$$

Set $\psi(\rho) = \int^\rho \frac{p(s)}{s^2}ds$. Multiply by $p(\rho)/\rho$ $(2.5)_1$ and integrate over $\Omega_t$, to get

$$\frac{d}{dt}\int_{\Omega_t} \rho\,\psi(\rho) dx = -\int_{\Omega_t} p(\rho)\nabla\cdot\mathbf{u} dx. \tag{3.23}$$

Adding (3.22) and (3.23), employing the Reynolds transport theorem, (2.12) and (3.13) we obtain

$$\begin{aligned}&\frac{d}{dt}\Big\{\int_{\Omega_t} \Big[\rho\frac{\mathbf{u}^2}{2} + \rho\psi(\rho) - \rho U\Big] dx + k\int_{\Sigma} \sqrt{1 + |\nabla_*\zeta|^2} dx_*\Big\}\\ &= -\frac{\nu}{2}\int_{\Omega_t} |S(\mathbf{u})|^2 dx + \mu\int_{\Omega_t} \mathbf{u}\cdot(\nabla\times\mathbf{H})\times\mathbf{H}\, dx - \int_{\Gamma_t} \mathbf{u}\cdot\Big[\mathbf{T}(\widetilde{\mathbf{H}})\Big]\cdot\mathbf{n}\, dS.\end{aligned} \tag{3.24}$$

Adding (3.24) to (3.9), using the identity (3.14), and operating as in the previous section we get

$$\frac{d}{dt}\Big\{\int_{\Omega_t}\Big[\rho\frac{\mathbf{u}^2}{2}+\rho\psi(\rho)+f\,z\Big]dx+\frac{\widetilde{\mu}}{2}\int_{\widetilde{\Omega}_t}\widetilde{\mathbf{H}}^2dx+\,k\int_{\Sigma}\sqrt{1+|\nabla_*\zeta|^2}dx_*\Big\}$$
$$=-\frac{\nu}{2}\int_{\Omega_t}|S(\mathbf{u})|^2\,dx-\frac{1}{\sigma}\int_{\Omega_t}\nabla\times\mathbf{H}^2dx+\mathcal{B}_S(t)+\Big[\widetilde{\mu}\Big]\mathtt{M}. \tag{3.25}$$

Clearly (3.25) **cannot provide a control** in time for the $L^2$ norms of solutions because the terms on the boundary have no definite sign. We remark that the boundary terms coincide with those deduced in the case of incompressible fluids.

Set

$$\mathcal{E}=\int_{\Omega_t}\Big[\rho\frac{\mathbf{u}^2}{2}+\rho\psi(\rho)+f\,z\Big]dx+\frac{\widetilde{\mu}}{2}\int_{\widetilde{\Omega}_t}\widetilde{\mathbf{H}}^2dx+\,k\int_{\Sigma}\sqrt{1+|\nabla_*\zeta|^2}dx_*,$$
$$D(t)=\frac{\nu}{2}\int_{\Omega_t}|S(\mathbf{u})|^2dx+\frac{1}{\sigma}\int_{\Omega_t}\nabla\times\mathbf{H}^2dx. \tag{3.26}$$

With notation (3.26), (3.25) furnishes again (3.18), see Remark 3.2.

In particular, under assumptions $\mathbf{j}_s=0$, $\mu-\widehat{\mu}=0$, (3.25) reduces to

$$\frac{d}{dt}\mathcal{E}(t)\;=\;-D(t). \tag{3.27}$$

Equation (3.27) implies that the total energy is not increasing in time. Hence (3.27) furnishes an "a priori estimate" for the solution for all initial data. We recall that $\mathcal{E}(t)$ is equivalent, up to a constant, to the $L^2$ norm of perturbations $\mathbf{u}$, $\rho-\rho_b$, $\eta:=\zeta-L$, if $\frac{dp}{d\rho}>0$, $k>0$. To prove this statement it is enough to follow the lines of [29].

### 3.6. Model rest state $S_b$

We write the rest state for a horizontal layer of heavy compressible fluid. Let the gravity force $\nabla U=-f\mathbf{k}$, with $f$ the positive constant gravity acceleration, and $\mathbf{j}_S$ be given. We know that there exists a basic equilibrium configuration $S_b=\{\mathbf{u}_b,\,\rho_b,\widetilde{\mathbf{H}}_b,\zeta_b\}$, with

$$\begin{array}{llll}
\mathbf{u}_b=0, & \psi(\rho_b)=-\,f\,z+c, & \zeta_b=L, & \displaystyle\int_{\Omega_b}\rho_b dx=M,\\
\mathbf{H}_b=H_b\mathbf{k}, & \mathbf{E}_b=0, & \text{in }\ \Omega_b, & \\
\widehat{\mathbf{H}}_b=H_b\dfrac{\mu L}{\widehat{\mu}\zeta}\mathbf{k}, & \widehat{\mathbf{E}}_b=\widehat{\mathbf{E}}_b(\mathbf{j}_S), & \text{in }\ \widehat{\Omega}_b, & \\
\mathbf{H}_b^-=H_b\dfrac{\mu L}{\mu^-\,z}\mathbf{k}, & \mathbf{E}_b^-=0, & \text{in }\ \Omega^-, &
\end{array} \tag{3.28}$$

with $\psi(\rho_b)=\int^{\rho_b}\frac{p'(s)}{s}ds$.

The energy $\mathcal{E}_b$ is not finite, thus we use the definitions of cut-off energies $\widehat{\mathcal{E}}_{bz}$, $\mathcal{E}_{bz}^-$ of Subsections 3.3, 3.4.

The energy and the dissipation in the basic state are given by

$$\widetilde{\mathcal{E}}_{bz}(t) = \int_{\widetilde{\Omega}_{bz}} \rho_b \psi(\rho_b)\, dx + \int_{\widetilde{\Omega}_{bz}} \frac{\widetilde{\mu}\widetilde{\mathbf{H}}_b^2}{2} dx + \Big(\frac{f\, L^2}{2} + \kappa\Big)|\Sigma| = \widetilde{\mathcal{E}}_{bz}(0),$$

$$\mathcal{D}_b = \frac{1}{\sigma} \int_{\Omega_b} (\nabla \times \mathbf{H}_b)^2 dx = 0.$$

### 3.7. True rest state $S_b$

All calculations given in the previous subsection hold true except the fact that $\widehat{\mathbf{H}}_b = \mathbf{H}_b = \text{const}$. All remarks made in subsection "True rest state" continue to hold. There are no further remarks.

### 3.8. Irreversible processes

Of course (3.18) is **not** sufficient to **provide a control** for the $L^2$ norms of solutions, because in general the right-hand side contains the unknown functions and has no definite sign. More precisely the term $\Big[\widetilde{\mu}\Big]\mathtt{M}$ takes into account the irreversibility of the process, see [9]. It is trivial that this term is not zero only when the sharp discontinuity surface $\Gamma_t$ is moving. Therefore we make the following assumption: *There exists a function $\alpha = \alpha(x', \zeta(x',t),t)$ coming from $\alpha = \alpha([\widetilde{\mu}], \mathbf{H}_n)$ such that*

$$-\frac{\widetilde{\mu}}{\mu} \int_{\Gamma_t} \Big[\widetilde{\mu}\Big] H_n^2\, u_n = -\frac{d}{dt} \int_{\Gamma_t} \alpha dS. \tag{3.29}$$

We call the coefficient $\alpha$ responsible for variations of $S$, in measure and in time, **the magnetic surface coefficient**.

Notice that $[\widetilde{\mu}]$ **has no definite sign!**

We stress that in Subsections 3.2, 3.5 the energy identity has been obtained in the general case.

## 4. Perturbations

In order to give a stability result for rest configuration of MHD fluids, we control, in the energy norm, the difference between the basic equilibrium configuration and the unsteady motion. To this end, we are led to define the perturbations to an equilibrium configuration. We give here two different definitions of perturbation to magnetic $\widetilde{\mathbf{H}}_b$, and electric $\widetilde{\mathbf{E}}_b$ in Eulerian coordinates.

As we shall show, the definition of perturbation is far to be trivial when the surface is unknown. Our approach starts from a criticism of previous definitions and shows that the *perturbations $\varphi$, $\psi$ to $\widetilde{\mathbf{H}}_b$, $\widetilde{\mathbf{E}}_b$, usually used in previous literature, don't satisfy the boundary conditions* (2.10) even when it is $\mu = \widehat{\mu}$ and in the linear approximation. Nevertheless conditions (2.10) appear in some papers on linear stability theory, e.g., [38]. Next we propose an alternative definition of perturbation. For the sake of generality, our definition of perturbation will be given for different magnetic permeabilities in the fluid and in a vacuum cf. [33], [26], [4, Section 5.2].

We explicitly remark that, in the linear case, stability results are independent of definition of perturbations. The difference between the perturbation fields for $\widetilde{\mathbf{H}}_b$, $\widetilde{\mathbf{E}}_b$ becomes sensible in the nonlinear case only.

In this section before introducing our definition $\widetilde{\mathbf{h}}$ of a perturbation to an electromagnetic field, we recall some previous definitions of perturbations $\varphi$, $\psi$ to magnetic and electric fields, [35], [4], and write the equations satisfied by these fields $\varphi$, $\psi$.

### 4.1. Definition I: perturbation $\widetilde{\varphi}$ to the rest state

Let there be given the domains $\Omega_b$, $\widehat{\Omega}_b$, $\Omega^-$, and the fields $\mathbf{H}_b = \mathbf{c}$ constant, $\widehat{\mathbf{H}}_b$ defined on $\Omega_b$, and $\widehat{\Omega}_t$, respectively. In order to define the perturbation to the field $\widetilde{\mathbf{H}}_b$ it is customary to extend to $\Omega_+ = \mathcal{P} - \Omega^-$ the fields $\mathbf{H}_b$, $\widehat{\mathbf{H}}_b$ in a natural way.

Here given a vector $\mathbf{w}$ defined in $\Omega_t$ or $\widehat{\Omega}_t$ with the exponent $^*$ we denote the extension of $\mathbf{w}$ in the whole $\mathcal{P}$. Now, given the decomposition of $\mathcal{P}$ as $\Omega^-$, $\Omega_t$, $\widehat{\Omega}_t$, we set

$$\begin{aligned}
\mathbf{H}(x_*, z, t) &= \mathbf{H}_b^* + \varphi(x_*, z, t), && (x_*, z, t) \in \Omega_t \times (0, \infty),\\
\widehat{\mathbf{H}}(x_*, z, t) &= \widehat{\mathbf{H}}_b^*(z) + \widehat{\varphi}(x_*, z, t), && (x_*, z, t) \in \widehat{\Omega}_t \times (0, \infty),\\
\mathbf{H}^-(x_*, z, t) &= \mathbf{H}_b^- + \varphi^-(x_*, z, t), && (x_*, z, t) \in \Omega^- \times (0, \infty),\\
\mathbf{E}(x_*, z, t) &= \mathbf{E}_b^* + \psi(x_*, z, t), && (x_*, z, t) \in \Omega_t \times (0, \infty),\\
\widehat{\mathbf{E}}(x_*, z, t) &= \widehat{\mathbf{E}}_b^*(z) + \widehat{\psi}(x_*, z, t), && (x_*, z, t) \in \widehat{\Omega}_t \times (0, \infty),\\
\mathbf{E}^-(x_*, z, t) &= \mathbf{E}_b^- + \psi^-(x_*, z, t), && (x_*, z, t) \in \Omega^- \times (0, \infty).
\end{aligned} \tag{4.1}$$

Notice that $\mathbf{H}_b^-$, $\mathbf{E}_b^-$ are not extended because the domain is fixed.

To compute boundary conditions of $\varphi$ on $\Gamma_t$ one starts from $(2.10)_{1,2,3}$. Developing vector and scalar products, one obtains on $\Gamma_t$,

$$\begin{aligned}
&\varphi_\tau(x_*, \zeta, t) + \mathbf{H}_{b,\tau}^*(\zeta) = \widehat{\varphi}_\tau(x_*, \zeta, t) + \widehat{\mathbf{H}}_{b,\tau}^*(\zeta),\\
&\mu\varphi_n(x_*, \zeta, t) + \mu H_{b,n}^*(\zeta) = \widehat{\mu}\widehat{\varphi}_n(x_*, \zeta, t) + \widehat{\mu}\widehat{H}_{b,n}^*(\zeta),\\
&\varphi_\tau(x_*, 0, t) + \mathbf{H}_{b,\tau}(0) = \varphi_\tau^-(x_*, 0, t) + \mathbf{H}_{b,\tau}^-(0).
\end{aligned} \tag{4.2}$$

**Definition I: perturbation $\widetilde{\varphi}$ to the model rest state.**

For the model rest state, recalling the expression of the basic magnetic field,

$$\begin{aligned}
&\varphi_\tau = \widehat{\varphi}_\tau + \frac{\nabla_*\eta}{\sqrt{g}} H_b\Big(\frac{\mu L}{\widehat{\mu}\zeta} - 1\Big),\\
&\mu\varphi_n = \widehat{\mu}\widehat{\varphi}_n - \frac{\mu}{\sqrt{g}} H_b \frac{\eta}{\zeta},\\
&\varphi_\tau(x_*, 0, t) = \varphi_\tau^-(x_*, 0, t),\\
&\mu\varphi_n(x_*, 0, t) = \mu^-\varphi_n^-(x_*, 0, t),\\
&\mathbf{n}^- = \mathbf{k}, \qquad \text{on } \pi
\end{aligned} \tag{4.3}$$

and the expected continuity conditions for $\varphi$ on $\Gamma_t$ are substituted by the jump conditions $(4.3)$[2]. We notice that

$$(\widehat{H}_b - H_b)(\zeta) = H_b\Big(\frac{\mu L}{\widehat{\mu}\zeta} - 1\Big) = H_b\Big(\frac{L}{\widehat{\mu}\zeta}(\mu - \widehat{\mu}) - \frac{\eta}{\zeta}\Big),$$
$$(\widehat{\mu}\widehat{H}_b - \mu H_b)(\zeta) = \mu H_b\Big(\frac{L}{\zeta} - 1\Big) = -\mu H_b\frac{\eta}{\zeta}$$

hold and the coefficient of $H_b$ in $(4.3)_1$ besides the term $\frac{L}{\widehat{\mu}\zeta}(\mu-\widehat{\mu})$, which vanishes for $\mu = \widehat{\mu}$, contains the term $-\nabla_*\eta\frac{\eta}{\zeta}$ which is at least quadratic in the perturbation. Also the last term $\mu H_b \frac{\eta}{\zeta}$ in $(4.3)_2$, is linear in $\eta$, and has no definite sign. Sometimes these boundary conditions are not used, cf. [38]. An analogous result holds in $\Omega^-$ and we omit it. Consequences of this choice in the energy computation will be given in the next section.

Notice that

$$\begin{aligned}\varphi_z &= \varphi\cdot\mathbf{k} = \varphi\cdot\Big(\mathbf{k}\cdot\mathbf{n}\,\mathbf{n} + \mathbf{k}\cdot\tau\,\tau\Big) = \frac{1}{\sqrt{g}}\varphi_n + \frac{1}{\sqrt{g}}\nabla_*\eta\cdot\varphi_\tau,\\ \widehat{\varphi}_z &= \widehat{\varphi}\cdot\mathbf{k} = \widehat{\varphi}\cdot\Big(\mathbf{k}\cdot\mathbf{n}\,\mathbf{n} + \mathbf{k}\cdot\tau\,\tau\Big) = \frac{1}{\sqrt{g}}\widehat{\varphi}_n + \frac{1}{\sqrt{g}}\nabla_*\eta\cdot\widehat{\varphi}_\tau.\end{aligned} \tag{4.4}$$

For the next computations it will be useful to compute the difference $\varphi_z - \widehat{\varphi}_z$; then it holds that

$$\varphi_z - \widehat{\varphi}_z = \frac{\widehat{\varphi}_n}{\sqrt{g}}\Big(\frac{\widehat{\mu}}{\mu} - 1\Big) - \frac{H_b}{g}\frac{\eta}{\zeta} + \frac{H_b}{g}\Big(\frac{\mu}{\widehat{\mu}}\frac{L}{\zeta} - 1\Big)|\nabla_*\eta|^2. \tag{4.5}$$

**Definition I: perturbation $\widetilde{\varphi}$ to the true rest state.**
All notations introduced in previous subsections still hold for the true rest state. Recalling the expression of the basic magnetic field, we deduce

$$\begin{aligned}\varphi_\tau &= \widehat{\varphi}_\tau + \frac{\nabla_*\eta}{\sqrt{g}}H_b\Big(\frac{\mu}{\widehat{\mu}} - 1\Big),\\ \mu\varphi_n &= \widehat{\mu}\widehat{\varphi}_n,\end{aligned} \tag{4.6}$$

and the expected continuity conditions on $\varphi$ on $\Gamma_t$ are replaced by the jump conditions $(4.6)$[3]. We notice that the coefficient of $H_b$ in $(4.6)_1$, linear in $\eta$ contains the term $\frac{(\mu-\widehat{\mu})}{\widehat{\mu}}$, which vanishes for $\mu = \widehat{\mu}$. Sometimes these boundary conditions

---

[2]In the linear case, for general field $\Phi$ these equations are reduced to equations on $\Gamma_b$ using the expansion of functions $\Phi(x_*, z, t)$ in terms of the normal direction to $\Gamma_t$, cf. [4, Section 5.2],

$$\Phi(x_*, \zeta, t) = \Phi(x_*, L, t) + \mathbf{n}\cdot\nabla\Phi(x_*, z, t)\Big|_{z=\overline{\zeta}}\eta.$$

[3]In linear case, for general field $\Phi$ these equations are reduced to equations on $\Gamma_b$ using the expansion of functions $\Phi(x_*, z, t)$ in terms of the normal direction to $\Gamma_t$, cf. [4, Section 5.2],

$$\Phi(x_*, \zeta, t) = \Phi(x_*, L, t) + \mathbf{n}\cdot\nabla\Phi(x_*, z, t)\Big|_{z=\overline{\zeta}}\eta.$$

are not used, cf. [38]. Consequences of this choice in the energy computation will be given in the next section.

In the wake of the model rest state, by (4.4) we compute the difference $\varphi_z - \widehat{\varphi}_z$. It holds that

$$\varphi_z - \widehat{\varphi}_z = \frac{\widehat{\varphi}_n}{\sqrt{g}}\Big(\frac{\widehat{\mu}}{\mu} - 1\Big) + \frac{H_b}{g}\Big(\frac{\mu}{\widehat{\mu}} - 1\Big)|\nabla_* \eta|^2. \tag{4.7}$$

**Perturbation equations for the magnetic field to both model and true rest state.** Of course in case $\eta = 0$ the usual definition of perturbation is recovered.

We end this subsection by writing the **perturbation equations for the magnetic field** $\widetilde{\varphi}$:

$$\begin{aligned} \mu\frac{\partial\varphi}{\partial t} &= -\nabla\times\Big((\mu\mathbf{H}_b + \mu\varphi)\times\mathbf{u} + \frac{1}{\sigma}\nabla\times\varphi\Big), && x\in\Omega_t \\ \widehat{\mu}\frac{\partial\widehat{\varphi}}{\partial t} &= -\nabla\times\widehat{\psi}, && x\in\widehat{\Omega}_t, \\ \mu^-\frac{\partial\varphi^-}{\partial t} &= -\frac{1}{\sigma^-}\nabla\times\nabla\times\varphi^-, && x\in\Omega^-. \end{aligned} \tag{4.8}$$

### 4.2. Definition II of perturbation $\widetilde{\mathbf{h}}$ to $\widetilde{\mathbf{H}}_b$, $\widetilde{\mathbf{E}}_b$

Our aim is a definition of perturbed fields, say $\widetilde{\mathbf{h}}$, $\widetilde{\boldsymbol{e}}$, to $\widetilde{\mathbf{H}}_b$, $\widetilde{\mathbf{E}}_b$ satisfying the boundary conditions (2.10) in the case $\mu = \widehat{\mu}$. We achieve this goal by introducing

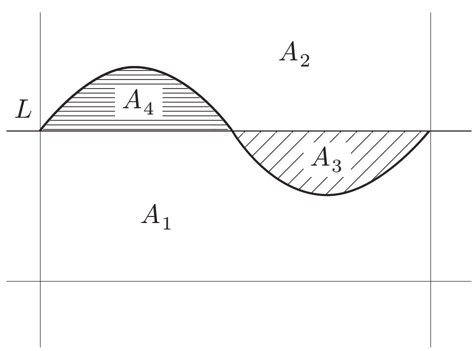


FIGURE 1

the following four sub-domains functions of time, see Fig. 1:

$$\begin{aligned} A_1(t) &= \{x\in R^3: && x\in\Omega_t\cap\Omega_b\}; \\ A_2(t) &= \{x\in R^3: && x\in\widehat{\Omega}_t\cap\widehat{\Omega}_b\}; \\ A_3(t) &= \{x\in\widehat{\Omega}_t: && 0 < z < L\}; \\ A_4(t) &= \{x\in\Omega_t: && z > L\}. \end{aligned}$$

We remark that Fig. 1 is a simplified version of reality. Actually we shall have a denumerable number of domains $A_{ji}$, $j = 1, \ldots, 4$ $i \in \mathbb{N}$, with analogous definitions as those already given. It holds

$$A_j = \cup_{i \in \mathbb{N}} A_{ji}, \qquad j = 1, \ldots, 4.$$

The set $\Sigma$ in the plane $z = 0$ is divided in two parts:

$$\begin{aligned} \Sigma_{-,t} &= \{x_* \in \Sigma : \quad \zeta(x_*, t) < L\}, \\ \Sigma_{+,t} &= \{x_* \in \Sigma : \quad \zeta(x_*, t) > L\}. \end{aligned}$$

The free boundary $\Gamma_t$ is given by the union of the following two subsets:

$$\Gamma_t = \{x_*, \zeta \in \Gamma_t : \quad x_* \in \Sigma_{-,t}\} \cup \{x_*, \zeta \in \Gamma_t : \quad x_* \in \Sigma_{+,t}\} = \Gamma_t^+ \cup \Gamma_t^- .$$

The boundaries of the subsets $A_i$, $i = 1, \ldots, 4$ are constituted by the union of the boundaries $\Gamma_b$ and $\Gamma_t$ as follows.

$$\begin{aligned} \partial A_1(t) &= \{(x_*, L) : x_* \in \Sigma_{+,t}\} \cup \{(x_*, \zeta) : x_* \in \Sigma_{-,t}\} =: \partial A_1(t)^L \cup \partial A_1(t)^\eta; \\ \partial A_2(t) &= \{(x_*, \zeta) : x_* \in \Sigma_{+,t}\} \cup \{(x_*, L) : x_* \in \Sigma_{-,t}\} =: \partial A_2(t)^\eta \cup \partial A_2(t)^L; \\ \partial A_3(t) &= \{(x_*, \zeta) : x_* \in \Sigma_{-,t}\} \cup \{(x_*, L) : x_* \in \Sigma_{-,t}\} =: \partial A_3(t)^\eta \cup \partial A_3(t)^L; \\ \partial A_4(t) &= \{(x_*, L) : x_* \in \Sigma_{+,t}\} \cup \{(x_*, \zeta) : x_* \in \Sigma_{+,t}\} =: \partial A_4(t)^L \cup \partial A_4(t)^\eta. \end{aligned}$$

Boundaries of the sets $A_i$ are oriented with normal $\mathbf{N}$ directed toward the exterior of $A_i$. Concerning the normals, denoting by $\mathbf{k}$ the normal to the plane oriented toward the vacuum region, and by $\mathbf{n}$ the normal to $\Gamma_t$ oriented toward the vacuum region, we have

$$\begin{aligned} &\mathbf{N} = \mathbf{k} \text{ normal to } \partial A_1(t)^L, && \mathbf{N} = \mathbf{n} \text{ normal to } \partial A_1(t)^\eta; \\ &\mathbf{N} = -\mathbf{n} \text{ normal to } \partial A_2(t)^\eta, && \mathbf{N} = -\mathbf{k} \text{ normal to } \partial A_2(t)^L; \\ &\mathbf{N} = -\mathbf{n} \text{ normal to } \partial A_3(t)^\eta, && \mathbf{N} = \mathbf{k} \text{ normal to } \partial A_3(t)^L; \\ &\mathbf{N} = -\mathbf{k} \text{ normal to } \partial A_4(t)^L, && \mathbf{N} = \mathbf{n} \text{ normal to } \partial A_4(t)^\eta. \end{aligned} \tag{4.9}$$

We are now in the position to define the perturbation field $\widetilde{\mathbf{h}}$ in each domain $\Omega_t$, $\widehat{\Omega}_t$. Here the basic electro-dynamical fields $\mathbf{H}_b$, $\widehat{\mathbf{H}}_b$, $\widehat{\mathbf{E}}_b$, are not extended in $\mathcal{P}$, and are just defined in $\Omega_b$, $\widehat{\Omega}_b$ respectively.

We define the perturbation to the magnetic field as

$$\widetilde{\mathbf{h}}(x, t) = \begin{cases} \mathbf{h}(x, t) & x \in \Omega_t, \\ \widehat{\mathbf{h}}(x, t) & \in \widehat{\Omega}_t. \end{cases}$$

Using this natural definition of $\widetilde{\mathbf{h}}$ we define $\widetilde{\mathbf{H}}$ as[4]

$$\widetilde{\mathbf{H}}(x,t) = \begin{cases} \mathbf{H}_b + \mathbf{h}(x,t) & x \in A_1(t), \\ \widehat{\mathbf{H}}_b + \widehat{\mathbf{h}}(x,t) & x \in A_2(t), \\ \mathbf{H}_b + \widehat{\mathbf{h}}(x,t) & x \in A_3(t), \\ \widehat{\mathbf{H}}_b + \mathbf{h}(x,t) & \in A_4(t). \end{cases} \tag{4.11}$$

Notice that for $\widetilde{\boldsymbol{e}}$ in $\widetilde{\Omega}_t$ it holds that

$$\widetilde{e}(x,t) = \begin{cases} \boldsymbol{e}(x,t) = (\widetilde{\mu}\widetilde{\mathbf{H}}_b + \mu\mathbf{h}) \times \mathbf{u} + \dfrac{1}{\sigma}\mathbf{h} & x \in A_1(t) \cup A_4(t) = \Omega_t, \\ \widehat{\boldsymbol{e}}(x,t) & x \in A_2(t) \cup A_3(t) = \widehat{\Omega}_t. \end{cases} \tag{4.12}$$

On both sides of $\Gamma_t$ the basic magnetic $\widetilde{\mathbf{H}}_b$, induction $\widetilde{\mu}\widetilde{\mathbf{H}}_b$, and electric $\widetilde{\mathbf{E}}_b$ fields are continuous. This fact allows us to find very simple boundary conditions on the perturbed fields $\widetilde{\mathbf{h}}$, $\widetilde{\mu}\widetilde{\mathbf{h}}$, $\widetilde{\boldsymbol{e}}$.

**Definition II: perturbation $\widetilde{\mathbf{h}}$ to the model rest state**
For the model rest state, developing these relations we deduce the boundary conditions on the perturbation $\widetilde{\mathbf{h}}$:

$$\mu h_n = \widehat{\mu}\widehat{h}_n - \left[\widetilde{\mu}\right]\widetilde{\mathbf{H}}_b \cdot \mathbf{n} \qquad \mathbf{h}_\tau = \widehat{\mathbf{h}}_\tau, \qquad \text{on } \Gamma_t, \tag{4.13}$$
$$\boldsymbol{e}_\tau = \widehat{\boldsymbol{e}}_\tau, \qquad \text{on } \Gamma_t,$$

and for $\mu = \widehat{\mu}$ we recover the expected continuity conditions on $\widetilde{\mathbf{h}}$, $\widetilde{\boldsymbol{e}}$ at the boundary[5].

Let us make a comparison between previous definitions of perturbations and definition (4.11). The field $\mathbf{H}$ in earlier papers was defined as $\mathbf{H}_b^* + \varphi$ in $\Omega_t$, while in our paper it has been defined as $\widetilde{\mathbf{H}}_b + \mathbf{h}$[6]. As example let us compare the two fields in the part $A_4(t)$ where $\mathbf{H}_b$ differs from $\widetilde{\mathbf{H}}_b$; it holds that

$$\mathbf{H}(x_*, z, t) = \mathbf{H}_b + \varphi(x_*, z, t) = \widehat{\mathbf{H}}_b(z) + \mathbf{h}(x_*, z, t). \tag{4.14}$$

[4]Different definitions of perturbation can be proposed. For example we may work using as unknown field the induction magnetic field $\mathbf{B} = \mu\mathbf{H}$. In this case we find

$$\widetilde{\mathbf{B}}(x,t) = \begin{cases} \mathbf{B}_b + \mathbf{b}(x,t) & x \in A_1(t), \\ \widehat{\mathbf{B}}_b + \widehat{\mathbf{b}}(x,t) & x \in A_2(t), \\ \mathbf{B}_b + \widehat{\mathbf{b}}(x,t) & x \in A_3(t), \\ \widehat{\mathbf{B}}_b + \mathbf{b}(x,t) & \in A_4(t). \end{cases} \tag{4.10}$$

The definition of perturbation strongly depends on the geometrical properties of the surface.
[5]For linear pinch, the equilibrium configuration is a cylinder of radius $R$. $\mathbf{H}_b$ is tangent to circumferences of radius $R$ in the cross sections of the cylinder. In the class of perturbations, axially symmetric $\widetilde{\mathbf{H}} \cdot \mathbf{n} = 0$, and we have continuity for the perturbation to $\widetilde{\mathbf{H}}_b$. Our guess is that the correct perturbation should depend on the basic motion.
[6]If one uses the perturbation of $\mathbf{B}$, then one can make an analogous comparison.

In general we deduce

$$\begin{aligned}
&\mathbf{h}(x_*, z, t) = \varphi(x_*, z, t), && \text{in } A_1(t),\\
&\widehat{\mathbf{h}}(x_*, z, t) = \widehat{\varphi}(x_*, z, t), && \text{in } A_2(t)\\
&\widehat{\mathbf{h}}(x_*, z, t) = \widehat{\varphi}(x_*, z, t) - H_b\Big(1 - \frac{\mu L}{\widehat{\mu} z}\Big)\mathbf{k}, && \text{in } A_3(t),\\
&\mathbf{h}(x_*, z, t) = \varphi(x_*, z, t) + H_b\Big(1 - \frac{\mu L}{\widehat{\mu} z}\Big)\mathbf{k}, && \text{in } A_4(t).
\end{aligned} \tag{4.15}$$

Hence our definition of perturbed magnetic field differs from that given in the previous papers even when $\mu = \widehat{\mu}$.

This difference may be expressed through other linear terms only if $\mu = \widehat{\mu}$. This claim becomes understandable if one observes that domains $A_{3,4}(t)$, where $\mathbf{h}$, and $\widehat{\mathbf{h}}$ are defined, are functions of $\eta$. In particular in $A_4(t)$ it holds that

$$\Big(1 - \frac{\mu}{\widehat{\mu}}\Big) < \Big(1 - \frac{\mu L}{\widehat{\mu} z}\Big) < \Big(1 - \frac{\mu L}{\widehat{\mu}(L + \eta)}\Big),$$

and analogously in $A_3(t)$.

We remark that *the perturbations $\varphi$, $\mathbf{h}$ coincide in domains with fixed boundary. A motion which is linearly stable in the sense of the classical definition* (*control of $\varphi$*) *is also linearly stable in the sense of our definition* (*control of* $\mathbf{h}$), *and vice versa.* However the two vector fields $\varphi$, and $\mathbf{h}$ are deeply different, indeed also the equation of motion are different.

**Definition II: perturbation $\widetilde{\mathbf{h}}$ to the true rest state**

Equations (4.13) are independent of the rest state, thus continue to hold when the basic magnetic field is constant. For the true rest state eqution (4.15) are simplified in the following ones:

$$\begin{aligned}
&\mathbf{h}(x_*, z, t) = \varphi(x_*, z, t), && \text{in } A_1(t),\\
&\widehat{\mathbf{h}}(x_*, z, t) = \widehat{\varphi}(x_*, z, t), && \text{in } A_2(t),\\
&\widehat{\mathbf{h}}(x_*, z, t) = \widehat{\varphi}(x_*, z, t) - H_b\Big(1 - \frac{\mu}{\widehat{\mu}}\Big)\mathbf{k}, && \text{in } A_3(t),\\
&\mathbf{h}(x_*, z, t) = \varphi(x_*, z, t) + H_b\Big(1 - \frac{\mu}{\widehat{\mu}}\Big)\mathbf{k}, && \text{in } A_4(t).
\end{aligned} \tag{4.16}$$

Hence our definition of perturbed magnetic field coincides with that given in the previous papers when $\mu = \widehat{\mu}$.

**Perturbation equations for the magnetic field**

We end this subsection by writing $(2.3)_2$ in the sets $A_1(t) \cup A_4(t)$, $A_2(t) \cup A_3(t)$ in terms of the perturbed magnetic field $\widetilde{\mathbf{h}}$ and obtain the **perturbation equations**

**for the magnetic field $\widetilde{\mathbf{h}}$:**

$$
\begin{aligned}
\mu\partial_t \mathbf{h} &= -\nabla \times \Big(\mu(\mathbf{H}_b + \mathbf{h}) \times \mathbf{u} + \frac{1}{\sigma}\nabla \times \mathbf{h}\Big), && x \in A_1(t),\\
\mu\partial_t \mathbf{h} &= -\nabla \times \Big(\mu(\widehat{\mathbf{H}}_b + \mathbf{h}) \times \mathbf{u} + \frac{1}{\sigma}\nabla \times \mathbf{h}\Big), && x \in A_4(t),\\
\widehat{\mu}\partial_t \widehat{\mathbf{h}} &= -\nabla \times \widehat{\boldsymbol{e}}, && x \in A_2(t) \cup A_3(t),\\
\mu^- \partial_t \mathbf{h}^- &= -\frac{1}{\sigma^-}\nabla \times \nabla \times \mathbf{h}^-, && x \in \Omega^-.
\end{aligned}
\tag{4.17}
$$

Comparing (4.17) with (4.8), one may find equations (4.8) simpler.

## 5. Nonlinear stability

In this section we use functions belonging to the following regularity classes:

$$
\mathcal{W}_i = \{(p, \mathbf{u}, \widetilde{\mathbf{H}}, \widetilde{\mathbf{E}}, \zeta) : \ \zeta \in L^\infty(0,\infty; W^{1,\infty}(\Sigma)),
$$
$$
(p, \mathbf{u}, \widetilde{\mathbf{H}}, \widetilde{\mathbf{E}}) \in L^\infty(0,\infty; L^\infty(\Omega_t)) \times [L^\infty(0,\infty; L^4(\Omega_t)) \cap L^2(0,\infty; W^{1,2}(\Omega_t))]^9\}.
$$
$$
\mathcal{W}_c = \{(\rho, \mathbf{u}, \widetilde{\mathbf{H}}, \widetilde{\mathbf{E}}, \zeta) : \ \zeta \in L^\infty(0,\infty; W^{1,\infty}(\Sigma)), \quad \rho \in L^\infty(0,\infty; L^\infty(\Omega_t)),
$$
$$
(\rho, \mathbf{u}, \widetilde{\mathbf{H}}, \widetilde{\mathbf{E}}) \in L^\infty(0,\infty; L^\infty(\Omega_t)) \times [L^\infty(0,\infty; L^4(\Omega_t)) \cap L^2(0,\infty; W^{1,2}(\Omega_t))]^9\}.
$$

In this section we develop a Lyapunov method using first the perturbation I, and secondly the new perturbation II. Our aim is to prove the following two main theorems

**Nonlinear Stability Theorem I.** *Let the external forces be conservative, $\mu = \widehat{\mu}$, and $\mathbf{j}_S = 0$. Then the rest state given by* (3.19) *is* **nonlinearly stable** *in the class of motions $(p, \mathbf{u}, \widetilde{\mathbf{H}}, \widetilde{\mathbf{E}}, \zeta) \in \mathcal{W}_i$, for every initial perturbation in $\Omega_0$ having the same volume as $\Omega_b$.*

**Nonlinear Stability Theorem II.** *Let the external forces be conservative, $\mu = \widehat{\mu}$, and $\mathbf{j}_S = 0$. Then the rest state given by* (3.28) *is* **nonlinearly stable** *in the class of motions $(\rho, \mathbf{u}, \widetilde{\mathbf{H}}, \widetilde{\mathbf{E}}, \zeta) \in \mathcal{W}_i$, for every initial perturbation having the same total mass $M$ as the basic rest state $M = \int_{\Omega_b} \rho_b dx$.*

*Proof.* In the proof we use a variant of the Lyapunov second method. We achieve this goal by computing the difference between the total energies of the perturbed motion and the basic rest state. The energy identities (3.18), (3.25) hold for the equilibrium configuration $S_b$ and for the unsteady motion $S(t)$ arising by perturbing initially $S_b$ and using (3.17), and (3.26) as energies of perturbations for incompressible and compressible fluids, respectively.

**Remark 5.1** In Theorems I and II above it is tacitly assumed that there exist global regular solutions to evolution equations both for incompressible and compressible fluids. However the problem of existence of global motions in the classes $\mathcal{W}_i$, $\mathcal{W}_c$, is still an open problem.

Before giving a proof of stability Theorems I and II, we describe the difficulties that arise by using the perturbation $\widetilde{\varphi}$. We limit ourselves to incompressible fluids, and we make complete calculations. To study the stability of compressible fluids it is enough to follow the lines of Subsection 5.2 below, together with those of Chapter 3 in [29]. We remark that the same obstacles are present in the study of nonlinear stability of compressible fluids.

### 5.1. Energy of perturbation $\widetilde{\varphi}$

We write (3.18) in terms of perturbations $\{\mathbf{u},\, \mathbf{H}-\mathbf{H}_b=\varphi,\, \widehat{\mathbf{H}}-\widehat{\mathbf{H}}_b=\widehat{\varphi},\, \zeta-h=\eta\}$, defined in Section 4. Since $\mathcal{E}(\mathbf{u})$ is already the $L^2$ norm of perturbation to the zero velocity, $\mathcal{E}(\zeta)$ is up to a constant equivalent to $k$ times the $L^2$ norm of the gradient $\nabla_*\eta$ plus $f$ times the $L^2$ norm of $\eta$, cf. [30][7]. It remains to show that $\mathcal{E}(\widetilde{\mathbf{H}})-\mathcal{E}(\widetilde{\mathbf{H}_b})$ is equivalent to the $L^2$ norm of perturbation to magnetic field $\widetilde{\mathbf{H}}_b$ in the sense just defined. To this end, we compute the difference between the two energies, paying attention to write the basic magnetic field in the new domain $\Omega_t$ as follows:

$$\begin{aligned}
2\big[\mathcal{E}(\widetilde{\mathbf{H}})-\mathcal{E}(\widetilde{\mathbf{H}}_b)\big] &= \int_{\Omega_t}\mu\mathbf{H}^2dx-\int_{\Omega_t}\mu\mathbf{H}_b^2dx+\int_{\widehat{\Omega}_t}\widehat{\mu}\widehat{\mathbf{H}}^2dx-\int_{\widehat{\Omega}_t}\widehat{\mu}\widehat{\mathbf{H}}_b^2dx\\
&+\int_{\Omega^-}\mu^-(\mathbf{H}^-)^2dx-\int_{\Omega^-}\mu^-(\mathbf{H}_b^-)^2dx = \mu\int_{\Omega_t}\varphi^2dx+2\mu\int_{\Omega_t}\varphi\cdot\mathbf{H}_b\,dx\\
&+\widehat{\mu}\int_{\widehat{\Omega}_t}\widehat{\varphi}^2dx+2\widehat{\mu}\int_{\widehat{\Omega}_t}\widehat{\varphi}\cdot\widehat{\mathbf{H}}_bdx+\int_{\Omega^-}\mu^-(\varphi^-)^2dx+2\mu^-\int_{\Omega^-}\varphi^-\cdot\mathbf{H}_b^-\,dx\\
&=: 2\mathcal{E}(\widetilde{\varphi})+\mathbb{J}
\end{aligned}\tag{5.1}$$

with

$$\mathbb{J} := 2\mu\int_{\Omega_t}\varphi\cdot\mathbf{H}_b\,dx+2\widehat{\mu}\int_{\widehat{\Omega}_t}\widehat{\varphi}\cdot\widehat{\mathbf{H}}_bdx+2\mu^-\int_{\Omega^-}\varphi^-\cdot\mathbf{H}_b^-\,dx$$

a nonlinear functional in $\widetilde{\varphi}$ and $\eta$. We notice at once that $2\mathcal{E}(\widetilde{\varphi})$ is the $L^2$ norm of the perturbed magnetic field. It remains to control the time derivative of term $\mathbb{J}$, and to this end we use the Reynolds transport theorem, to find

$$\begin{aligned}
\frac{d}{dt}\mathbb{J} &= \frac{d}{dt}\Big\{2\mu\int_{\Omega_t}\mathbf{H}_b\cdot\varphi\,dx+2\widehat{\mu}\int_{\widehat{\Omega}_t}\widehat{\mathbf{H}}_b\cdot\widehat{\varphi}dx+2\mu^-\int_{\Omega^-}\mathbf{H}_b^-\cdot\varphi^-\,dx\Big\} \qquad (5.2)\\
&= 2\mu\int_{\Omega_t}\partial_t(\mathbf{H}_b\cdot\varphi)dx+2\widehat{\mu}\int_{\widehat{\Omega}_t}\partial_t(\widehat{\mathbf{H}}_b\cdot\widehat{\varphi})dx\,+2\mu^-\int_{\Omega^-}\partial_t(\mathbf{H}_b^-\cdot\varphi^-)dx+\mathbb{B},
\end{aligned}$$

$$\mathbb{B} = 2\mu\int_{\Gamma_t}\mathbf{H}_b\cdot\varphi\,u_n\,dS+2\widehat{\mu}\int_{\Gamma_t}\widehat{\mathbf{H}_b}\cdot\widehat{\varphi}\,u_{\widehat{n}}dS. \tag{5.3}$$

---

[7]We recall that for the potential energy $\Pi(\zeta)$ the potential term, in case of gravitational force, is represented by $U=-f\,z=-\pi$, with $f$ the gravity acceleration, and using the incompressibility assumption, implies

$$\Pi(\zeta) := \int_\Sigma\int_0^\zeta f\,z\,dz\,dx_* = \int_\Sigma\frac{(L+\eta)^2}{2}dx_* = \int_\Sigma\frac{\eta^2}{2}dx_*+c\,\equiv\,\Pi(\eta),$$

with $c$ a constant.

We analyze the boundary terms specifying the sign of the normals $\mathbf{n}$, $\widehat{\mathbf{n}}$. In (5.3) the normals $\mathbf{n}$, $\widehat{\mathbf{n}}$ differ in sign, since they are directed exterior to their respective domains $\Omega_t$, $\widehat{\Omega}_t$. The time derivative of $\mathbb{J}$ will be computed once we compute the volume integrals in (5.2). To this end we multiply $(4.8)_1$ times $\mathbf{H}_b$, and $(4.8)_2$ times $\widehat{\mathbf{H}}_b$, $(4.8)_3$ times $\mathbf{H}_b^-$ integrate over the respective sets $\Omega_t$, $\widehat{\Omega}_t$, $\Omega^-$, recalling the definition of $\psi$ and that $\mathbf{H}_b$ and $\widehat{\mathbf{H}}_b$, $\mathbf{H}_b^-$ are constant in time, we get

$$
\begin{aligned}
\mu \int_{\Omega_t} \partial_t(\mathbf{H}_b \cdot \varphi) dx &= -\int_{\Omega_t} \mathbf{H}_b \cdot \nabla \times \psi \, dx; \\
\widehat{\mu} \int_{\widehat{\Omega}_t} \partial_t(\widehat{\mathbf{H}}_b \cdot \widehat{\varphi}) \, dx &= -\int_{\widehat{\Omega}_t} \widehat{\mathbf{H}}_b \cdot \nabla \times \widehat{\psi} dx; \qquad (5.4)\\
\mu^- \int_{\Omega^-} \partial_t(\mathbf{H}_b^- \cdot \varphi^-) dx &= -\int_{\Omega^-} \mathbf{H}_b^- \cdot \nabla \times \psi^- \, dx.
\end{aligned}
$$

Since $\nabla \times \widehat{\mathbf{H}}_b = \nabla \times \mathbf{H}_b = \nabla \times \mathbf{H}^-{}_b = 0$, using the boundary conditions on the electric field, adding (5.4), by (5.2) it is easy to check that

$$
\begin{aligned}
\frac{1}{2}\Big(\frac{d}{dt}\mathbb{J} - \mathbb{B}\Big) &= -\int_{\Omega_t} \mathbf{H}_b \cdot \nabla \times \psi \, dx - \int_{\widehat{\Omega}_t} \widehat{\mathbf{H}}_b \cdot \nabla \times \widehat{\psi} dx - \int_{\Omega^-} \mathbf{H}_b^- \cdot \nabla \times \psi^- \, dx \\
&= \int_{\Gamma_t} (\widehat{\mathbf{H}}_b - \mathbf{H}_b) \cdot \mathbf{n} \times \widehat{\psi} dS + \int_{\Sigma} (\mathbf{H}_b - \mathbf{H}_b^-) \cdot \mathbf{k} \times \psi^- \, dS =: \frac{1}{2}\mathbb{C}. \qquad (5.5)
\end{aligned}
$$

In particular, the continuity of the tangential component of $H_b$ yields

$$
\begin{aligned}
\mathbb{C} &= -2\int_{\Gamma_t} H_b\Big(\frac{\mu}{\widehat{\mu}}\frac{L}{\zeta} - 1\Big)\mathbf{k} \cdot \mathbf{n} \times \widehat{\psi} dS \quad \text{model rest state,} \\
\mathbb{C} &= -2\int_{\Gamma_t} H_b\Big(\frac{\mu}{\widehat{\mu}} - 1\Big)\mathbf{k} \cdot \mathbf{n} \times \widehat{\psi} dS \quad \text{true rest state.}
\end{aligned}
$$

Gathering previous identities, we get

$$
\frac{d}{dt}\mathbb{J} = \frac{d}{dt}\Big\{ \int_{\Omega_t} 2\mu \mathbf{H}_b \cdot \varphi \, dx + \int_{\widehat{\Omega}_t} 2\widehat{\mu}\widehat{\mathbf{H}}_b \cdot \widehat{\varphi} dx \Big\} + 2\mu^- \int_{\Omega^-} \mathbf{H}_b^- \cdot \varphi^- \, dx = \mathbb{B} + \mathbb{C}. \quad (5.6)
$$

From (5.6), and the expression of $\mathbb{B}$, $\mathbb{C}$ it follows that *the time derivative of* $\mathbb{J}$ in general is not zero. Specifically, if $\mu = \widehat{\mu}$, in the model rest state, it *furnishes a nonlinear term in perturbations*, while in the true rest state $\mathbb{B} + \mathbb{C} = 0$, see (5.8), (5.9) below.

From (3.26) we have

$$
\mathcal{E}(\eta) = \int_{\Sigma} \frac{1}{\sqrt{(1 + |\nabla_* \overline{\eta}|^2)^3}} \left\{ |\nabla_* \eta|^2 + |\nabla_* \overline{\eta}|^2 |\nabla_* \eta|^2 - (\nabla_* \overline{\eta} \cdot \nabla_* \eta)^2 \right\} dx_*,
$$

where $\overline{\eta}$ denotes a point between $\eta$ and 0, for regular $\eta$ we deduce that $\mathcal{E}(\eta)$ is equivalent to the $L^2$ norm of $\nabla_* \eta$, cf. [30]. Moreover, since $\nabla \times \widetilde{\mathbf{H}} = 0$ it holds that

$$
\mathcal{D}(\mathbf{H}) = \mathcal{D}(\varphi).
$$

Finally, in absence of surface currents, and for $\mu = \widehat{\mu}$, which is a typical assumption in plasmas, the energy equation reduces to

$$\frac{d}{dt}\left\{\mathcal{E}(\mathbf{u}) + \mathcal{E}(\widetilde{\varphi}) + \mathcal{E}(\eta)\right\} + \mathcal{D}(\mathbf{u}) + \mathcal{D}(\varphi) = \mathbb{B} + \mathbb{C}. \tag{5.7}$$

For the *model problem* using (4.5) after some calculations we deduce

$$\begin{aligned}\mathbb{B} + \mathbb{C} = 2H_b \int_{\Gamma_t} \frac{H_b}{g}\Big[\frac{\eta}{\zeta} + + (\mu - \widehat{\mu})\frac{\widehat{\varphi}_n}{\sqrt{g}} + \Big(\frac{\mu}{\widehat{\mu}}\frac{L}{\zeta} - 1\Big)|\nabla_*\eta|^2\Big]u_n\, dS \\ + 2\, H_b \int_{\Gamma_t}\Big[\frac{L}{\widehat{\mu}\zeta}(\mu - \widehat{\mu}) - \frac{\eta}{\zeta}\Big]\mathbf{k}\cdot\mathbf{n}\times\widehat{\psi} dS.\end{aligned} \tag{5.8}$$

For the *true problem*, (5.8) reduces to

$$\mathbb{B} + \mathbb{C} = 2H_b\,\frac{(\mu - \widehat{\mu})}{\widehat{\mu}}\int_{\Gamma_t}\Big[-\widehat{\mu}\frac{\widehat{\varphi}_n}{\sqrt{g}}\,u_n + \frac{|\nabla_*\eta|^2}{g}u_n + \mathbf{k}\cdot\mathbf{n}\times\widehat{\psi}\Big]\,dS, \tag{5.9}$$

which vanishes for $\mu = \widehat{\mu}$.

In general the term (5.8) has no definite sign even when $\mu = \widehat{\mu}$, and the control of $\mathbb{B} + \mathbb{C}$ constitutes an open problem.

### 5.2. Energy of new perturbation $\widetilde{\mathbf{h}}$

The energy identity (3.18) holds for the equilibrium configuration $S_b$ and for any unsteady motion $S(t) = \{\mathbf{u}, \widetilde{\mathbf{H}} = \widetilde{\mathbf{H}}_b + \widetilde{\mathbf{h}}, \widetilde{\mathbf{E}} = \widetilde{\mathbf{E}}_b + \widetilde{\boldsymbol{e}}, \zeta = L + \eta\}$ that develops, perturbing initially $S_b$ with conditions $S(0) = \{\mathbf{u}_0, \widetilde{\mathbf{H}}_b + \widetilde{\mathbf{h}}_0, \widetilde{\mathbf{E}}_b + \widetilde{e}_0, L + \eta_0\}$.

We write (3.18) in terms of perturbations $\{\mathbf{u}, \widetilde{\mathbf{H}} - \widetilde{\mathbf{H}}_b = \widetilde{\mathbf{h}}, \zeta - h = \eta\}$, defined in Subsection 4.2. Since $\mathcal{E}(\mathbf{u})$ is already the $L^2$ norm of perturbation to velocity, $\mathcal{E}(\zeta)$ is up to a constant equivalent to $k$ times the norm of the gradient $\nabla_*\eta$ in a Orlicz space, plus $f$ times the $L^2$ norm of $\eta$, cf.[30], it remains to show that $\mathcal{E}(\widetilde{\mathbf{H}})$ up to a constant is equivalent to the $L^2$ norm of perturbation to magnetic field $\widetilde{\mathbf{H}}_b$ in the sense just defined. To this end, we compute the difference

$$\begin{aligned}\mathcal{E}(\widetilde{\mathbf{H}}) - \mathcal{E}(\widetilde{\mathbf{H}}_b) &= \int_{\widetilde{\Omega}_t}\widetilde{\mu}\widetilde{\mathbf{H}}^2 dx - \int_{\widetilde{\Omega}_b}\widetilde{\mu}\widetilde{\mathbf{H}}_b^2 dx \\ &= \mu\int_{A_1(t)}\Big((\mathbf{H}_b + \mathbf{h})^2 - \mathbf{H}_b^2\Big)dx + \widehat{\mu}\int_{A_2(t)}\Big((\widehat{\mathbf{H}}_b + \widehat{\mathbf{h}})^2 - \widehat{\mathbf{H}}_b^2\Big)dx \\ &\quad + \int_{A_3(t)}\Big(\widehat{\mu}(\mathbf{H}_b + \widehat{\mathbf{h}})^2 - \mu\mathbf{H}_b^2\Big)dx + \int_{A_4(t)}\Big(\mu(\widehat{\mathbf{H}}_b + \mathbf{h})^2 - \widehat{\mu}\widehat{\mathbf{H}}_b^2\Big)dx \\ &\quad + \mu^-\int_{\Omega^-}\Big((\mathbf{H}_b^- + \mathbf{h}^-)^2 - (\mathbf{H}_b^-)^2\Big)dx \\ &= \widetilde{\mu}\int_{\widetilde{\Omega}_t}\widetilde{\mathbf{h}}^2 dx + 2\mu\int_{A_1(t)}\mathbf{h}\cdot\mathbf{H}_b\, dx + 2\widehat{\mu}\int_{A_2(t)}\widehat{\mathbf{h}}\cdot\widehat{\mathbf{H}}_b dx \\ &\quad + 2\widehat{\mu}\int_{A_3(t)}\widehat{\mathbf{h}}\cdot\mathbf{H}_b dx + 2\mu\int_{A_4(t)}\mathbf{h}\cdot\widehat{\mathbf{H}}_b dx\end{aligned}$$

$$
\begin{aligned}
&+ (\widehat{\mu} - \mu)\Big(\int_{A_3} H_b^2\, dx - \int_{A_4} \widehat{H}_b^2 dx\Big) + 2\mu^- \int_{\Omega^-} \mathbf{h}^- \cdot \mathbf{H}_b^-\, dx \\
&= \mathcal{E}(\widetilde{\mathbf{h}}) + \mathbb{J} + (\widehat{\mu} - \mu)\mathbb{G}, \qquad (5.10)
\end{aligned}
$$

where

$$
\mathbb{G} = \Big(\int_{A_3} H_b^2 dx - \int_{A_4} \widehat{H}_b^2\, dx\Big), \tag{5.11}
$$

$$
\begin{aligned}
\mathbb{J} &= 2\mu \int_{A_1(t)} \mathbf{h} \cdot \mathbf{H}_b\, dx + 2\widehat{\mu} \int_{A_2(t)} \widehat{\mathbf{h}} \cdot \widehat{\mathbf{H}}_b dx \\
&+ 2\widehat{\mu} \int_{A_3(t)} \widehat{\mathbf{h}} \cdot \mathbf{H}_b dx + 2\mu \int_{A_4(t)} \mathbf{h} \cdot \widehat{\mathbf{H}}_b dx + 2\mu \int_{\Omega^-} \mathbf{h}^- \cdot \mathbf{H}_b^-\, dx.
\end{aligned} \tag{5.12}
$$

The contribution of $\mathbb{G}$ appears to be new in literature.

For the model problem (5.11) furnishes

$$
\mathbb{G} = H_b^2\Big(\int_{A_3} dx - \int_{A_4} \frac{\mu^2}{\widehat{\mu}^2}\frac{L^2}{\zeta^2} dx\Big),
$$

while for the true problem (5.11) yields

$$
\mathbb{G} = H_b^2\Big(\int_{A_3} dx - \int_{A_4} \frac{\mu^2}{\widehat{\mu}^2}\, dx\Big) = -H_b^2 \frac{\mu^2 - \widehat{\mu}^2}{\widehat{\mu}^2} \int_{\Sigma_{+t}} \eta dx_*
$$

Multiply $(4.17)_1$ times $\mathbf{H}_b$ in $A_1(t)$, and $(4.17)_2$ times $\widehat{\mathbf{H}}_b$ in $A_4(t)$, $(4.17)_3$ times $\mathbf{H}_b$ in $A_3(t)$, and times $\widehat{\mathbf{H}}_b$ in $A_2(t)$. Integrating over the respective sets, recalling that the basic magnetic fields $\mathbf{H}_b$ and $\widehat{\mathbf{H}}_b$ are constant in time, we get

$$
\begin{aligned}
\mu \int_{A_1(t)} \partial_t(\mathbf{h} \cdot \mathbf{H}_b)\, dx &= -\int_{A_1(t)} \mathbf{H}_b \cdot \nabla \times \boldsymbol{e}\, dx =: \mathcal{F}_1; \\
\widehat{\mu} \int_{A_2(t)} \partial_t(\widehat{\mathbf{h}} \cdot \widehat{\mathbf{H}}_b)\, dx &= -\int_{A_2(t)} \widehat{\mathbf{H}}_b \cdot \nabla \times \widehat{\boldsymbol{e}}\, dx =: \mathcal{F}_2; \\
\widehat{\mu} \int_{A_3(t)} \partial_t(\widehat{\mathbf{h}} \cdot \mathbf{H}_b)\, dx &= -\int_{A_3(t)} \mathbf{H}_b \cdot \nabla \times \widehat{\boldsymbol{e}}\, dx =: \mathcal{F}_3; \\
\mu \int_{A_4(t)} \partial_t(\mathbf{h} \cdot \widehat{\mathbf{H}}_b)\, dx &= -\int_{A_4(t)} \widehat{\mathbf{H}}_b \cdot \nabla \times \boldsymbol{e}\, dx =: \mathcal{F}_4; \\
\mu^- \int_{\Omega^-} \partial_t(\mathbf{h}^- \cdot \mathbf{H}_b^-)\, dx &= -\int_{\Omega^-} \mathbf{H}_b^- \cdot \nabla \times \boldsymbol{e}^-\, dx =: \mathcal{F}_5.
\end{aligned} \tag{5.13}
$$

Here we have used the hypothesis that the basic state is steady. Using the Reynolds theorem, we deduce the identities

$$
\begin{aligned}
\int_{A_i(t)} \partial_t(\widetilde{\mathbf{h}} \cdot \widetilde{\mathbf{H}}_b) dx &= \frac{d}{dt}\int_{A_i(t)} \widetilde{\mathbf{H}}_b \cdot \widetilde{\mathbf{h}}\, dx - \int_{\partial A_1(t)} \widetilde{\mathbf{H}}_b \cdot \mathbf{h}\, V_n^{(i)} dS \\
&=: \frac{dJ_i}{dt} - \int_{\partial A_i(t)} \widetilde{\mathbf{H}}_b \cdot \mathbf{h}\, V_n^{(i)} dS.
\end{aligned} \tag{5.14}
$$

We study the boundary terms at right-hand sides of (5.14), because we have to specify the values of the velocities $V_n^{(i)}$, $i = 1, \dots, 4$. Recalling the definitions of $\partial A_i$, since $V_n^{(i)} = 0$ on $\partial A_i(t)^L$ we get

$$\begin{aligned}
\int_{\partial A_1(t)} \mathbf{H}_b \cdot \mathbf{h} \mathbf{V}_n^{(1)} dS &= \int_{\Sigma_{-,t}} \mathbf{H}_b \cdot \mathbf{h}(\zeta, t)\, \partial_t \eta dx_*; \\
\int_{\partial A_4(t)} \widehat{\mathbf{H}}_b \cdot \mathbf{h} \mathbf{V}_n^{(4)} dS &= \int_{\Sigma_{+,t}} \widehat{\mathbf{H}}_b(\zeta) \cdot \mathbf{h}(\zeta, t)\, \partial_t \eta dx_*; \\
\int_{\partial A_2(t)} \widehat{\mathbf{H}}_b \cdot \widehat{\mathbf{h}} \mathbf{V}_n^{(2)} dS &= -\int_{\Sigma_{+,t}} \widehat{\mathbf{H}}_b(\zeta) \cdot \widehat{\mathbf{h}}(\zeta, t)\, \partial_t \eta dx_*; \\
\int_{\partial A_3(t)} \mathbf{H}_b \cdot \widehat{\mathbf{h}} \mathbf{V}_n^{(3)} dS &= -\int_{\Sigma_{-,t}} \mathbf{H}_b \cdot \widehat{\mathbf{h}}(\zeta, t)\, \partial_t \eta dx_*.
\end{aligned} \tag{5.15}$$

Here the normals to $\partial A_1^\eta$, and $\partial A_4^\eta$ have been taken toward $\widehat{\Omega}_t$, while the normals to $\partial A_2^\eta$, and $\partial A_3^\eta$ are taken toward $\Omega_t$. Next we multiply terms at r.h.s. of $(5.15)_{1,2}$ times $\mu$ those at r.h.s. of $(5.15)_{2,3}$ times $\widehat{\mu}$, and sum over $i$, $i = 1, \dots, 4$. We get

$$\begin{aligned}
&\mu \int_{\Sigma_{-,t}} \mathbf{H}_b \cdot \mathbf{h}(\zeta, t) \partial_t \eta \, dx_* + \mu \int_{\Sigma_{+,t}} \widehat{\mathbf{H}}_b(\zeta) \cdot \mathbf{h}(\zeta, t) \partial_t \eta \, dx_* \\
&\quad - \widehat{\mu} \int_{\Sigma_{+,t}} \widehat{\mathbf{H}}_b(\zeta) \cdot \widehat{\mathbf{h}}(\zeta, t) \partial_t \eta \, dx_* - \widehat{\mu} \int_{\Sigma_{-,t}} \mathbf{H}_b \cdot \widehat{\mathbf{h}}(\zeta, t) \partial_t \eta \, dx_* \\
&= \int_{\Sigma_{-,t}} \Big[ H_{bn}(\mu\, h_n - \widehat{\mu} \widehat{h}_n) + H_{b\tau}(\mu h_\tau - \widehat{\mu} \widehat{h}_\tau) \Big] \partial_t \eta \, dx_* \\
&\quad + \int_{\Sigma_{+,t}} \Big[ \widehat{H}_{bn}(\mu\, h_n - \widehat{\mu} \widehat{h}_n) + \widehat{H}_{b\tau} \widehat{h}_\tau (\mu - \widehat{\mu}) \Big] \partial_t \eta \, dx_* =: \frac{\mathcal{I}}{2}.
\end{aligned} \tag{5.16}$$

Recalling $(4.13)_{1,2}$ one has

$$\mathcal{I} = \int_{\Gamma_t} \Big[ (\widehat{\mu} - \mu) \widetilde{H}_{bn}^2 + \widetilde{H}_{b\tau} \widehat{h}_\tau (\mu - \widehat{\mu}) \Big] \mathbf{u} \cdot \mathbf{n} \, dS =: -(\mu - \widehat{\mu}) \mathtt{A},$$

with

$$\mathtt{A} := \int_{\Gamma_t} \Big[ \widetilde{H}_{bn}^2 - \widetilde{H}_{b\tau} \widehat{h}_\tau \Big] \mathbf{u} \cdot \mathbf{n} \, dS.$$

Notice that $\mathtt{A}$ depends on $\partial_t \eta$, therefore it vanishes for fixed boundaries.

Finally, we multiply $(5.14)_{1,4}$ times $2\mu$, $(5.14)_{2,3}$ times $2\widehat{\mu}$, and add the resulting equations over $i = 1, \dots, 4$, thus we use (5.4), (5.12), (5.13), (5.16) to get

$$\frac{d\,\mathbb{J}}{dt} = \mathtt{F} + \Big( \widehat{\mu} - \mu \Big) \mathtt{A} \,, \tag{5.17}$$

where

$$\mathtt{F} := 2\mathcal{F}_1 + 2\mathcal{F}_2 + 2\mathcal{F}_3 + 2\mathcal{F}_4.$$

In order to compute the sum at r.h.s. of (5.17) we notice that they reduce to boundary terms, because $\mathbf{H}_b = const.$ and $\nabla \times \widehat{\mathbf{H}}_b = 0$, and it yields

$$\begin{aligned} \mathtt{F} = & -2\int_{\partial A_1(t)^h} \mathbf{H}_b \cdot (\mathbf{k} \times \boldsymbol{e}) dS - 2\int_{\partial A_1(t)^\eta} \mathbf{H}_b \cdot (\mathbf{n} \times \boldsymbol{e}) dS \\ & + 2\int_{\partial A_4(t)^h} \widehat{\mathbf{H}}_b \cdot (\mathbf{k} \times \boldsymbol{e}) dS - 2\int_{\partial A_4(t)^\eta} \widehat{\mathbf{H}}_b \cdot (\mathbf{n} \times \boldsymbol{e}) dS \\ & + 2\int_{\partial A_2(t)^h} \widehat{\mathbf{H}}_b \cdot (\mathbf{k} \times \widehat{\boldsymbol{e}}) dS + 2\int_{\partial A_2(t)^\eta} \widehat{\mathbf{H}}_b \cdot (\mathbf{n} \times \widehat{\boldsymbol{e}}) dS \\ & - 2\int_{\partial A_3(t)^h} \mathbf{H}_b \cdot (\mathbf{k} \times \widehat{\boldsymbol{e}}) dS + 2\int_{\partial A_3(t)^\eta} \mathbf{H}_b \cdot (\mathbf{n} \times \widehat{\boldsymbol{e}}) dS. \end{aligned} \tag{5.18}$$

Since $\mathbf{H}_b \cdot \mathbf{k} \times \mathbf{a} = 0$, for any vector $\mathbf{a}$, and $\mathbf{n} \times \boldsymbol{e} = \mathbf{n} \times \widehat{\boldsymbol{e}}$, adding $(5.18)_1 + (5.18)_3 + (5.18)_4 + (5.18)_2$, we deduce

$$\begin{aligned} \mathtt{F} = & -2\int_{\Sigma_{-,t}} \mathbf{H}_b \cdot (\mathbf{n} \times \widehat{\boldsymbol{e}}) dx_* + 2\int_{\Sigma_{-,t}} \mathbf{H}_b \cdot (\mathbf{n} \times \widehat{\boldsymbol{e}}) dx_* \\ & - 2\int_{\Sigma_{+,t}} \widehat{\mathbf{H}}_b \cdot (\mathbf{n} \times \widehat{\boldsymbol{e}}) dx_* + 2\int_{\Sigma_{+,t}} \widehat{\mathbf{H}}_b \cdot (\mathbf{n} \times \widehat{\boldsymbol{e}}) dx_* = 0\,. \end{aligned} \tag{5.19}$$

Hence,

$$\frac{d}{dt}\mathbb{J} = (\widehat{\mu} - \mu)\mathtt{A},$$

which for $\mu = \widehat{\mu}$ yields

$$\frac{d\,\mathbb{J}}{dt} = 0\,. \tag{5.20}$$

Also from previous calculations it follows that

$$\frac{d}{dt}\mathcal{E}(\widetilde{\mathbf{H}}) = \frac{d}{dt}\mathcal{E}(\widetilde{\mathbf{h}}) + \frac{d}{dt}\mathbb{J} + (\widehat{\mu} - \mu)\frac{d}{dt}\mathbb{G} = \frac{d}{dt}\mathcal{E}(\widetilde{\mathbf{h}}) + \Big(\widehat{\mu} - \mu\Big)\frac{d}{dt}\mathbb{G} + \Big(\widehat{\mu} - \mu\Big)\,\mathtt{A}.$$

We recall that up to a constant $\mathcal{E}(\eta)$ is equivalent to the $W^{1,2}$ norm of $\eta$, cf. [30]. Moreover, since $\nabla \times \widetilde{\mathbf{H}} = 0$ it holds that

$$\mathcal{D}(\mathbf{H}) = \mathcal{D}(\mathbf{h}),$$

and it yields

$$\frac{d}{dt}\Big\{\mathcal{E}(\mathbf{u}) + \mathcal{E}(\widetilde{\mathbf{h}}) + \Pi(\eta) + \mathcal{E}(\eta) - [\widetilde{\mu}]\mathbb{G}\Big\} + \mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{h}) = \mathcal{B}_S(t) + \Big[\widetilde{\mu}\Big]\mathtt{M} + \Big[\widetilde{\mu}\Big]\,\mathtt{A}. \tag{5.21}$$

Finally, in absence of surface currents, and for $\mu = \widehat{\mu}$, the energy equation reduces to

$$\frac{d}{dt}\Big\{\mathcal{E}(\mathbf{u}) + \mathcal{E}(\widetilde{\mathbf{h}}) + \mathcal{E}(\eta)\Big\} + \mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{h}) = 0. \tag{5.22}$$

Equation (5.22) furnishes the wanted nonlinear stability theorem which states a global-in-time a priori estimate for the $L^2$ norm of perturbations to the rest, however large are the initial data.

## 6. Conclusions

We end the paper with some comments concerning our theory, and some comparisons with old methods and the one here developed.

In case $\mu \neq \widehat{\mu}$ the term $[\widetilde{\mu}](\mathtt{A}+\mathtt{M})$ has no definite sign, and it must be estimated in terms of dissipation. Also the term $\mathbb{G}$ defined by (5.11) is dimensionally an energy whose physical meaning has to be clarified through the thermodynamics of non-equilibrium. These problems constitute very challenging open problems related to non-equilibrium thermodynamics.

The present stability result continues to hold, in the class of regular solutions, for inviscid fluids, in absence of magnetic diffusivity, and of surface tension, provided the fluid is heavy and above a horizontal layer, cf. [30], [31].

In the presence of kinematic, magnetic, and surface viscosities it is possible to prove an exponential decay to zero for the perturbation; this may be proved with the free work identity, and constitutes the subject of a paper in preparation.

Even when $\mu = \widehat{\mu}$, the two definitions of perturbations to the basic magnetic field in model rest state, *do not* coincide. In particular in the proof of stability of the model constant rest state one is lead to choose different perturbations. Up to now we don't know a proof of nonlinear stability with perturbations $\widetilde{\varphi}$.

If $\mu \neq \widehat{\mu}$, the two definitions of perturbations to a constant magnetic field of the true rest state, *do not coincide*. However the proof of stability of a true constant rest state is still unknown whatever one chooses the perturbations.

If $\mu = \widehat{\mu}$, the two definitions of perturbations to a constant magnetic field of the true rest state, *do coincide*. Therefore in the proof of stability of true constant rest state one is free to choose different perturbations.

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# Viscous Flows in Domains with a Multiply Connected Boundary

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**Abstract.** In this paper we consider stationary Navier-Stokes equations in a bounded domain with a boundary, which has several connected components. The velocity vector is given on the boundary, where the fluxes differ from zero on its components. In the general case, the solvability of this problem has been an open question up to now. We provide a survey of previous results, which deal with partial versions of the problem. We construct an a priori estimate of the Dirichlet integral for a velocity vector in the case when the flow has an axis of symmetry and a plane of symmetry perpendicular to it, which also intersects each component of the boundary. Having available this estimate, we prove an existence theorem for the axially symmetric problem in a domain with a multiply connected boundary. We consider also the problem in a curvilinear ring and formulate a conditional result concerning its solvability.

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**Keywords.** Navier–Stokes equations, incompressible liquid, multiply connected boundary, Dirichlet integral, a priori estimate.

## 1. Introduction

Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ be a bounded domain with smooth boundary $\partial\Omega$ consisting of $m$ disjoint components $\Sigma_1, \ldots, \Sigma_m$. The stationary problem for the Navier-Stokes equations in a zero external force field

$$\mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nu \Delta \mathbf{v}, \quad \nabla \cdot \mathbf{v} = 0, \quad x \in \Omega, \tag{1.1}$$

$$\mathbf{v} = \mathbf{a}_i(x), \quad x \in \Sigma_i \ (i = 1, \ldots, m) \tag{1.2}$$

is considered. We introduce values

$$q_i = \int_{\Sigma_i} \mathbf{n}_i \cdot \mathbf{a}_i d\Sigma_i \quad (i = 1, \ldots, m) \tag{1.3}$$

where $\mathbf{n}_i$ is a unit vector of an exterior normal to the surface $\Sigma_i$. In view of the continuity equation,

$$q_i + \cdots + q_m = 0. \tag{1.4}$$

Let there be fulfilled a stronger condition

$$q_i = 0 \ (i = 1, \dots, m) \tag{1.5}$$

instead of (1.4). In this case, under corresponding smoothness conditions, the global existence theorem for the problem (1.1), (1.2) holds (J. Leray, [1]). The proof is based on finiteness of the Dirichlet integral

$$I = \int_\Omega \nabla \mathbf{v} : \nabla \mathbf{v} \, dx \tag{1.6}$$

for all possible solutions of the problem (1.1), (1.2), (1.5). Leray's demonstration used argument by contradiction and did not contain an *a priori estimate* of $I$ in terms of the problem data. E. Hopf [2] first obtained an effective estimate of the Dirichlet integral. His construction is based on the following lemma.

**Lemma 1.** *Assume that* $\Sigma_i \in C^{2+\alpha}, \ 0 < \alpha < 1,$ *and* $\mathbf{a}_i \in \mathbf{C}^{2+\alpha}(\Sigma_i), \ i = 1, \dots, m.$ *If condition* (1.5) *is satisfied, then for arbitrary* $\varepsilon > 0$ *there exists a solenoidal continuation* $\mathbf{b}_i(x) \in \mathbf{C}^{2+\alpha}(\bar{\Omega})$ *of vector* $\mathbf{a}_i(x)$ *into domain* $\Omega$ *such that for any* $\mathbf{u}(x) \in \mathbf{H}(\Omega)$,

$$\left| \int_\Omega \mathbf{b}_i \cdot \mathbf{u} \cdot \nabla \mathbf{u} \, dx \right| \le \varepsilon ||\nabla \mathbf{u}||^2_{L_2}, \quad i = 1, \dots, m. \tag{1.7}$$

Here $\mathbf{H}(\Omega)$ is the functional space introduced by O.A. Ladyzhenskaya [3]. Everywhere below the smoothness conditions formulated in Lemma 1 are assumed to be fulfilled. A. Takeshita proved [4] that the condition (1.5) is not only a sufficient but also a necessary one for the possibility of continuation of vector field $\mathbf{a}_i$ so that inequality (1.7) is satisfied for any $\varepsilon > 0$.

We will consider problem (1.1), (1.2) under general outflow conditions. It means that $q_i \neq 0$ at least for one $i \in 1, \dots, m$. It should be noted that violation of condition (1.5) does not lead to principal difficulties for the non-stationary problem for the Navier-Stokes equations [3]. As for the stationary one, there have been no general results about its solvability in case $q_i \neq 0$ up to now. On the other hand, there are a number of papers in which the existence theorems are proved under some additional conditions on the problem data. The next section is devoted to a description of results obtained in this direction.

## 2. Survey of previous results

As shown by R. Finn [5], the existence theorem for the problem (1.1), (1.2) remains valid if one assumes that $|q_i| < c_* \nu, \ i = 1, \dots, m$, and $c_*$ is small enough. G.P. Galdi [6, 7] has given the bound $c_*$ in terms of imbedding constants depending on the domain $\Omega$ and properties of solutions of a non-uniform divergence equation. The constant was computed explicitly in the flow in an annulus. For special cases, if domain $\Omega$ is confined by concentric spheres (or circles as $n = 2$ ) with radii $R_1$ and $R_2 > R_1$, W. Borchers and K. Pileckas [8] have obtained effective estimates of admissible $|q_i|$ bounds in terms of $R_1, \ R_2$ and $\nu$.

C.J. Amick [9] showed how to relax condition (1.5) without the smallness assumption on $|q_i|$ quantities. He studied two-dimensional flow under a specific symmetry assumption. Following [9], let us introduce

**Definition 1.** A bounded domain $\Omega \subset \mathbb{R}^2$ is said to be *admissible* if (a) $\partial\Omega$ is of class $C^{2+\alpha}$, (b) $\partial\Omega$ consists of $m \geq 2$ components $\Sigma_i$, (c) $\Omega$ is symmetric about the line $\{x_2 = 0\}$ and (d) each component $\Sigma_i$ intersects the line $\{x_2 = 0\}$.

A function $\mathbf{h} = (h_1, h_2)$ mapping $\Omega$ or $\partial\Omega$ into $\mathbb{R}^2$ is said to be *symmetric about the line* $\{x_2 = 0\}$ if $h_1$ is an even function of $x_2$ while $h_2$ is an odd function of $x_2$.

**Definition 2.** A vector $\mathbf{a}$ is said to be *admissible data* if (a) $\mathbf{a} \in \mathbf{C}^{2+\alpha}(\partial\Omega \to \mathbb{R}^2)$ and (b) $\mathbf{a}$ is symmetric about the line $\{x_2 = 0\}$.

It is well known that the Navier-Stokes equations are invariant with respect to reflection about the coordinate axis. This property allows us to seek symmetric solutions $(\mathbf{v},\ p)$ of this system, with $\mathbf{v}$ being symmetric about $\{x_2 = 0\}$ and the corresponding pressure $p$ being an even function of $x_2$ .

**Theorem 1** [9]. *Let $\Omega \subset \mathbb{R}^2$ be an admissible domain and let $(\mathbf{a_1}, \dots, \mathbf{a}_m)$ be admissible data. Then for every $\nu > 0$ there exists a solution $(\mathbf{v}, p) \in \mathbf{C}^{2+\alpha}(\bar{\Omega} \to \mathbb{R}^2) \times C^{1+\alpha}(\bar{\Omega} \to \mathbb{R})$ of (1.1), (1.2). The function $\mathbf{v}$ is symmetric about $\{x_2 = 0\}$ and the pressure $p$ is an even function of $x_2$.*

Similarly to Leray's basic work [1], Amick's result was obtained via a proof by contradiction. The next important step was done by H. Fujita [10], who presented a constructive proof of an existence of symmetric solutions via an a priori estimate of the Dirichlet integral (1.6). Fujita's construction is based on the concept of a *virtual drain* introduced by him.

**Definition 3.** Vector field $\mathbf{c}_i(x)$ is said to be a virtual drain if (a) $\mathbf{c}_i \in \mathbf{C}^{2+\alpha}(\bar{\Omega})$ is solenoidal and parallel to the $x_1$-axis, (b) the outflow of $\mathbf{c}_i$ from each $\Sigma_i$ coincides with that of $\mathbf{a}_i$; namely,

$$\int\limits_{\Sigma_1} \mathbf{n}_i \cdot \mathbf{c}_i \, d\Sigma_i = \int\limits_{\Sigma_1} \mathbf{n}_i \cdot \mathbf{a}_i \, d\Sigma_i \quad (i = 1, \dots, m) \tag{2.1}$$

and (c) $\mathbf{c}_i$ contains a positive free parameter $\kappa$, and by choosing $\kappa$ sufficiently small, we can make $\sup(|x_2||c_{i1}(x)|)$ arbitrarily small.

Another method for obtaining an a priori estimate of the Dirichlet integral under conditions of Theorem 1 was proposed by H. Morimoto [11]. Her construction exploits the stream function $\psi$ of plane flow defined by relations

$$v_1 = \frac{\partial\psi}{\partial x_2}, \quad v_2 = -\frac{\partial\psi}{\partial x_1}. \tag{2.2}$$

H. Fujita and H. Morimoto [12] studied problem (1.1), (1.2) in a domain $\Omega$ with two components of the boundary $\Sigma_1$ and $\Sigma_2$. Functions $\mathbf{a}_i$ in (1.2) were taken in the form $\mu\nabla\varphi + \tilde{\mathbf{a}}_i$ where $\mu \in \mathbb{R}$, $\varphi$ is a fundamental solution of the Laplace operator and $\tilde{\mathbf{a}}_i$ $(i = 1, 2)$ satisfy the condition (1.5). The authors proved that there is a countable subset $N$ of $\mathbb{R}$ such that if $\mu \notin N$ and $\tilde{\mathbf{a}}_i$ are small (in a suitable norm), then problem (1.1), (1.2) has a weak solution. Moreover, if $\Omega \in \mathbb{R}^2$ is an annulus, then $N$ is empty.

In the conclusion of this section, we mention results of papers [13–15] dedicated to flows in an annular domain $\Omega = \{x \in \mathbb{R}^2;\ R_1 < |x| < R_2\}$ under boundary conditions with non-vanishing outflow. H. Morimoto [13] considered this problem in the case

$$\mathbf{a}_i = \mu R_i^{-1}\mathbf{e}_r + b_i\mathbf{e}_\theta \quad \text{on} \quad \Sigma_i = \{x \in \mathbb{R}^2;\ |x| = R_i\}, \quad i = 1, 2 \tag{2.3}$$

where $\mu, b_1, b_2$ are constants and $\mathbf{e}_r$, $\mathbf{e}_\theta$ are the unit vectors in polar coordinates $\{r, \theta\}$. Problem (1.1), (2.3) has an exact rotationally symmetric solution, in which $\mathbf{v} = \mathbf{v}(r)$, $p = p(r)$ are given by explicit formulae. As $\mu = 0$, this solution describes the well-known Couette flow. In [13] it was proved that if $|\mu|$, $|b_1 - b_2|$ are sufficiently small and $\mu \neq -2\nu$ then the solution of problem (1.1), (2.3) is unique. The uniqueness theorem is valid also in case $\mu = -2\nu$ and $|\mu|$, $|b_1|$, $|b_2|$ are sufficiently small. Besides, for sufficiently large $\nu$, the above exact solutions are exponentially stable.

Let now the boundary condition have the form

$$\mathbf{a}_i = \{\mu R_i^{-1} + \varphi_i(\theta)\}\mathbf{e}_r + \{\omega_i R_i + \psi_i(\theta)\}\mathbf{e}_\theta \ \text{ on } \ \Sigma_i = \{x \in \mathbb{R}^2;\ |x| = R_i\}, \ \ i = 1, 2 \tag{2.4}$$

where $\varphi_i(\theta)$, $\psi_i(\theta)$ are smooth functions of $\theta$ with a zero mean value. Problem (1.1), (2.4) was studied by H. Morimoto and S. Ukai [14]. The main result of the paper [14] is

**Theorem 2.** *Suppose the inequality*

$$|\omega_1 - \omega_2| \frac{R_1^2 R_2^2}{R_2^2 - R_1^2} \Big(\log \frac{R_2}{R_1}\Big)^2 < 2\nu \tag{2.5}$$

*holds. Then there exists at most a discrete countable set $M$ such that for each $\mu \in \mathbb{R}\backslash M$ the boundary problem* (1.1), (2.4) *has a solution for sufficiently small $\varphi_i(\theta)$, $\psi_i(\theta)$ $(i = 1, 2)$.*

We note that under the condition of Theorem 2 the quantity $|\mu|$ can be large in comparison with $\nu$. It is interesting to distinguish a class of conditions (2.4), as the set $M$ is empty. H. Fujita, H. Morimoto and H. Okamoto [15] established that this is true as $\omega_1 = \omega_2$; in this case, inequality (2.5) is fulfilled automatically. The special case $b_1 = b_2 = 0$ in (2.3) corresponds to a radial flow with velocity field $v_r = \mu r^{-1}$, $v_\theta = 0$. As it is shown in [15], the radial flow in an annulus is stable to perturbation of the steady state, whatever the Reynolds number $\mu/\nu$ or the aspect ratio $R_1/R_2$ are. At the same time, the precise calculations carried out in [15] provide numerical evidence that Hopf's bifurcations occur for the case $b_1 R_1 = b_2 R_2$. In this case, the solution of problem (1.1), (2.3) is self-similar; the corresponding velocity field is $v_r = \mu r^{-1}$, $v_\theta = \lambda r^{-1}$ where $\lambda = b_1 R_1$.

## 3. Axially symmetric flows

In this section we consider problem (1.1), (1.2) in the case, when domain $\Omega \in \mathbb{R}^3$ has an axis of symmetry and a plane of symmetry, which is perpendicular to this axis.

**Definition 4.** A bounded domain $\Omega \in \mathbb{R}^3$ is said to be *admissible* is (a) $\partial\Omega$ is of class $C^{2+\alpha}$, (b) $\partial\Omega$ consists of $m \geq 2$ simply connected components $\Sigma_i$, (c) $\Omega$ has the axis of symmetry $\{x_1 = x_2 = 0\}$ and the plane of symmetry $\{x_3 = 0\}$, and (d) each component $\Sigma_i$ intersects the plane $\{x_3 = 0\}$.

Let us introduce cylindrical coordinates $r = (x_1^2 + x_2^2)^{1/2}$, $\theta = \operatorname{arctg}(x_2/x_1)$, $z = x_3$ and denote as $v_r, v_\theta, v_z$ projections of vector $\mathbf{v}$ on the axes $r, \theta, z$.

A function $\mathbf{h} = (h_r, h_\theta, h_z)$ mapping $\Omega$ or $\partial\Omega$ into $\mathbb{R}^3$ is said to be *axially symmetric* if $h_\theta = 0$ while $h_r$ and $h_z$ do not depend on $\theta$. A function $\mathbf{h} = (h_r, 0, h_z)$ mapping $\Omega$ or $\partial\Omega$ into $\mathbb{R}^3$ is said to be *symmetric about the planes* $\{z = 0\}$ if $h_r$ is an even function of $z$ while $h_z$ is an odd function of $z$.

**Definition 5.** A vector $\mathbf{a}$ is said to be *admissible data* if (a) $\mathbf{a} \in \mathbf{C}^{2+\alpha}(\partial\Omega \to \mathbb{R}^3)$ and (b) $\mathbf{a}$ is axially symmetric and symmetric about the plane $\{z = 0\}$.

Our purpose is to prove the existence theorem for problem (1.1), (1.2) in the class of axially symmetric flows. It means that the vector $\mathbf{v} = (v_r, 0, v_z)$ is axially symmetric and symmetric about plane $\{z = 0\}$, moreover the corresponding pressure $p$ does not depend on $\theta$. In consequence (1.1), functions $v_r, v_z$ and $p$ satisfy the following system:

$$\begin{aligned}
&v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} = -\frac{\partial p}{\partial r} + v \left( \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2} \right), \\
&v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = -\frac{\partial p}{\partial z} + v \left( \frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r} \frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2} \right), \\
&\frac{1}{r} \frac{\partial (r v_r)}{\partial r} + \frac{\partial v_z}{\partial z} = 0.
\end{aligned} \tag{3.1}$$

**Lemma 2.** *Let $\Omega \to \mathbb{R}^3$ be an admissible domain and let $(\mathbf{a}_1, \ldots, \mathbf{a}_m)$ be admissible data. Then the Dirichlet integral* (1.6) *is finite for all possible solutions of problem* (1.1)*,* (1.2)*.*

*Proof.* It is based on a special construction of the virtual drain, which generalizes the Fujita construction [9].

According to the conditions of Lemma 2, the boundary of domain $\Omega$ consists of $m \geq 2$ disjoint smooth simply connected components $\Sigma_1, \ldots, \Sigma_m$. We assume that surface $\Sigma_m$ encloses the other components $\Sigma_1, \ldots, \Sigma_{m-1}$. Each of surfaces $\Sigma_1, \ldots, \Sigma_{m-1}$ is a surface of revolution. Let us denote as $S_1, \ldots, S_m$ plane curves, which are meridian sections of $\Sigma_1, \ldots, \Sigma_{m-1}$. Further, we notice that the ray $r > 0$ in semi-plane $\{r,\ z : r > 0,\ z \in \mathbb{R}\}$ and each $S_i$ intersects orthogonally at two different points $P_i = (y_i, 0)$ and $P_i^* = (y_i^*, 0)$ of which we can assume that $y_i > y_i^*$ $(i = 1, \ldots, m)$. Moreover, we may assume that $y_m > y_{m-1} > \cdots > y_1 > 0$.

Now we define the domain $\Omega^+ = \{r,\ \theta,\ z : r > 0,\ z > 0,\ (r,\theta,z) \in \Omega\}$. In other words, $\Omega^+$ is "a half of $\Omega$". It should be noted that the domain $\Omega^+$ has a simply connected boundary $\partial\Omega^+$, as domain $\Omega$ is admissible. Let us designate $C_\delta = \{r,\ \theta,\ z : r \in \mathbb{R}^+,\ 0 < z < \delta\}$. Then we choose a small positive number $\delta$ so that the domain $\Omega^+ \cap C_\delta$ consists of $m$ disjoint components $K_i$. Each of the domains $K_i$ will be a support of the $i$th component of the virtual drain.

Let us consider domain $K_1$. Its boundary consists of two basements, belonging to planes $z = 0$, $z = \delta$, and two lateral parts $L_1^l$ and $L_1^r$, which are surfaces of revolution. The lower basement of $K_1$ is a ring $y_1 < r < y_2^*$, $z = 0$. The first virtual drain $\mathbf{c}_i(x)$ we take in the form

$$\mathbf{c}_1 = -\frac{1}{4\pi r} q_1(\eta(z), 0, 0), \tag{3.2}$$

where $\eta(t) = \eta(t;\ \delta, \kappa)$ is the Hopf-type cutting function [16] and $\kappa \in (0, 1/2)$ is a free parameter. Here we use a small modification of function $\eta(t)$ construction given in [9], namely:

$$\eta(t) = \frac{1}{\gamma_\kappa \delta} \zeta_\kappa\left(\frac{t}{\delta}\right), \quad \zeta_\kappa \in C_0^\infty(\mathbb{R}), \quad \zeta_\kappa(t) \geq 0\ (\forall t \geq 0),$$

$$\zeta_\kappa(-t) = \zeta_\kappa(t), \quad \zeta_\kappa(t) = 0\ (t \geq 1),$$

$$\zeta_\kappa(t) \leq \frac{1}{t}\ (0 < t < \infty), \quad \zeta_\kappa(t) = \frac{1}{t}\ \left(\kappa \leq t \leq \frac{1}{2}\right),$$

$$\gamma_\kappa = \int_{-1}^{1} \zeta_\kappa(t)\, dt, \quad \gamma_\kappa \geq 2 \int_{\kappa}^{1/2} \frac{dt}{t} \to \infty\ (\kappa \to 0),$$

$$\int_{-\infty}^{\infty} \eta(t)\, dt = \int_{-\delta}^{\delta} \eta(t)\, dt = 1. \tag{3.3}$$

If $m = 2$, the structure of a virtual drain is completed. In fact, vector $\mathbf{c}_i(x)$ is solenoidal and smooth. In view of (3.2), (3.3), the equality

$$\int_{L_2^l} \mathbf{n}_1 \cdot \mathbf{c}_1\, dL_2^l = \frac{1}{2}\, q_1 \tag{3.4}$$

holds (we recall that $\mathbf{n}_1$ is unit vector of an *exterior* normal to the surface $\Sigma_1$). Taking into account (3.4) and extending function $\mathbf{c}_1(x)$ on negative values of $z$ as an even function of these variables, we guarantee fulfillment of equality (2.1) as $i = 1$. At last, choosing parameter $\kappa$ small, we can provide $\sup(|z||c_{1,r}(z)|)$, $x \in \bar{K}_1$, arbitrarily small. In view of (1.4), $q_2 = -q_1$ if $m = 2$. Replacing $y_2^*$ by $y_2$ in the case $m = 2$ and identifying $\mathbf{c}_2$ with $\mathbf{c}_1$, we arrive at relation

$$\int_{L_2^r} \mathbf{n}_2 \cdot \mathbf{c}_2\, dL_2^r = \frac{1}{2}\, q_2,$$

which ensures (2.1) for $i = 2$.

Let now $m = 3$. In this case, we consider domain $K_2$ connecting surfaces $\Sigma_2$ and $\Sigma_3$. We define as before function $\mathbf{c}_1(x)$ by formula (3.2). Further, let us denote as $L_2^l$ and $L_2^r$ the left and right lateral sides of surface $\partial K_2$, which intersects orthogonally the plane $z = 0$, and set

$$\mathbf{c}_2 = -\frac{1}{4\pi r}(q_1 + q_2)(\eta(z), 0, 0).$$

By (3.4), the liquid flux through surface $L_1^l$ equals to $q_1/2$ , hence the flux through surface $L_2^r \subset \Sigma_2$ is $-q_1/2$. At the same time, the flux through surface $L_2^l \subset \Sigma_2$ is $(q_1 + q_2)/2$. It implies equality (2.1) for $i = 2$. Thus, the second component of the virtual drain in the case $m = 3$ is constructed. The third component is defined by relation

$$\mathbf{c}_3 = \frac{1}{4\pi r} q_3(\eta(z), 0, 0) \tag{3.5}$$

in an annular layer $K_2$ and by continuation of the function given by (3.5) on negative values of $z$. Relation (2.1) for $i = 3$ will be satisfied on account of equality $q_1 + q_2 + q_3 = 0$ (1.4). When $m > 3$, we continue the described procedure until its completion for $m - 1$ steps.

Due to the symmetry condition, this is sufficient to estimate the integral

$$I^+ = \int_{\Omega^+} \nabla\mathbf{v} : \nabla\mathbf{v}\, dx \tag{3.6}$$

to prove Lemma 2, since $I = 2I^+$ where $I$ is a Dirichlet integral (1.6). To this end, we represent $\mathbf{v}(x)$ in the form

$$\mathbf{v} = \mathbf{u} + \sum_{i=1}^{m}(\mathbf{b}_i + \mathbf{c}_i). \tag{3.7}$$

Here $\mathbf{u} \in \mathbf{H}(\Omega)$, $\{\mathbf{c}_i\}$, is the set of virtual drains, each of the solenoidal vector-functions $\mathbf{b}_i$ $(i = 1, \ldots, m)$ satisfies the zero flux condition

$$\int_{\Sigma_i} \mathbf{n}_i \cdot \mathbf{b}_i\, d\Sigma_i = 0 \tag{3.8}$$

and

$$\mathbf{a}_i(x) = \mathbf{b}_i(x) + \mathbf{c}_i(x),\ x \in \Sigma_i,\ i = 1, \ldots, m. \tag{3.9}$$

Moreover, the support of each function $\mathbf{b}_i(x)$ is a narrow strip near the surface $\Sigma_i$. Functions $\mathbf{c}_i$ (virtual drains) are determined previously. Equality (3.9) means that vector $\mathbf{b}_i + \mathbf{c}_i$ is a solenoidal continuation of vector $\mathbf{a}_i$ into domain $\Omega$. As far as $(\mathbf{a}_1, \ldots, \mathbf{a}_m)$ are admissible data and vectors $\mathbf{c}_i$ $(i = 1, \ldots, m)$ satisfy the symmetry condition we can consider that vectors $\mathbf{b}_i$ $(i = 1, \ldots, m)$ satisfy the same condition too. Then (3.7) implies that vector $\mathbf{u}$ is symmetric also. There is a freedom in the choice of $\mathbf{b}_i$ vectors. In view of (3.8) we can apply Lemma 1 and realize the construction of functions $\mathbf{b}_i$ so that inequalities (1.7) hold with a positive constant $\varepsilon$, which will be chosen later.

To provide symmetry properties of vectors $\mathbf{b}_i$, we define their components in the form

$$b_{i,r} = -\frac{1}{r}\frac{\partial(\eta(n)\Psi_i)}{\partial z}, \quad b_{i,z} = \frac{1}{r}\frac{\partial(\eta(n)\Psi_i)}{\partial r}, \quad (i = 1, \dots, m).$$

Here $n$ is the distance of current point $(r, z) \in \Omega$ from $\partial\Omega$, $\eta(n)$ is the cutting function defined by formula (3.3) and $\Psi_i(r, z)$ is the stream function of an axially symmetric solenoidal vector field admitted given boundary values $\mathbf{b}_i = \mathbf{a}_i - \mathbf{c}_i$ on the surface $\Sigma_i$.

For any smooth axially symmetric vector $\mathbf{h}$ we can define its strain tensor $D = D(\mathbf{h})$ with elements

$$D_{rr} = \frac{\partial h_r}{\partial r}, \ \ D_{\theta\theta} = \frac{h_r}{r}, \ \ D_{zz} = \frac{\partial h_z}{\partial z}, \ \ D_{rz} = \frac{1}{2}\left(\frac{\partial h_r}{\partial z} + \frac{\partial h_z}{\partial r}\right), \ \ D_{r\theta} = D_{\theta z} = 0.$$

If function $\mathbf{h}$ is symmetric about the planes $\{z = 0\}$ then the following equalities hold:

$$h_z = 0, \quad D_{rz} = 0, \quad (r, z) \in \partial\Omega^+ \cap \{z = 0\}. \tag{3.10}$$

The next step of our consideration is obtaining an integral relation for the sought vector $\mathbf{u}$. To get it, we substitute representation (3.7) into system (3.1), multiply its first and second equation by $u_r, u_z$ respectively and integrate the result over domain $\Omega^+$. We note that each of the functions on the right-hand side of (3.7) satisfies conditions (3.10). Having applied these conditions and the well-known Green identity for the Stokes operator [3], we come after simple calculations to the required relation:

$$2\nu\int_{\Omega^+} D(\mathbf{u}) : D(\mathbf{u})\, dx - \sum_{i=1}^{m}\int_{\Omega^+} \mathbf{b}_i \cdot \mathbf{u} \cdot \nabla\mathbf{u}\, dx - \sum_{i=1}^{m}\int_{\Omega^+} \mathbf{c}_i \cdot \mathbf{u} \cdot \nabla\mathbf{u}\, dx \tag{3.11}$$

$$= -2\nu\sum_{i=1}^{m}\int_{\Omega^+} D(\mathbf{b}_i + \mathbf{c}_i) : D(\mathbf{u})\, dx + \sum_{i=1}^{m}\int_{\Omega^+} \mathbf{u} \cdot (\mathbf{b}_i + \mathbf{c}_i) \cdot \nabla(\mathbf{b}_i + \mathbf{c}_i)\, dx.$$

Since function $\mathbf{u}$ is symmetric, the following equalities are valid:

$$\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{u}\, dx = 2\int_{\Omega^+} \nabla\mathbf{u} : \nabla\mathbf{u}\ dx, \ \int_\Omega D(\mathbf{u}) : D(\mathbf{u})\, dx = 2\int_{\Omega^+} D(\mathbf{u}) : D(\mathbf{u})\ dx,$$

$$\int_\Omega |\mathbf{u}^2|\ dx = 2\int_{\Omega^+} |\mathbf{u}|^2\ dx. \tag{3.12}$$

For any $\mathbf{u} \in \mathbf{H}(\Omega)$, the Korn inequality [7, 17]

$$\int_\Omega D(\mathbf{u}) : D(\mathbf{u})\, dx \geq M_1\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{u}\, dx, \tag{3.13}$$

and the Poincare inequality [3, 7]

$$\int_\Omega |\mathbf{u}^2|\ dx \leq M_2\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{u}\, dx \tag{3.14}$$

are true with positive constants $M_k = M_k(\Omega),\ \ k = 1, 2$. Relations (3.12) allow us to replace the integration domain $\Omega$ in inequalities (3.13), (3.14) by domain $\Omega^+$.

This gives the desired estimates of the right-hand side in relation (3.11) (upper estimate) and the first term of its left side (lower estimate). To estimate the second term on the left-hand side of (3.11) above, we apply Lemma 1 with $\varepsilon = \nu M_1/2m$ that gives

$$\Big|\int_{\Omega^+} \mathbf{b}_i \cdot \mathbf{u} \cdot \nabla \mathbf{u}\, dx\Big| \le \frac{\nu M_1}{2m} \int_{\Omega^+} \nabla \mathbf{u} : \nabla \mathbf{u}\; dx,\ i = 1, \dots, m. \tag{3.15}$$

The most crucial point of our examination is the derivation of the same estimate for the third term on the left-hand side of (3.11),

$$\Big|\int_{\Omega^+} \mathbf{c}_i \cdot \mathbf{u} \cdot \nabla \mathbf{u}\, dx\Big| \le \frac{\nu M_1}{2m} \int_{\Omega^+} \nabla \mathbf{u} : \nabla \mathbf{u}\; dx,\ i = 1, \dots, m. \tag{3.16}$$

The integral on the left-hand side of (3.16) can be written as

$$\int_{\Omega^+} \mathbf{c}_i \cdot \mathbf{u} \cdot \nabla \mathbf{u}\, dx = J_1 + J_2, \tag{3.17}$$

where

$$J_1 = 2\pi \int_{K_i} c_{i,r} u_r \frac{\partial u_r}{\partial r}\; r dr dz,$$

$$J_2 = 2\pi \int_{K_i} c_{i,r} u_z \frac{\partial u_r}{\partial z}\; r dr dz.$$

Here we took into account that vector $\mathbf{c}_i$ has only a nonzero component $c_{i,r}$ and its support is $K_i$. Evaluating integral $J_2$, we note that

$$\sup_t |t| \eta(t; \delta, \kappa) \to 0 \quad \text{as} \quad \kappa \to 0 \tag{3.18}$$

as follows from (3.3). Next, component $u_z$ of the symmetric vector $\mathbf{u}$ vanishes on the plane $z = 0$ in the sense of trace. Hence, we can apply the Hardy-type inequality [3, 7]

$$\int_{K_i} \frac{u_z^2}{z^2}\, dx \le 4 \int_{K_i} |\nabla u_z|^2\, dx. \tag{3.19}$$

Remembering the expressions for virtual drains (3.1), (3.4) and similar to them, we obtain inequality

$$|J_2| \le \frac{1}{8\pi}(m-1) q_* \int_{K_i} \sup_{K_i}(|z| \eta(z)) \frac{|u_z|}{z} \Big|\frac{\partial u_r}{\partial z}\Big|\, dx,$$

where $q_* = \max|q_i|,\ i = 1, \dots, m$. Choosing sufficiently small $\kappa$, we arrive from this inequality and (3.18), (3.19) at the desired estimate (3.16). As for integral $J_1$, it is equal to zero because $r c_{i,r}$ does not depend on $r$ and function $u_r$ vanishes on the end-wall parts of $K_i$ boundary, which belongs to $\partial\Omega$.

Combining equalities (3.12), inequalities (3.13), (3.14) and estimates (3.15), (3.16) we conclude from (3.11) that

$$\int_{\Omega^+} \nabla \mathbf{u} : \nabla \mathbf{u}\, dx \le \frac{1}{\nu} M_3, \tag{3.20}$$

where $M_3 = M_3(\Omega, ||\mathbf{b}_i||_{H^1})$ is a positive constant. Inequality (3.20) and representation (3.7) lead to the required estimates of integral $I^+$ (3.6) and consequently of Dirichlet integral $I = 2I^+$,

$$\int_{\Omega^+} \nabla\mathbf{v} : \nabla\mathbf{v}\, dx \leq M_4 \tag{3.21}$$

with constant $M_4 = M_4(\Omega, \nu, ||\mathbf{b}_i||_{H^1},\ ||\mathbf{c}_i||_{H^1}) > 0$. This completes the proof of Lemma 2. □

**Theorem 3.** *Under assumptions as in Lemma 2, there exists a solution* $\mathbf{v}(x) \in \mathbf{C}^{2+\alpha}(\bar{\Omega}),\ p(x) \in C^{1+\alpha}(\bar{\Omega})$ *of the problem* (1.1), (1.2).

The proof of Theorem 3 is omitted here. It is based on estimate (3.20) and follows the classical scheme given in [3] or [18].

## 4. Flow in a curvilinear ring

Here we return to the problem mentioned in Section 2. Let $\Omega \in \mathbf{R}^2$ be a curvilinear ring bounded by smooth curves $\Sigma_1$ (interior boundary) and $\Sigma_2$ (exterior boundary). All previous results on the solvability of problem (1.1), (1.2) for this case had either a local character or dealt with an annular geometry of $\Omega$ [5–8, 12–15]. The point is to obtain an a priori estimate of the Dirichlet integral for an arbitrarily large flux. This problem is still open. We postpone a discussion to the next section and now propose some construction, which can be useful for the problem treatment.

It is well known that any plane flow of an incompressible liquid is characterized by the stream function $\psi$ defined by relations (2.2). Level lines of function $\psi$ coincide with trajectories of liquid particles in the case of steady-state flow. If domain $\Omega$ is simply connected, the stream function is a single-valued one. In the opposite case, this property holds only under condition (1.5). We consider the situation when this condition is violated. If $\Omega$ is a curvilinear ring, it means that $q_1 = -q_2 \neq 0$. In this case, function $\psi(x_1, x_2)$ is a multi-valued one, which receives the increment $q_1$ after going around $\Sigma_1$. If $q_1 \neq 0$, there is at least one stream line $l_1$ which intersects both components $\Sigma_1$ and $\Sigma_2$ of $\partial\Omega$. This assertion can be proved by contradiction. We assume that the line about the above-mentioned intersection is transversal and denoted by $P_1$ and $P_2$ the points of intersection of $l_1$ with $\Sigma_1$ and $\Sigma_2$, respectively.

Further, we assume that there exists another stream line $l_2$, which also intersects curves $\Sigma_1$ and $\Sigma_2$. Thus, both curves $\Sigma_i$ $(i = 1, 2)$ are divided by lines $l_i$ on two parts $\Sigma_i^-$ and $\Sigma_i^+$. Respectively, the domain $\Omega$ is divided into two simply connected domains $\Omega^+$ and $\Omega^-$. We suppose additionally that the flux through components $\Sigma_1^-$ and $\Sigma_1^+$ are equal to $q_1/2$. Choosing a single-valued branch of function $\psi$ and denoting it by $\Psi(x_1, x_2)$, we can consider that $\Psi \to q_1/2$ when one tends to point $P_1$ along curve $\Sigma_1^+$ and $\Psi \to -q_1/2$ when one tends to point $P_2$ along curve $\Sigma_1^-$. Now we will construct a virtual drain for the flow in domain $\Omega$.

The idea of the construction is close to the structure proposed in paper [11] for symmetric flow, but we are not able to apply this structure word for word. Our idea consists in construction of a virtual drain with support in a narrow curvilinear strip near the line $l_1$. To this end, we pass in system (1.1) to curvilinear orthogonal coordinates following [19].

Let us denote the curvilinear orthogonal coordinates as $s_1$ and $s_2$, and the corresponding Lame coefficients as $H_1$ and $H_2$. We preserve notations $v_1$ and $v_2$ for projections of velocity vectors on the axes $s_1$ and $s_2$ because this will not lead to misunderstanding. System (1.1) written in curvilinear coordinates takes the form

$$
\begin{aligned}
&\frac{v_1}{H_1}\frac{\partial v_1}{\partial s_1}+\frac{v_2}{H_2}\frac{\partial v_1}{\partial s_2}+\frac{v_2}{H_1H_2}\left(v_1\frac{\partial H_1}{\partial s_2}-v_2\frac{\partial H_2}{\partial s_1}\right) \qquad (4.1)\\
&=-\frac{1}{H_1}\frac{\partial p}{\partial s_1}+\nu\left[\frac{1}{H_1^2}\frac{\partial^2 v_1}{\partial s_1^2}+\frac{1}{H_2^2}\frac{\partial^2 v_1}{\partial s_2^2}+\frac{1}{H_1H_2}\frac{\partial(H_1^{-1}H_2)}{\partial s_1}\frac{\partial v_1}{\partial s_1}\right.\\
&\quad+\frac{1}{H_1H_2}\frac{\partial(H_2^{-1}H_1)}{\partial s_2}\frac{\partial v_1}{\partial s_2}+\frac{2}{H_1^2H_2}\frac{\partial H_1}{\partial s_2}\frac{\partial v_2}{\partial s_1}-\frac{2}{H_1H_2^2}\frac{\partial H_2}{\partial s_1}\frac{\partial v_2}{\partial s_2}\\
&\quad+\frac{1}{H_1}\frac{\partial}{\partial s_1}\left(\frac{1}{H_1H_2}\frac{\partial H_2}{\partial s_1}\right)v_1+\frac{1}{H_2}\frac{\partial}{\partial s_2}\left(\frac{1}{H_1H_2}\frac{\partial H_1}{\partial s_2}\right)v_1\\
&\quad\left.+\frac{1}{H_1}\frac{\partial}{\partial s_1}\left(\frac{1}{H_1H_2}\frac{\partial H_1}{\partial s_2}\right)v_2-\frac{1}{H_2}\frac{\partial}{\partial s_2}\left(\frac{1}{H_1H_2}\frac{\partial H_2}{\partial s_1}\right)v_2\right],
\end{aligned}
$$

$$
\begin{aligned}
&\frac{v_1}{H_1}\frac{\partial v_2}{\partial s_1}+\frac{v_2}{H_2}\frac{\partial v_2}{\partial s_2}-\frac{v_1}{H_1H_2}\left(v_1\frac{\partial H_1}{\partial s_2}-v_2\frac{\partial H_2}{\partial s_1}\right)\\
&=-\frac{1}{H_2}\frac{\partial p}{\partial s_2}+\nu\left[\left(\frac{1}{H_1^2}\frac{\partial^2 v_2}{\partial s_1^2}+\frac{1}{H_2^2}\frac{\partial^2 v_2}{\partial s_2^2}+\frac{1}{H_1H_2}\frac{\partial(H_1^{-1}H_2)}{\partial s_1}\frac{\partial v_2}{\partial s_1}\right.\right.\\
&\quad+\frac{1}{H_1H_2}\frac{\partial(H_2^{-1}H_1)}{\partial s_2}\frac{\partial v_2}{\partial s_2}-\frac{2}{H_1^2H_2}\frac{\partial H_1}{\partial s_2}\frac{\partial v_1}{\partial s_1}+\frac{2}{H_1H_2^2}\frac{\partial H_2}{\partial s_1}\frac{\partial v_1}{\partial s_2}\\
&\quad+\frac{1}{H_1}\frac{\partial}{\partial s_1}\left(\frac{1}{H_1H_2}\frac{\partial H_2}{\partial s_1}\right)v_2+\frac{1}{H_2}\frac{\partial}{\partial s_2}\left(\frac{1}{H_1H_2}\frac{\partial H_1}{\partial s_2}\right)v_2\\
&\quad\left.-\frac{1}{H_1}\frac{\partial}{\partial s_1}\left(\frac{1}{H_1H_2}\frac{\partial H_1}{\partial s_2}\right)v_1+\frac{1}{H_2}\frac{\partial}{\partial s_2}\left(\frac{1}{H_1H_2}\frac{\partial H_2}{\partial s_1}\right)v_1\right],
\end{aligned}
$$

$$
\frac{\partial(H_2v_1)}{\partial s_1}+\frac{\partial(H_1v_2)}{\partial s_2}=0.
$$

Let us choose now a special coordinates system proposed at first by R. von Mises in the paper [20] devoted to boundary layer theory (see also [19]). Taking point $P_1$ as the origin, we will determine the position of point $P\in\Omega$ by coordinates $s_1=s$ and $s_2=n$, where $s$ is the arc length of line $l_1$ and $n$ is the length of the normal to this line taken with an appropriate sign. Then the first quadratic form is written as

$$d\sigma^2 = \left[1 + \frac{n}{\rho(s)}\right]^2 ds^2 + dn^2$$

and therefore

$$H_1 = 1 + \frac{n}{\rho(s)}, \ H_2 = 1. \tag{4.2}$$

Here $\rho(s)$ is the curvature radius of curve $l_1$ in the point with coordinate $s$. We will suppose that the curve $l_1$ is smooth enough so that $\rho(s) \in C^1[0, L]$ where $L$ is the length of $l_1$.

Let us denote as $S_\delta$ the strip $S_\delta = \{s, n \, : \, s \in \mathbf{R}, \ |n| < \delta\}$ and define the domain $K_1$ by relation $K_1 = \Omega \cap S_\delta$ . The virtual drain $\mathbf{c}_1$ is defined by formula

$$\mathbf{c}_1 = -\lambda q_1(\eta(n), \ 0) \tag{4.3}$$

where $\eta(n)$ is the cutting function introduced in [10] and $\lambda = \lambda(\Omega, l_1, \delta)$ is a positive constant. In view of (4.2) and the last equation of (4.1), vector $\mathbf{c}_1$ is smooth and solenoidal. Choosing a suitable correcting multiplier $\lambda$, we are able to provide the required flux $q_1/2$ through the curves $\Sigma_1^+$ and $\Sigma_1^-$.

Having available two simply connected domains $\Omega^+, \ \Omega^-$ and the virtual drain (4.3), we can repeat almost literally the procedure described in the proof of Lemma 2. It consists in obtaining an identity like (3.10) for vector $\mathbf{u} \in \mathbf{H}(\Omega)$ defined by an analogue of formula (3.6). As a result, we come to an a priori estimate of type (3.20). Unfortunately, now constant $M_4$ depends not only on $\Omega, v, ||\mathbf{b}_1||_{H^1}$ and $||\mathbf{c}_1||_{H^1}$ but also on the $C^3$-norm of the function, which parametrizes curve $l_1$. Because of this reason, our result has a conditional character.

## 5. Discussion

a) An a priori estimate of the Dirichlet integral (1.6) for the solution of problem (1.1), (1.2) has not only a theoretical interest but also allows us to justify approximate methods, in particular, the Galerkin method [18]. The result of Lemma 2 guarantees such justification for symmetric flows in $\mathbb{R}^3$.

b) Detailing the proof of Lemma 2, we may conclude that dependence of value $M_4$ in (3.20) on norms $||\mathbf{c}_i||_{H^1}$ $(i = 1, \ldots, m)$ is not more than a linear one. It means that norm $||\mathbf{v}||_{H^1}$ has maximum linear growth in $q_* = \max|\mathrm{q_i}|$ $(i = 1, \ldots, m)$ because norms $||\mathbf{b}_i||_{H^1}$ $(i = 1, \ldots, m)$ and value $M_3$ in (3.19) do not depend on $q_*$. This assertion is compatible with results of article [13] where a number of exact solutions to the problem are studied.

c) During our treatment of the problem, functions $\mathbf{a}_i$ in boundary condition (1.2) were supposed to be smooth. It is possible to relax this condition up to inclusion $\mathbf{a}_i \in \mathbf{H}^{1/2}(\Sigma_i), \ (i = 1, \ldots, m)$ as it was done in [10, 11]. The statement of Lemma 2 holds in this case.

d) We restrict our analysis to the case of absence of an external body force acting on a liquid. The case of potential force is reduced to the previous case with the help of a pressure transform. Let us consider the general situation where an acceleration of body force is $\mathbf{f}(x)$, where $\mathbf{f}$ is a given admissible function. Following the arguments of [10, 11], we can prove an analogue of Lemma 2 if $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and analogues of Theorem 3 and Theorem 4 if $\mathbf{f} \in \mathbf{C}^\alpha(\bar{\Omega})$.

e) The conditional result declared in Section 4 stimulates the study of stream lines structure in two-dimensional stationary incompressible viscous flows. For classical symmetric solutions of the problem (1.1), (1.2) in admissible domain $\Omega \in \mathbb{R}^2$, we can apply the Kronrod theorem [21]. In particular, this theorem implies the following conclusion.

*Let us consider a set of level lines*

$$\psi(x_1, x_2) = c \tag{5.1}$$

*where $\psi \in C^2(\bar{\Omega})$ and $c \in \mathbb{R}$. There exists a set $N$ of zero measure such that for any $c \in \mathbb{R} \setminus N$ the corresponding level line* (5.1) *comes out on $\partial\Omega$ or this line is closed.*

This illuminates the situation with the structure of stream lines in a plane symmetric case. On the basis of Theorem 4, a similar statement is true for axially symmetric flows. As for a general two-dimensional flow, we know almost nothing about the structure of the set of stream lines.

Let us consider a flow in a curvilinear ring $\Omega$ under additional conditions

$$\mathbf{a}_1 \cdot \mathbf{n}_1 < 0, \ x \in \Sigma_1; \quad \mathbf{a}_2 \cdot \mathbf{n}_2 > 0, \ x \in \Sigma_2 \tag{5.2}$$

or

$$\mathbf{a}_1 \cdot \mathbf{n}_1 > 0, \ x \in \Sigma_1; \quad \mathbf{a}_2 \cdot \mathbf{n}_2 < 0, \ x \in \Sigma_2. \tag{5.3}$$

In other words, each point of curve $\Sigma_1$ is an input (output) point of a stream line inside (outside) domain $\Omega$ and the same property is valid for curve $\Sigma_2$. The following conjecture (C) seems to be likely.

*Let $\mathbf{v}, p$ be a solution to problem* (1.1), (1.2), (5.2) *or* (1.1), (1.2), (5.3) *in a curvilinear ring $\Omega$. Then each stream line connects $\Sigma_1$ with $\Sigma_2$ and intersects transversally these curves.*

This conjecture is the most plausible if the Reynolds number $\mathrm{Re} = |q_1|/\nu$ is sufficiently large.

f) Let us consider the flow in a curvilinear ring, assuming that the Reynolds number $\mathrm{Re} \to \infty$. In this case, a formal asymptotic solution of the problem (1.1), (1.2), a solution can be constructed by a certain modification of the Vishik-Lyusternik method [22]. In contrast to a boundary layer near a solid wall, the thickness of the boundary layer in problem (1.1), (1.2) has an order of $\mathrm{Re}^{-1}$. This boundary layer is localized near the curve $\Sigma_1$ if $q_1 > 0$ and near the curve $\Sigma_2$ in the opposite case.

Unfortunately, we are not able to establish closeness of approximate and exact solutions of the problem as $\mathrm{Re} \to \infty$. A natural approach based on the linearization of the problem in the approximate solution and consequent application of the Kantorovich theorem on convergence of the Newton method does not lead to success, since we do not have sufficient information concerning the linearized operator.

g) In the conclusion, we discuss briefly how to weaken symmetry assumptions in the solution to problem (1.1), (1.2). One of the reasonable ways is to preserve the symmetry of the flow domain but to cancel the symmetry property of the boundary conditions.

For simplicity, let us consider a curvilinear ring $\Omega$, which is symmetric about the line $\{x_2 = 0\}$. Now we will not suppose the symmetry of functions $\mathbf{a}_i$ $(1 = 1, 2)$ in boundary condition (1.2). Let us decompose functions $\mathbf{a}_i$ into symmetric and antisymmetric parts,

$$\mathbf{a}_i = \mathbf{h}_i + \mathbf{g}_i, \quad i = 1, 2. \tag{5.4}$$

Here $\mathbf{h}_i$ is a symmetric function with respect to the line $\{x_2 = 0\}$ in the sense of Definition 1, while $\mathbf{g}_i$ is an antisymmetric one with respect to this line. The latter means that $g_1$ is an odd function of $x_2$ while $g_2$ is an even function of $x_2$.

The solution to problem (1.1), (1.2) is sought in the form

$$\mathbf{v} = \mathbf{u} + \mathbf{w}, \quad p = p_s + p_a. \tag{5.5}$$

Here $\mathbf{u}$ is a symmetric function, $\mathbf{w}$ is an antisymmetric function, $p_s$ is even in variable $x_2$ and $p_a$ is odd in this variable. Substituting (5.4), (5.5) into the system (1.1) and boundary condition (1.2) we obtain as a result of a decomposition procedure:

$$\mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{w} \cdot \nabla \mathbf{w} = -\nabla p_s + \nu \Delta \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0, \quad x \in \Omega, \tag{5.6}$$

$$\mathbf{u} = \mathbf{h}_i(x), \quad x \in \Sigma_i, \ (i = 1, 2), \tag{5.7}$$

$$\mathbf{u} \cdot \nabla \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{u} = -\nabla p_a + \nu \Delta \mathbf{w}, \quad \nabla \cdot \mathbf{w} = 0, \quad x \in \Omega, \tag{5.8}$$

$$\mathbf{w} = \mathbf{g}_i(x), \quad x \in \Sigma_i, \ (i = 1, 2). \tag{5.9}$$

At given $\mathbf{u}$, function $\mathbf{w}$ is determined as the solution of the linear problem (5.8), (5.9). If the corresponding linear operator is convertible and an appropriate norm of $\mathbf{g}_i$ is small, we can prove the solvability of problem (5.6), (5.7). Unfortunately, there are no sufficient conditions for the existence of a unique solvability of the problem (5.8), (5.9). It would be interesting to prove the following statement in view of the result, obtained in paper [12]:

*Let $\Omega$ be a symmetric curvilinear ring with a smooth boundary $\Sigma_1 \cup \Sigma_2$. Let $\mathbf{h}_i \in \mathbf{C}^{2+\alpha}(\Sigma_i)$ be symmetric functions while $\mathbf{g}_i \in \mathbf{C}^{2+\alpha}(\Sigma_i)$ are antisymmetric functions $(i = 1, 2)$. There is a countable subset $N$ of $\mathbf{R}$ such that if $q_1 \notin N$ and $||\mathbf{g}_i||_{C^{2+\alpha}}$ are small, then problem (5.6)–(5.9) has at least one classical solution.*

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# Problems with Insufficient Information about Initial-boundary Data

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**Abstract.** Problems with insufficient information about initial-boundary data are studied in terms of irreducible Chapman–Enskog projection. The existence conditions for Chapman–Enskog projection are formulated in terms of the solvability of matrix equations for which necessary and sufficient existence conditions are obtained.

**UDC.** 517.95

**Keywords.** hyperbolic regularizations, state equation, attracting invariant manifold, the Chapman projection, the Navier-Stokes approximation, matrix equation

## Introduction

In this paper, we consider mathematical aspects arising in the study of the hyperbolic regularizations for the system of conservation laws [4]

$$\partial_t u_i + \mathrm{div}_x\ f^i(u,v) = 0,\ i = 1,\dots,m, \tag{1}$$

$$\partial_t v_k + \mathrm{div}_x\ g^k(u,v) + b^k(u)v = 0,\ k = m+1,\dots,N \tag{2}$$

where $x \in R^n$, $u \in R^m$, $v \in R^{N-m}$, $b$ is an $(N-m)\times(N-m)$ relaxation matrix, $f(u,v)$ and $g(u,v)$ respectively $m\times N$ and $(N-m)\times N$ flow matrices, $u$ are conservative variables, $v$ are not-in-equilibrium variables, and $m$ is the number of conservative variables.

In dealing with from kinetic problems for not-in-equilibrium processes [4], one must do so with insufficient information about initial-boundary data for most not-in-equilibrium variables, which have no intuitive physical sense – they can not be determined from experiment. Furthermore, a number of boundary conditions that could be reasonable from the physical point of view are not sufficient for useful formulations of boundary-value problems [5]. Therefore, it is necessary to understand how the initial and boundary conditions should be interpreted.

The Chapman–Enskog conjecture [3] reads: for well–posed models in continuum mechanics (from the physical point of view) the influence of most not-in-equilibrium variables (higher order moments) is inessential. We do not list different versions of the notion of "well-posedness" from the physical point of view, but discuss the expression "the influence of most not-in-equilibrium variables is inessential". Following [7] this means the following.

**1. Projection**

Nonequilibrium variables are expressed in terms of conservative variables. I.e., there exists an operator correspondence(state equation)

$$v = \mathcal{Q}u, \tag{3}$$

such that the system of projection

$$\partial_t w + \partial_x f(w, \mathcal{Q}u(w)) = 0 \tag{4}$$

into the phase space of conservation variables remains in the class of hyperbolic systems with relaxation (may be pseudodifferential) and solutions $w$ to the Cauchy problem for the system (4) with initial data

$$w|_{t=0} = w^0,$$

define an attracting invariant manifold $\mathcal{M}_{\text{ChEns}}$ of special solutions to the Cauchy problem (1), (2)

$$U_{\text{ChEns}} = (w, \mathcal{Q}w),$$

i.e., there is an operator connecting the initial data of the original system with those of the system of projection

$$w_0 = \mathcal{T}(u_0, v_0)$$

in such a way that the special solutions $U_{\text{ChEns}}$ to the Cauchy problem for (2) determined by solutions to the Cauchy problem for (4) satisfies (in some norm) the following:

$$\|U - U_{\text{ChEns}}\| \ \to \ 0, \ t \to \infty.$$

Moreover, if in the phase space of conservative variables there is

$$w \to 0, \text{ when } t \to \infty \quad \text{(in some norm on the section } t = \text{const)},$$

then the residual $U - U_{\text{ChEns}}$ tends to zero faster than $U_{\text{ChEns}}$.

**2. Separation of dynamics**

Projections that are not representable as the composition of projections must correspond to the basic (characteristic) dynamics of the simulated process. Moreover, if relations for relaxation times are different (i.e., the so-called time relaxation ranges are different, see [6]), then the corresponding attracting invariant manifolds are also different.

By virtue of these two statements the Cauchy problem will be called **correct by Chapman–Enskog** with respect to the projection in the phase space of conservative variables.

In the case of a mixed problem, such researches are more complicated since not all well-posed mixed problems in original system allow us to define a state equation (3). In general, only some subclass of well-posed mixed problems will determine an attracting invariant manifold such as $\mathcal{M}_{\mathrm{ChEns}}$.

Problems such as (1), (2) in which some variables can be treated as inessential are sufficiently more and they are more than just moment approximations of kinetic equations [13], [14]. For example, multiphase mediums yield such problems. In the Bio model for a saturated porous media, dissipation is determined by the interphase friction. Over a long time period, pore pressure plays the role of a conservative variable. The projection in the phase space of this variable is derived from the Darcy law as an approximation to the operator state equation. This allows us to describe fluid injection induced microseismicity in porous rock [16].

**Navier–Stokes approximation.** There are sufficiently more justifications of the Chapman–Enskog conjecture on a physical level (see, for example, [19]). First, the substantive justification to the Chapman–Enskog conjecture (from the mathematical point of view) for $2 \times 2$ systems (1) is made in [4]. In the general case, we obtain an interesting artifact, which reduces to so-called ultraviolet catastrophe [19], [20]. Following [4] consider the regular asymptotic expansion

$$u = u_0 + \varepsilon u_\varepsilon^1 + \cdots$$
$$v = \varepsilon v_\varepsilon^1 + \cdots$$

for the solution of (1), (2) with rigid relaxation

$$\begin{gathered} \partial_t u + \partial_{x_j} f^j(u, v) = 0, \\ \partial_t v + \partial_{x_j} g^j(u, v) + \frac{1}{\varepsilon}\, b(u) v = 0, \end{gathered} \tag{5}$$

where the small parameter $\varepsilon = 1/Kn$, $Kn$ is the Knudsen number. We assume that the matrix $b(u)$ is invertible for all $u \in R^m$. The zero order approximation to a state equation is

$$v_e = 0$$

and the local equilibrium approximation to (4) is:

$$\partial_t u_e + \partial_{x_k} f_k(u_e,\, 0) = 0,$$

The system $(\varepsilon^1)$,

$$\begin{gathered} v_1 = -\varepsilon\, b^{-1}(u_\varepsilon)\partial_{x_k}\, g_k(u_\varepsilon, 0), \\ \partial_t u_\varepsilon + \partial_{x_k} f_k(u_\varepsilon, -\varepsilon\, b^{-1}(u_\varepsilon)\partial_{x_k}\, g_k(u_\varepsilon, 0)) = 0, \end{gathered} \tag{6}$$

is called the Navier–Stokes approximation of (5). It is easy to see that the stability condition for the linearizations of (5) on constants can be expressed as follows: the system (6) is parabolic and the linearizations of (6) are stable on constants. However the next approximations ($\varepsilon^k$, $k \geq 2$), so-called post-Navier–Stokes approximations, are unstable.We emphasize that this happens in spite of the stability of the linearizations of the original system (5) on constants. This phenomenon is referred to as the ultraviolet catastrophe. For the Boltzmann kinetic equation this approximation is the Navier–Stokes equations for incompressible fluid exactly.

What is a reason for this phenomenon? Whether the conjecture of the existence of a projection to the phase space of conservative variables fails or the Navier–Stokes approximations are not sufficiently well justified?

**Example. Hyperbolic regularization of the Hopf equation.** Consider the artifact considered above for the example of the hyperbolic regularization of the Hopf equation:

$$\begin{gathered}\partial_t u + \partial_x v = -u\partial_x u,\\ \partial_t v + \alpha_1 \partial_x u + \partial_x w + \frac{1}{\varepsilon}\beta_1\, v = 0,\\ \partial_t w + \alpha_2 \partial_x v + \frac{1}{\varepsilon}\beta_2\, w = 0;\end{gathered} \tag{7}$$

$1/\beta_j$ are the relaxation times, $\beta_2 > \beta_1 > 0$. The dispersion equation of the linearization of (22) on $u = u_e = 0, v = v_e = 0, w = w_e = 0$, is

$$\begin{aligned} D(\omega,\xi) &= \det\begin{pmatrix} \omega & \xi & 0\\ \alpha_1\xi & \omega - i\beta_1 & \xi\\ 0 & \alpha_2\xi & \omega - i\beta_2\end{pmatrix}\\ &= \omega(\omega - i\beta_1)(\omega - i\beta_2) - \alpha_1\xi^2(\omega - i\beta_2) - \alpha_2\xi^2\omega\\ &= P_0(\omega,\xi) - i\,P_1(\omega,\xi) - P_2(\omega,\xi) = 0\end{aligned} \tag{8}$$

$$P_0 = \omega(\omega^2 - (\alpha_1+\alpha_2)\xi^2),\ P_1 = (\beta_1+\beta_2)\left(\omega^2 - \frac{\alpha_1\beta_1}{(\beta_1+\beta_2)}\xi^2\right),\ P_2 = \beta_1\beta_2\,\omega.$$

The polynomial $D(\omega,\xi)$ under $\omega$ satisfies the following conditions for the stability of a hyperbolic pencil [2]):

1. The Hurwitz condition – the leading coefficients of the homogeneous polynomials $P_j$ are positive when
$$\beta_1 + \beta_2 > 0,\ \beta_1\beta_2 > 0.$$
2. The Sturm condition – the roots of neighboring polynomials of the pencil are strictly separated if
$$\alpha_1 + \alpha_2 > \frac{\alpha_1\beta_1}{\beta_1+\beta_2} > 0.$$

In this case the Navier–Stokes approximation is

$$v = -\varepsilon\frac{\alpha_1}{\beta_1}\partial_x u_\varepsilon,\ w = 0, \qquad \partial_t u_\varepsilon + u_\varepsilon\partial_x u_\varepsilon = \varepsilon\partial_x\left(\frac{\alpha_1}{\beta_1}\,\partial_x u_\varepsilon\right),$$

which dispersion equation

$$\omega = i\varepsilon\frac{\alpha_1}{\beta_1}\,\xi^2$$

is stable, with imaginary part of the root $\operatorname{Im}\omega > 0,\ \forall\xi \neq 0$.

The post-Navier–Stokes approximation ($\varepsilon^2$) is determined as

$$v = -\varepsilon\frac{\alpha_1}{\beta_1}\partial_x u_\varepsilon + \varepsilon^2\frac{\alpha_1}{\beta_1^2}\partial_t\partial_x u_\varepsilon,$$

$$w = \varepsilon^2\frac{\alpha_1\alpha_2}{\beta_1\beta_2}\partial_x^2 u_\varepsilon,$$

$$\partial_t u_\varepsilon + u_\varepsilon\partial_x u_\varepsilon = \varepsilon\partial_x\left(\frac{\alpha_1}{\beta_1}\,\partial_x u_\varepsilon\right) - \varepsilon^2\frac{\alpha_1}{\beta_1^2}\partial_t\partial_x^2 u_\varepsilon$$

such that the dispersion equation

$$\left(1 - \varepsilon^2\frac{\alpha_1}{\beta_1^2}\omega\xi^2\right)\omega = i\varepsilon\frac{\alpha_1}{\beta_1}\,\xi^2$$

is unstable. Such a situation is often observed in quantum mechanics and statistical physics. It is clear that we should take sufficiently many terms in the regular asymptotic expansion. However, the main question is: How many?

## State equation. Linear analysis

This section is devoted to the analysis of the Chapman–Enskog conjecture in the linear case(or for linearized problems). Consider the linear hyperbolic system of $N$ equations with relaxation [4]:

$$\partial_t u + \sum_{j=1}^{n} A_j\partial_{x_j} u + Bu = 0, \tag{9}$$

where $A$ and $B$ are matrices with constant entries such that $b_{ij} = 0$, $i = 1,\dots,m_c$, $j = 1,\dots,N$; $i = 1,\dots,N$, $j = 1,\dots,m_c$, and $m_c$, $1 \le\ m_c <\ N$, is the number of conservative variables, or coinciding with conservative variables are said to be consolidated.

We look for a projection to the phase space of $m$ first equations ($m \ge\ m_c$) in the form of the state equation

$$u = Pu_c,\ P^2 = P. \tag{10}$$

The variables of projection $u_c = (u_1,\dots,u_m,0,\dots,0)^T$ including with conservative variables are said to be consolidated. The matrix pseudodifferential operator $P(\partial_x)$, corresponding to the projection we look for in the form

$$P(\partial_x) = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}, \tag{11}$$

where $P_{11} = E_m$ is the identity matrix of order $m$, $P_{22} = 0_{N-m}$, $P_{12} = 0_{m\times(N-m)}$ is the zero quadratic matrix of order $(N-m)\times(N-m)$ and $m\times(N-m)$ respectively. The resolvent matrix of the system (9)

$$\Lambda(\xi) = \sum_{j=1}^{n} A_j i\xi + B$$

has the form

$$\Lambda = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix}, \tag{12}$$

where $\Lambda_{ij}$ have the same size as the matrices $P_{ij}$. Let's show that the matrix symbol $\Pi(\xi)$ of the projection is a solution of the matrix equation

$$\Pi_{21}(\xi)(\Lambda_{11}(\xi) + \Lambda_{12}(\xi)\Pi_{21}(\xi)) = \Lambda_{21}(\xi) + \Lambda_{22}(\xi)\Pi_{21}(\xi),\ \xi \in R^n \tag{13}$$

which completely determines the projection $P$. Observe that the matrix equation

$$XAX + BX + XC + Q = 0$$

is a nontrivial object(see [28], [29]). For example, we will consider two special $2 \times 2$ cases (13):

$$X^2 = 0,\ X^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

There are infinitely many such matrices in the first case, and they form a two-dimensional cone in $C^4(\det X = 0,\ \mathbf{tr} X = 0)$. There are no solutions to the second equation, since a matrix has only the zero eigenvalue if the squared matrix possesses this property, i.e., $X$ is nilpotent and the squared nilpotent matrix of second-order vanishes.

Observe, that the study of problems with insufficient information about initial-boundary data for partial differential equations is reduced to the solvability problem of the matrix equation depending on many-dimensional parameters. This demands an additional investigation of the solvability problem when a solution depends smoothly on a parameter and satisfies the conditions of the theory of pseudodifferential operators (see [30]).

**The problem reduction to a quadratic matrix equation.** We formally construct a Chapman–Enskog projection in the linear case (9). Since $P$ is a projection, $P^2 = P$, one has

$$P\partial_t u_c + \sum_{j=1}^{n} AP\partial_{x_j} u_c + BP\, u_c = 0, \tag{14}$$

and

$$P\partial_t u_c + P\sum_{j=1}^{n} AP\partial_{x_j} u_c + P\, BP\, u_c = 0. \tag{15}$$

Subtracting (15) from (14), we find:

$$(E - P)(\sum_{j=1}^{n} A\partial_{x_j} + B)P\, u_c = 0. \tag{16}$$

We denote by $\Pi(\xi)$ the Fourier image of $P(\partial_x)$ with respect to $x$. Then (16) can be written in the terms of Fourier images as:

$$(E - \Pi)\Lambda\, \Pi v_c = 0,$$

i.e., $\Lambda(\xi)\,\Pi(\xi)v_c \in Ker(E-P)$, $\forall \xi \in R^n$. Representing $\forall v \in Ker(E-\Pi)$ as $v^T = (v_m^T, v_{N-m}^T)$, $v_k \in \mathbf{R}^k$, we obtain the equality

$$v_{N-m} = \Pi_{21} v_m.$$

Hence we arrive at the system:

$$\Pi_{21}(\Lambda_{11} + \Lambda_{12}\Pi_{21}) = \Lambda_{21} + \Lambda_{22}\Pi_{21} \tag{17}$$

which completely determines the projection $P$. If the Chapman–Enskog projection is to be useful to us for study of mixed problems in the half-plane $x_n > 0$, it needs to express the derivatives $\partial_{x_n} u$ and to pass to the Fourier image from $(t, \xi')$ to $x = (x_n, \xi')$.

**Existence of a matrix solution. Necessary and sufficient conditions.** This section is devoted to the solvability condition for the matrix equation.

**Proposition 1.** *Let a matrix* $\Pi_{21}$ *be a solution*

$$\Pi_{21}\Lambda_{12}\Pi_{21} - \Lambda_{22}\Pi_{21} + \Pi_{21}\Lambda_{11} - \Lambda_{21} = 0 \tag{18}$$

*and* $X = \Lambda\,\Pi$, *where* $\Pi = \begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{pmatrix}$ *is a quadratic matrix of order* $N$, $\Pi_{11}$ *is the identity matrix of order* $m$, *and* $\Pi_{12}$, $\Pi_{22}$ *are zero matrices. Then* $X$ *is a solution to the quadratic matrix equation*

$$X^2 - \Lambda\, X = 0. \tag{19}$$

As we will show below, the matrix equation (19) is simpler than the general matrix equation and it is not difficult to describe one completely. Solutions of the matrix equation (18) correspond to a part of the set of solutions for equation (19) only. So we must define the selection rule. We restrict ourselves to the simple case, when $|\Lambda| \neq 0$.

**Theorem 1.** *Let* $\det \Lambda \neq 0$. *The matrix equation* (18) *is solved then and only then, when there exist two solutions* $X_1$, $X_2$ *for the equation* (19) *such that*

$$\begin{gathered} X_1 e_j = 0 \quad \forall j > m, \\ e_j^T X_2 = e_j^T \Lambda \quad \forall j = 1, \ldots, m, \\ \Lambda\, X_2 = X_1 \Lambda. \end{gathered} \tag{20}$$

*Then* $\Pi = \Lambda^{-1}\ X$.

Our study is based on the following assertion.

**Lemma 1.** *Let* $|\Lambda| \neq 0$. *The matrix* $\Pi$ *has the form*

$$\Pi = \begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{pmatrix},$$

*where* $\Pi_{11}$ *is the identity matrix of order* $m$, $P_{12}$, $\Pi_{22}$ *are zero matrices. Then the quadratic matrix equation* (18) *is solved when and only when there exists a* *matrix* $\Pi$ *of the earlier described form, which is a solution of the quadratic matrix equation*

$$(E - \Pi)\Lambda\,\Pi = 0 \tag{21}$$

*where* $E$ *is the identity matrix.*

Concretely, let us first solve the matrix equation (18). We compute explicitly the product $(E-\Pi)\Lambda\,\Pi$. Since the first m rows of the matrix $(E-P)$ are zero, the first m rows of the product $(E-\Pi)\Lambda\,\Pi$ are also zero. Since the last $N-m$ columns of the matrix $P$ are zero, the last $N-m$ columns of the product $(E-\Pi)\Lambda\,\Pi$ are also zero. For $((E-\Pi)\Lambda\,\Pi)_{21}$ we have

$$((E-\Pi)\Lambda\,\Pi)_{21} = -P_{21}\Lambda_{11} - P_{21}\Lambda_{12}P_{21} + \Lambda_{21} + \Lambda_{22}P_{21} = 0.$$

Conversely, let $\Pi$ be a solution of (21), having the view described above. Thus, $(E-\Pi)\Lambda\,\Pi = 0$. Consequently, the relation (21) is a necessary and sufficient condition for the solvability of the matrix equation (18).

*Proof of Theorem* 1. Let at first the matrix equation (18) be solved, then the matrix equation (21) will be solved. Observe that the matrix $P$ corresponds to the set described above when and only when $Pe_j = 0\ \forall j > m$, $e_j^T P = e_j^T\ \forall j \le m$. Multiplying (21) from the left by $\Lambda$, we see that $X_1 = \Lambda\,P$ is a solution to (19). Analogously, multiplying (21) from the right by $\Lambda$ , we see that $X_2 = P\Lambda$ is a solution to (19). The first and second equations in (20) follow from the structure of the matrix $P$. Further, since $X_1 = \Lambda\,P$, $X_2 = P\Lambda$ the third equation in (20) is fulfilled also.

Now let $X_1$, $X_2$ be two solutions to (19), satisfying the equations 1–3 in (20). Set $P = \Lambda^{-1}X_1 = X_2\Lambda^{-1}$. Then from the equations 1, 2 (20) it follows that the matrix $P$ has the desired view. Next, setting $X_1 = \Lambda\,P$ in (19) and multiplying the obtained expression from the left by $\Lambda^{-1}$, we see that $P$ is a solution to (21).

Describe the solution to $X^2 = \Lambda\,X$. Below we bring some results of [8]:

**Lemma 2.** *Let* $det(\Lambda) \neq 0$ *and the vectors* $h_1, \dots, h_n$ *form a Jordan basis for the matrix* $X$ *which is a solution to* (19). *Then there exists* $K \ge 0$ *such that* $h_1, \dots, h_K$ *are a part of a Jordan basis for the matrix* $\Lambda$ *that preserves the adjunction order* (*i.e., if* $h_j$ *is such that* $Xh_j = \lambda\,h_j + h_{j-1}$, *then* $\Lambda\,h_j = \lambda\,h_j + h_{j-1}$) *and* $h_{K+1}, \dots, h_n$ *are the eigenvectors to the zero eigenvalue.*

Look once more at the geometrical formulation of the necessary and sufficient conditions of the solvability of the quadratic matrix equation (18), when $\det\Lambda \neq 0$. The case $\det\Lambda = 0$ was considered in [10].

**Theorem 2.** *Let* $|\Lambda| \neq 0$. *Suppose besides that there exist vectors* $v_1, \dots, v_m$ *such that:*

1. $V = \mathrm{Lin}\{v_j\}_1^m$ *is a proper subspace to the matrix* $\Lambda$, *i.e.,* $\Lambda\,V = V$.
2. *Vectors* $v_1, \dots, v_m, e_{m+1}, \dots, e_n$ *form a basis.*

*Then and only then the quadratic matrix equation* (18) *is solved.*

**Hyperbolic regularization to the Hopf equation. Reduction to upper block-triangular form.** We continue the study of the system

$$\begin{aligned} \partial_t u + \partial_x v &= -u\partial_x u, \\ \partial_t v + \alpha_1 \partial_x u + \partial_x w + \beta_1\, v &= 0, \\ \partial_t w + \alpha_2 \partial_x v + \beta_2\, w &= 0. \end{aligned} \tag{22}$$

This section is devoted to the proof of the following proposition.

**Proposition 2.** *The Cauchy problem for the system* (22) *is $L_2$-Chapman–Enskog correct, if the following condition for the relaxation time is fulfilled:*

$$\frac{\beta_1}{\beta_2} > \max\left(\frac{2\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2},\ \frac{1}{\alpha_1 + \alpha_3}\right). \tag{23}$$

**Reduction to the block form.** The projection makes possible the reduction of the linearized system (22) to a block-triangular form (existence of the projection and separation of the system (9) by putting it into block-triangular form are equivalent (see [8])). Set

$$S = \begin{pmatrix} E_m & 0 \\ P_{21} & E_{n-m} \end{pmatrix}.$$

We will find the projection conditions in the phase space of the variable $u$(the projection to one equation). Let us show that

$$P_{21} = (q_1(-\partial_x^2)\partial_x, q_2(-\partial_x^2))^\top.$$

Making the change

$$U = S\ (z_1, z_2, z_3)^\top$$

we transform the linearized system (22) to the block-triangular form. Here we have

$$S = \begin{pmatrix} 1 & 0 & 0 \\ q_1(-\partial_x^2)\partial_x & 1 & 0 \\ q_2(-\partial_x^2) & 0 & 1 \end{pmatrix},\ S^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -q_1(-\partial_x^2)\partial_x & 1 & 0 \\ -q_2(-\partial_x^2) & 0 & 1 \end{pmatrix}.$$

Whence, the state equation $U = Z + PZ$ can be written as

$$u = z_1,\ v = q_1(-\partial_x^2)\partial_x z_1 + z_2,\ w = q_2(-\partial_x^2)z_1 + z_3.$$

In the new variables the resolvent matrix $\Lambda(\xi)$ appears as follows:

$$\begin{pmatrix} 1 & 0 & 0 \\ -q_1(-\partial_x^2)\partial_x & 1 & 0 \\ -q_2(-\partial_x^2) & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & \partial_x & 0 \\ \alpha_1\partial_x & \beta_1 & \partial_x \\ 0 & \alpha_2\partial_x & \beta_2 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ q_1(-\partial_x^2)\partial_x & 1 & 0 \\ q_2(-\partial_x^2) & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} q_1\partial_x^2 & \partial_x & 0 \\ -q_1^2\partial_x^3 + \alpha_1\partial_x + \beta_1 q_1\partial_x + q_2\partial_x & -q_1\partial_x + \beta_1 & \partial_x \\ -q_1q_2\partial_x^2 + \alpha_2 q_1\partial_x^2 + \beta_2 q_2 & -q_2\partial_x + \alpha_2\partial_x & \beta_2 \end{pmatrix}.$$

The conditions of the block form are

$$-q_1^2\partial_x^3 + \alpha_1\partial_x + \beta_1 q_1\partial_x + q_2\partial_x = 0, \qquad -q_1q_2\partial_x^2 + \alpha_2 q_1\partial_x^2 + \beta_2 q_2 = 0.$$

In terms of Fourier images we have

$$q_1^2(\xi^2)\xi^2 + \alpha_1 + q_2(\xi^2) + \beta_1 q_1(\xi^2) = 0,$$

$$q_2(\xi^2) = \frac{\alpha_2 \xi^2 q_1(\xi^2)}{(\xi^2 q_1(\xi^2) + \beta_2)}.$$

We introduce the production function

$$Q = \xi^2 q_1(\xi^2),\ Q(0) = 0;$$

then the system for the symbols $q_1, q_2$ reduces to the equation

$$(Q(\xi^2) + \beta_2)(Q(\xi^2) + \beta_1)Q(\xi^2) + \xi^2[\alpha_1(Q(\xi^2) + \beta_2) + \alpha_2 Q(\xi^2)] = 0.$$

The existence condition of a smooth root $Q(|\xi|^2)$(so-called diffusion root) is defined by the inequality

$$\frac{\beta_2}{\alpha_1 + \alpha_3} < \beta_1.$$

From the properties of the diffusion root $Q$ we obtain

$$Q(\xi^2) \ \to\ -\frac{\beta_2}{\alpha_1 + \alpha_3}\ |\xi| \to \infty,$$

$$Q(\xi^2) + \beta_2 \geq c_0 > 0 \quad \text{for any } \xi \in R \text{ for some constant } c_0 > 0.$$

**Block system.** In accordance with the above calculations, we transform the linearized system (22) in the neighborhood of the equilibrium state to the form

$$\partial_t z_1 - Q(-\partial_x^2)z_1 + \partial_x z_2 = 0, \tag{24}$$

$$\partial_t z_2 + (-q_1\partial_x + \beta_1)z_2 + \partial_x z_3 = 0, \tag{25}$$

$$\partial_t z_3 + (-q_2 + \alpha_2)\partial_x z_2 + \beta_2 z_3 = 0$$

with initial data

$$z_1^0 = u_0,\ z_2^0 = v_0 - q_1(-\partial_x^2)\partial_x u^0,\ z_3^0 = w_0 - q_2(-\partial_x^2)u^0.$$

The conditions

$$v_0 - q_1(-\partial_x^2)\partial_x u^0 = 0,\ w_0 - q_2(-\partial_x^2)u^0 = 0$$

define the invariant manifold

$$\mathcal{M}_{\mathrm{ChEns}} = \{Z = (z_1, 0, 0),\ z_1|_{t=0} = u^0,\ z_2|_{t=0} = 0,\ z_3|_{t=0} = 0\}$$

of the solutions to the Cauchy problem for the system (24), (25). Then there are

$$v_0 - q_1(-\partial_x^2)\partial_x u^0 \neq 0,\ w_0 - q_2(-\partial_x^2)u^0 \neq 0.$$

We must find the existence conditions of a corrector

$$z_{\mathrm{cor}}^1 = \mathcal{T}(v_0, w_0)$$

such that the solution $Z_{\mathrm{cor}} \in \mathcal{M}_{\mathrm{ChEns}}$ to the Cauchy problem (24), (25) with the initial data

$$Z_{\mathrm{cor}}|_{t=0} = (z_{\mathrm{cor}}^1, 0, 0)$$

is compatible with the solutions $U$ for (24), (25) with the initial data

$$z_1|_{t=0} = 0,\ z_2|_{t=0} = v_0 - q_1(-\partial_x^2)\partial_x u^0,\ z_3|_{t=0} = w_0 - q_2(-\partial_x^2)u^0,$$

i.e., in a appropriate norm we have

$$\|U - Z_{\rm cor}\| \to\ 0,\ t \to \infty.$$

**The solution structure.** The general solution is

$$\begin{aligned} U &= (z_1,\ q_1(-\partial_x^2)\partial_x z_1 + z_2,\ q_2(-\partial_x^2)z_1 + z_3)^\top \\ &= (z_1,\ q_1(-\partial_x^2)\partial_x z_1,\ q_2(-\partial_x^2)z_1)^\top + (0,\ z_2,\ z_3)^\top. \end{aligned}$$

Now compare $U_1 = (z_1,\ q_1(-\partial_x^2)\partial_x z_1,\ q_2(-\partial_x^2)z_1)^\top$ and the spatial solution

$$U_{\rm ChEns}(x,t) = (z,\ q_1(-\partial_x^2)\partial_x z,\ q_2(-\partial_x^2)z)^\top$$

where $z$ is the solution to the Cauchy problem for the projection equation

$$\begin{aligned} \partial_t z - Q(-\partial_x^2)z &= 0, \\ z|_{t=0} &= z^0. \end{aligned} \tag{26}$$

Nonequilibrium variables are unessential if the solution to the Cauchy problem for (22) is attracted to a special solution. In the Fourier image one has

$$\begin{aligned} \widetilde{z_1}(t,\xi) &= e^{Q(\xi^2)\,t}[\widetilde{u_0} - i\int_0^t\ e^{-Q(\xi^2)\,s}\xi\widetilde{z_2}(s,\xi)ds] \\ &= e^{Q(\xi^2)\,t}[\widetilde{u_0} - i\int_0^\infty\ e^{-Q(\xi^2)\,s}\xi\widetilde{z_2}(s,\xi)ds] + i\int_t^\infty\ e^{Q(\xi^2)\,(t-s)}\xi\widetilde{z_2}(s,\xi)ds \end{aligned}$$

if the integral

$$\int_0^\infty\ e^{-Q(\xi^2)\,s}\xi\widetilde{z_2}(s,\xi)ds$$

is finite (the crack condition). Then

$$e^{Q(\xi^2)\,t}[\widetilde{u_0} - i\int_0^\infty\ e^{-Q(\xi^2)\,s}\xi\widetilde{z_2}(s,\xi)ds]$$

belongs to the invariant manifold

$$\mathcal{M}_{\rm ChEns} = \{U_{\rm ChEns}(x,t) = (z,\ q_1(-\partial_x^2)\partial_x z,\ q_2(-\partial_x^2)z)^\top\}$$

of spatial solutions

$$U_{\rm ChEns}(x,t) = (z,\ q_1(-\partial_x^2)\partial_x z,\ q_2(-\partial_x^2)z)^\top, \tag{27}$$

and is defined by initial data

$$z^0 = u_0 - i\int_{R^1} e^{i(x,\xi)}\int_0^\infty\ e^{-Q(\xi^2)\,s}\xi\widetilde{z_2}(s,\xi)dsd\xi.$$

So we must find the condition when

$$\begin{aligned} U^\perp &= (0,\ z_2,\ z_3)^\top + (z^\perp,\ q_1(-\partial_x^2)\partial_x z^\perp,\ q_2(-\partial_x^2)z^\perp)^\top \\ &= e^{Q\,t}\Big[(0,\ e^{-Q\,t}z_2,\ e^{-Q\,t}z_3)^\top + (1,\ q_1(-\partial_x^2)\partial_x,\ q_2(-\partial_x^2))^\top\, e^{-Q\,t}z^\perp\Big] \end{aligned}$$

tends to zero faster than (27), where

$$e^{-Q\,t}z^{\perp} = i\int_t^{\infty} e^{-Q(\xi^2)\,s}\xi\widetilde{z_2}(s,\xi)ds.$$

It is sufficient that functions

$$e^{-Q\,t}z_2,\ e^{-Q\,t}z_3,\ i\int_t^{\infty} e^{-Q(\xi^2)\,s}\xi\widetilde{z_2}(s,\xi)ds$$

tend to zero in an appropriate norm.

**The crack condition.** Note that $\omega = -iQ$ is the purely imaginary root of the dispersion equation (33). One has

$$D(-iQ) = Q(Q^2 + (\alpha_1+\alpha_2)\xi^2) + ((\beta_1+\beta_2)Q^2 + \alpha_1\beta_1\xi^2) + \beta_1\beta_2 Q = 0. \quad (28)$$

As may be seen, we obtained the equation for the production function $Q$. Let us show that two other roots (28), different from the real root $Q$, are found to the left from the vertical line $\operatorname{Re} Q = -\frac{\alpha_1\beta_2}{\alpha_1+\alpha_2}$ for any $\xi \in R$. Set

$$Q = -\frac{\alpha_1\beta_2}{\alpha_1+\alpha_2} + iZ,\ \forall Z \in R.$$

The equality to zero of the imaginary part $\operatorname{Im} P(-\frac{\alpha_1\beta_2}{\alpha_1+\alpha_2)} + iZ) = 0$ reduces to the equation

$$Z^2\left(\beta_1+\beta_2 - \frac{3\alpha_1\beta_2}{\alpha_1+\alpha_2}\right) + \frac{\alpha_1\alpha_2\beta_2^2}{(\alpha_1+\alpha_2)^2}\left(\beta_1 - \frac{\alpha_1\beta_2}{\alpha_1+\alpha_2}\right) = 0$$

which has purely imaginary roots, if

$$\beta_1+\beta_2 - \frac{3\alpha_1\beta_2}{\alpha_1+\alpha_2} > 0. \quad (29)$$

Then two other roots of (28), different from the real root $Q$, lie on the left from the vertical $\operatorname{Re} Q = -\frac{\alpha_1\beta_2}{\alpha_1+\alpha_2}$ for all $\xi \in R$. So that the condition (29) is the sufficient condition when the set of special solution (27) is attracting and the nonequilibrium variables $(v,w)$ are unessential to the Cauchy problem for the system (22) linearized in a neighborhood of the equilibrium state $u_e = v_e = w_e = 0$.

**Observation 1.** *Now we can explain the mathematical nature of so-called ultraviolet catastrophe. Making the transformation $\beta_1 \to \beta/\varepsilon$, $\beta_1 \to \beta/\varepsilon$ in the hyperbolic regularization for the Hopf equation we reduce the operator $Q(-\Delta)$ in the projection equation* (26) *to the operator $Q(-\varepsilon^2\Delta)$. Also, we obtain $U_\varepsilon = S_\varepsilon Z_\varepsilon$, whose regular expansions lead to the Navier–Stokes and post-Navier–Stokes approximation. We now have*

$$S_\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ q_1(-\varepsilon^2\partial_x^2)\partial_x & 1 & 0 \\ q_2(-\varepsilon^2\partial_x^2) & 0 & 1 \end{pmatrix}.$$

*The instability problem of the post-Navier–Stokes approximations connects with the bad approximation of the symbol $Q(\xi^2)$, which is the so-called kink function with an inner layer, by polynomials.*

## Chapman–Enskog projection and Schrödinger approximation

Now we consider the hyperbolic regularization [21,22] to the Maxwell system [25] which possesses a number of remarkable properties:

$$i\alpha\frac{\partial U}{\partial t} = \operatorname{rot} U - \beta\nabla\ \varrho - f, \tag{30}$$
$$i\alpha_1\frac{\partial \rho}{\partial t} = \operatorname{div} U - \gamma_1\rho,$$

or

$$i\partial_t\begin{pmatrix} U \\ \varrho \end{pmatrix} + A_j\partial_{x_j}\begin{pmatrix} U \\ \varrho \end{pmatrix} + B\begin{pmatrix} U \\ \varrho \end{pmatrix} = 0$$

where $\alpha,\ \alpha_1,\ \beta$ are positive real constants and $\gamma_1 = \mu_1 + i\mu_2,\ \mu_j > 0.$

**Connection of the hyperbolic regularization to the Maxwell system with basic equations in quantum mechanics.** Setting $\alpha_1 = \beta = 0$ in (30) we obtain the Maxwell system

$$i\alpha\frac{\partial U}{\partial t} = \text{rot}\ \ U - f, \quad \text{div}\ \ U = \gamma_1\rho.$$

Indeed, we introduce the notation $\alpha = \sqrt{\varepsilon\mu}/c$,

$$U = U_1 + iU_2, \quad U_1 = \sqrt{\varepsilon}E, \quad U_2 = \sqrt{\mu}H;$$

$$f = f_1 + if_2, \quad f_1 = -\frac{4\pi}{c}\ \sqrt{\varepsilon}\ j_m, \quad f_2 = \frac{4\pi}{c}\ \sqrt{\mu}\ j_e;$$

where

$$\rho = \rho_1 + i\rho_2,\ \ \rho_1 = \frac{4\pi}{\sqrt{\varepsilon}}\ \rho_e,\ \ \rho_2 = \frac{4\pi}{\sqrt{\mu}}\ \rho_m.$$

Thus we obtain a "symmetrized" Maxwell system, which differs from the classical one [25] by the presence of the "magnetic" charge $\rho_m$, introduced by Dirac and "magnetic" flow $j_m$, introduced by Schwinger. The system (30) has the same number of equations as the Dirac system, but they are not equivalent nevertheless. We show that the former system is closely connected with the Schrödinger equation. Depending on the ratio $\gamma_1 = 1/\varepsilon,\ 0 < \varepsilon \ll 1$, we introduce approximations, called Schrodinger approximations, which are similar to Navier–Stokes approximations [4]. We use the method of regular asymptotic expansions. Then from the second equation (30) we find for $\varrho$ the state equation in the first approximation, connecting the nonequilibrium variable and the conservation variables

$$\varrho = \varepsilon \operatorname{div} U. \tag{31}$$

From the first three equations (30) for the divergence to the potential part of the solution we find the first approximation

$$i\alpha\partial_t \operatorname{div} U + \varepsilon\beta\Delta \operatorname{div} U + \operatorname{div} f = 0. \tag{32}$$

The system (31), (32) is the Schrödinger approximation. From this case arise problems analogous to the problems of the ultraviolet catastrophe [6, 19].

**Reduction to block form.** Prove the existence of a projection in the phase space of the variables $U$. The dispersion equation of the system (30) is

$$D_4 = ((\alpha\omega)^2 - |\xi|^2)\;(\alpha\omega(\alpha_1\omega - \gamma_1) - \beta|\xi|^2) = 0. \tag{33}$$

1. The second factor

$$P_2 = \alpha\alpha_1\omega^2 - \alpha\gamma_1\omega - \beta|\xi|^2 = (\alpha\alpha_1\omega^2 - \alpha\mu_1\omega - \beta|\xi|^2) - i\alpha\mu_2\omega = 0$$

is stable (see [2], i.e., the imaginary part of roots $\operatorname{Im}\omega_j(|\xi|^2) > 0$, $j = 3, 4$), since this is a nonstrict hyperbolic pencil, whose homogeneous parts are strictly hyperbolic, and whose roots are separated from each other:

$$\begin{aligned}\omega_3(|\xi|^2) &= \frac{1}{2\alpha\alpha_1}\left[\alpha\mu_1 - \sqrt{\alpha^2\mu_1^2 + 4\alpha\alpha_1\beta|\xi|^2}\right] < 0\\ < \omega_4(|\xi|^2) &= \frac{1}{2\alpha\alpha_1}\left[\alpha\mu_1 + \sqrt{\alpha^2\mu_1^2 + 4\alpha\alpha_1\beta|\xi|^2}\right],\ \forall|\xi| > 0.\end{aligned}$$

For $\xi = 0$ we have

$$\omega_3(0) = 0 < \omega_4(0) = \frac{\mu_1}{\alpha_1}.$$

The second factor does not have multiple roots since

$$(\alpha\gamma_1)^2 + 4\alpha\alpha_1\beta|\xi|^2 \neq 0,$$

if $\mu_1 \neq\ 0$.

2. There are two wave roots $\omega_\pm(|\xi|) = \pm|\xi|/\alpha$ of the first factor.

3. The factors are the common root $\omega(0) = 0$ when $|\xi| = 0$. For any $|\xi| \neq 0$ there is a common root $\omega_*$, if

$$(\alpha_1\omega_* - \gamma_1) - \beta\alpha\omega_* = (\alpha_1 - \beta\alpha)\omega_* - \gamma_1 = 0.$$

which is impossible, if $\alpha_1 - \beta\alpha = 0$. When $\alpha_1 - \beta\alpha \neq 0$ we obtain

$$|\xi_*|^2 = (\alpha\omega_*)^2 = \frac{\alpha^2\gamma_1^2}{(\alpha_1 - \beta\alpha)^2}$$

that contradicts $\operatorname{Im}\gamma_1 > 0$.

So that, for any $\xi \in R^3$ there are four eigenvectors to the resolvent matrix

$$\Lambda(\xi) = A_j i\xi_j + B = \begin{pmatrix} 0 & \dfrac{i\xi_3}{\alpha} & -\dfrac{i\xi_2}{\alpha} & \dfrac{i\beta\xi_1}{\alpha}\\ -\dfrac{i\xi_3}{\alpha} & 0 & \dfrac{i\xi_1}{\alpha} & \dfrac{i\beta\xi_2}{\alpha}\\ \dfrac{i\xi_2}{\alpha} & -\dfrac{i\xi_1}{\alpha} & 0 & \dfrac{i\beta\xi_3}{\alpha}\\ -i\dfrac{\xi_1}{\alpha_1} & -i\dfrac{\xi_2}{\alpha_1} & -i\dfrac{\xi_3}{\alpha_1} & \dfrac{\gamma_1}{\alpha_1}\end{pmatrix}, \tag{34}$$

corresponding to four roots of the dispersion equation. The results shown above about the solvability of the matrix equation say that in this case there can be a Chapman–Enskog projection in the phase space of the variables $U$. Our goal is the proof of such projector existence.

**Proposition 3.** *The Cauchy problem for the system* (30) *is $L_2$-Chapman–Enskog correct with respect to the projection in the phase space of the variables $U$.*

We will look for the operator transformation

$$(U, \varrho)^\top = S(\nabla_x)(V, r)^\top,$$

which reduces (30) to an upper block-triangular form. Then the symbol

$$S_{21}(\xi) = (i\xi_1 R_1(\xi),\ i\xi_2 R_2(\xi),\ i\xi_3 R_3(\xi)),$$

where $S_{11} = E$ is the identity matrix of $3 \times 3$ order and $S_{22} = 1$, $S_{12} = 0$ is the zero matrix of $3 \times 1$ order. We introduce the production function $Q(\xi) = \sum\limits_{j=1}^{3} \xi_j^2 R_j$. Then in terms of Fourier images we can write

$$(S^{-1}(\xi)\,\Lambda\,S)_{22} = \frac{Q + \frac{\alpha}{\alpha_1}\gamma_1}{\alpha},$$

$$(S^{-1}\,\Lambda\,S)_{11} = \begin{pmatrix} -\frac{\beta}{\alpha}\xi_1^2 R_1 & \frac{i\xi_3}{\alpha} - \frac{\beta}{\alpha}\xi_1\xi_2 R_2 & -\frac{i\xi_2}{\alpha} - \frac{\beta}{\alpha}\xi_1\xi_3 R_3 \\ -\frac{i\xi_3}{\alpha} - \frac{\beta}{\alpha}\xi_1\xi_2 R_1 & -\frac{\beta}{\alpha}\xi_2^2 R_2 & \frac{i\xi_1}{\alpha} - \frac{\beta}{\alpha}\xi_2\xi_3 R_3 \\ \frac{i\xi_2}{\alpha} - \frac{\beta}{\alpha}\xi_1\xi_3 R_1 & -\frac{i\xi_1}{\alpha} - \frac{\beta}{\alpha}\xi_2\xi_3 R_2 & -\frac{\beta}{\alpha}\xi_3^2 R_3 \end{pmatrix}, \tag{35}$$

$$(S^{-1}(\xi)\,\Lambda\,S)_{12} = \begin{pmatrix} i\frac{\beta\xi_1}{\alpha} & i\frac{\beta\xi_2}{\alpha} & i\frac{\beta\xi_3}{\alpha} \end{pmatrix}^\top, \tag{36}$$

$$\begin{aligned} (S^{-1}(\xi)\,\Lambda\,S)_{21} &= (\pi_1,\ \pi_2,\ \pi_3), \\ \pi_1 &= -i\xi_1 R_1\left(-\frac{\beta}{\alpha}\xi_1^2 R_1\right) - i\xi_2 R_2\left(-i\frac{\xi_3}{\alpha} - \frac{\beta}{\alpha}\xi_1\xi_2 R_1\right) \\ &\quad - i\xi_3 R_3\left(i\frac{\xi_2}{\alpha} - \frac{\beta}{\alpha}\xi_1\xi_3 R_1\right) - i\frac{\xi_1}{\alpha_1} + i\frac{\gamma_1}{\alpha_1}\xi_1 R_1, \\ \pi_2 &= -i\xi_1 R_1\left(i\frac{\xi_3}{\alpha} - \frac{\beta}{\alpha}\xi_1\xi_2 R_2\right) - i\xi_2 R_2\left(-\frac{\beta}{\alpha}\xi_2^2 R_2\right) \\ &\quad - i\xi_3 R_3\left(-i\frac{\xi_1}{\alpha} - \frac{\beta}{\alpha}\xi_2\xi_3 R_2\right) - i\frac{\xi_2}{\alpha_1} + i\frac{\gamma_1}{\alpha_1}\xi_2 R_2, \\ \pi_3 &= -i\xi_1 R_1\left(-i\frac{\xi_2}{\alpha} - \frac{\beta}{\alpha}\xi_1\xi_3 R_3\right) - i\xi_2 R_2\left(i\frac{\xi_1}{\alpha} - \frac{\beta}{\alpha}\xi_3\xi_2 R_3\right) \\ &\quad - i\xi_3 R_3\left(-\frac{\beta}{\alpha}\xi_3^2 R_3\right) - i\frac{\xi_3}{\alpha_1} + i\frac{\gamma_1}{\alpha_1}\xi_3 R_3. \end{aligned}$$

The system (30) is reduced to the upper block-triangular form, if

$$(S^{-1}(\xi)\,\Lambda\,S)_{21} = 0. \tag{37}$$

Then we obtain the projector in the phase space of the variables $U$, when the system (30) is transformed to

$$i\alpha\,\partial_t V - \operatorname{rot} V + \beta\, B_1 V + \beta\nabla r + f = 0, \tag{38}$$

$$i\alpha_1\partial_t r + (\frac{\beta\alpha_1}{\alpha}\, Q(-\Delta) + \gamma_1(-\Delta))\, r = \frac{\alpha_1}{\alpha}(\partial_{x_1} R_1 f_1 + \partial_{x_2} R_2 f_2 + \partial_{x_3} R_3 f_3). \tag{39}$$

Here the Fourier image of $B_1$ has the form

$$\xi^T\xi \begin{pmatrix} R_1 & 0 & 0 \\ 0 & R_2 & 0 \\ 0 & 0 & R_3 \end{pmatrix}. \tag{40}$$

Making the change $X_j = \xi_j R_j$ we reduce the system (37) to the equations

$$\begin{aligned} X_1\left(\frac{\beta}{\alpha}\xi_1 X_1 + \frac{\gamma_1}{\alpha_1}\right) - X_2\left(-i\frac{\xi_3}{\alpha} - \frac{\beta}{\alpha}\xi_2 X_1\right) - X_3\left(i\frac{\xi_2}{\alpha} - \frac{\beta}{\alpha}\xi_3 X_1\right) &= \frac{\xi_1}{\alpha_1}, \\ -X_1\left(i\frac{\xi_3}{\alpha} - \frac{\beta}{\alpha}\xi_1 X_2\right) + X_2\left(\frac{\beta}{\alpha}\xi_2 X_2 + \frac{\gamma_1}{\alpha_1}\right) - X_3\left(-i\frac{\xi_1}{\alpha} - \frac{\beta}{\alpha}\xi_3 X_2\right) &= \frac{\xi_2}{\alpha_1}, \\ -X_1\left(-i\frac{\xi_2}{\alpha} - \frac{\beta}{\alpha}\xi_1 X_3\right) - X_2\left(i\frac{\xi_1}{\alpha} - \frac{\beta}{\alpha}\xi_2 X_3\right) + X_3\left(\frac{\beta}{\alpha}\xi_3 X_3 + \frac{\gamma_1}{\alpha_1}\right) &= \frac{\xi_3}{\alpha_1}. \end{aligned} \tag{41}$$

Whence, for the production function $Q = \xi_1 X_1 + \xi_2 X_2 + \xi_3 X_3$ we obtain

$$P(Q) = \beta\alpha_1\, Q^2 + \gamma_1(|\xi|^2)\alpha Q - \alpha\,|\xi|^2 = 0,\ Q(0) = 0. \tag{42}$$

Then

$$\begin{aligned} X_1\left(\frac{\beta}{\alpha}Q + \frac{\gamma_1}{\alpha_1}\right) + iX_2\frac{\xi_3}{\alpha} - iX_3\frac{\xi_2}{\alpha} &= \frac{\xi_1}{\alpha_1}, \\ -iX_1\frac{\xi_3}{\alpha} + X_2\left(\frac{\beta}{\alpha}Q + \frac{\gamma_1}{\alpha_1}\right) + iX_3\frac{\xi_1}{\alpha} &= \frac{\xi_2}{\alpha_1}, \\ +iX_1\frac{\xi_2}{\alpha} - iX_2\frac{\xi_1}{\alpha} + X_3\left(\frac{\beta}{\alpha}Q + \frac{\gamma_1}{\alpha_1}\right) &= \frac{\xi_3}{\alpha_1} \end{aligned} \tag{43}$$

where

$$X_j = \frac{\xi_j}{\alpha_1\left(\frac{\beta}{\alpha}Q + \frac{\gamma_1}{\alpha_1}\right)} \ \Rightarrow\ R_j = \frac{1}{\alpha_1\left(\frac{\beta}{\alpha}Q + \frac{\gamma_1}{\alpha_1}\right)},\ j = 1, 2, 3,$$

are symbols of order zero. The denominator is

$$\left(\frac{\beta}{\alpha}Q + \frac{\gamma_1}{\alpha_1}\right) \neq\ 0,\ \forall|\xi| \geq 0,$$

on solutions of (42), since we have

$$P(Q)|_{Q=-\frac{\gamma_1\alpha}{\alpha_1\beta}} = \beta\alpha_1\left(\frac{\gamma_1\alpha}{\alpha_1\beta}\right)^2 - \gamma_1\alpha\frac{\gamma_1\alpha}{\alpha_1\beta} - \alpha\,|\xi|^2 = -\alpha\,|\xi|^2 \neq 0,\ \xi \neq 0,$$

where $\gamma_1(0) \neq 0$. The dispersion equation of (38), (39) is

$$D_4(\omega) = D_3(\omega)\left(\alpha_1\omega - \left(\frac{\beta\alpha_1}{\alpha}\, Q(-\Delta) + \gamma_1\right)\right),$$

where

$$D_3(\omega) = ((\alpha\omega)^2 - |\xi|^2)\,\left(\omega - \frac{1}{2\alpha\alpha_1}\left(\alpha\gamma_1 - \sqrt{\alpha^2\gamma_1^2 + 4\alpha\alpha_1\beta|\xi|^2}\right)\right).$$

Concretely, $Q(\xi^2) = \frac{\alpha}{\beta}\,\omega_4(\xi) - \frac{\beta\gamma_1}{\alpha_1\alpha}$, where $\omega_4$ is the boundary layer root of the dispersion equation (33). Setting this expression in the equation for the product function we have

$$P(Q)|_{Q=\frac{\alpha}{\beta}\,\omega_4 - \frac{\beta\gamma_1}{\alpha_1\alpha}} = \frac{\alpha_1\alpha^2}{\beta}\omega_4^2 - \frac{\alpha^2\gamma_1}{\beta}\,\omega_4 - \alpha\,|\xi|^2$$
$$= \frac{\alpha}{\beta}[\alpha\alpha_1\omega_4^2 - \alpha\gamma_1\omega_4 - \beta|\xi|^2].$$

So we have obtained the second factor of the dispersion equation (33).

**Observation 2.** *Therefore, we have proved the existence of the smooth transformation $S(\partial_x)$, which reduces* (30) *to the block-triangular form* (38), (39). *Whence, it follows that the eigenvectors $R_\pm(\xi), R_3(\xi)$ of the resolvent matrix $\Lambda(\xi)$, corresponding to the roots $\omega_\pm(|\xi|), \omega_3(|\xi|^2)$ of the dispersion equation* (33), *satisfy the solvability condition*

$$\mathrm{Lin}\{V, e_4\} = \mathbb{R}^4,$$

*for the matrix equation of the projection in the phase space of the variables $U$, where $V$ is the proper subspace of the eigenvectors $R_\pm(\xi), R_3(\xi)$*

**Unessentialness of the nonequilibrium variable $r$.** Now we prove the unessentialness of the nonequilibrium variable $r$. For this we need to prove that the set of special solutions for (30) is an **attracted invariant manifold**. For simplicity consider the case when there are no exterior forces $f = 0$. Then we have invariant manifolds:

1. $\mathcal{M}_{\mathrm{ChEns}}$ is determined by special solutions $(U_{\mathrm{ChEns}}, \rho_{\mathrm{ChEns}})$ to the Cauchy problem (30):

$$(U_{\mathrm{ChEns}}, \rho_{\mathrm{ChEns}}) = (W_1, W_2, W_2, \partial_{x_1}R_1(\partial_x)W_1 + \partial_{x_2}R_1(\partial_x)W_2 + \partial_{x_3}R_3(\partial_x)W_3),$$

where

$$i\alpha\;\partial_t W - \mathrm{rot}\, W + \beta\, B_1 W = 0, \tag{44}$$

$$W|_{t=0} = U^0. \tag{45}$$

Observe, that $B_1 W = -i\nabla_x R(-\Delta)\,\mathrm{div}\, W$ is a potential field. Whence we have $W = \nabla_x\Psi,\; W^0 = \nabla_x\Psi^0$, where

$$i\alpha\;\partial_t\Psi + \beta Q(-\Delta)\Psi = 0, \tag{46}$$

$$\Psi|_{t=0} = \Psi^0 \tag{47}$$

and the solution to the Cauchy problem is described as

$$\Psi = e^{it\frac{\beta}{\alpha}Q(-\Delta)}\Psi^0. \tag{48}$$

2. Next, the invariant manifold $\mathcal{M} = \{(U,\rho) = S(V,r)\}$ coincides with the solutions of (38), (39)

$$i\alpha\, \partial_t V - \operatorname{rot} V + \beta\, B_1 V + \beta \nabla r = 0, \tag{49}$$

$$i\alpha_1 \partial_t r + \left(\frac{\beta\alpha_1}{\alpha}\, Q(-\Delta) + \gamma_1\right) r = 0, \tag{50}$$

$$V^0 = 0,\ r^0 = \varrho^0 - \partial_{x_1} R_1(\partial_x) U_1^0 + \partial_{x_2} R_1(\partial_x) U_2^0 + \partial_{x_3} R_3(\partial_x) U_3^0 \tag{51}$$

with initial data

$$V^0 = 0,\ r^0 = \varrho^0 - \partial_{x_1} R_1(\partial_x) U_1^0 + \partial_{x_2} R_1(\partial_x) U_2^0 + \partial_{x_3} R_3(\partial_x) U_3^0.$$

Hence, the solution of (49) is potential, i.e.:

$$V = \nabla_x \Psi,$$

where

$$i\alpha\, \partial_t \Psi + \beta Q(-\Delta)\Psi + \beta\, r = 0, \tag{52}$$

$$i\alpha_1 \partial_t r + \left(\frac{\beta\alpha_1}{\alpha}\, Q(-\Delta) + \gamma_1\right) r = 0, \tag{53}$$

with the initial data

$$\Psi^0 = 0,\ r^0 = \varrho^0 - \partial_{x_1} R_1(\partial_x) U_1^0 + \partial_{x_2} R_1(\partial_x) U_2^0 + \partial_{x_3} R_3(\partial_x) U_3^0.$$

Then for $\operatorname{div} V$ we have

$$i\alpha\, \partial_t \operatorname{div} V - \beta Q(-\Delta) \operatorname{div} V + \beta \Delta r = 0, \tag{54}$$

$$i\alpha_1 \partial_t r + \left(\frac{\beta\alpha_1}{\alpha}\, Q(-\Delta) + \gamma_1\right) r = 0. \tag{55}$$

Here we used that $\Delta R(-\Delta) = Q(-\Delta)$.

The question is when the manifold $\mathcal{M}_{\mathrm{ChEns}}$ of the special solutions will be attractive to the solutions of the Cauchy problem $(U,\rho) = S(\nabla_x \Psi, r)$. Observe, that for potential solutions of the projection system (44) we have

$$i\alpha\, \partial_t \operatorname{div} W + \beta Q(-\Delta) \operatorname{div} W = 0, \tag{56}$$

$$\operatorname{div} W|_{t=0} = \operatorname{div} U^0. \tag{57}$$

Whence, it follows that to solve this problem we need to show that the solution of the nonhomogeneous Cauchy problem (52) is represented in the form

$$\operatorname{div} V = e^{it\frac{\beta}{\alpha}Q(-\Delta)} \left(\operatorname{div} V^0_{\mathrm{cor}} + o\left(\frac{1}{t}\right)\right). \tag{58}$$

Also we have

$$r = e^{it(\frac{\beta}{\alpha}\, Q(-\Delta) + \frac{\gamma_1}{\alpha_1})} r^0 = e^{it\frac{\beta}{\alpha}Q(-\Delta)} \left[e^{it\frac{\gamma_1}{\alpha_1}} r^0\right].$$

Using the Duhamel principle we write $V$ – components of the solution to the Cauchy problem (51) –

$$V = e^{it\frac{\beta}{\alpha}Q(-\Delta)} \int_0^t e^{is\frac{\gamma_1}{\alpha_1}} r^0 ds = -ie^{it\frac{\beta}{\alpha}Q(-\Delta)} \frac{\alpha_1}{\gamma_1} \left( e^{it\frac{\gamma_1}{\alpha_1}} - 1 \right)$$

$$= e^{it\frac{\beta}{\alpha}Q(-\Delta)} \left[ i\frac{\alpha_1}{\gamma_1} r^0 - i\frac{\alpha_1}{\gamma_1} e^{it\frac{\gamma_1}{\alpha_1}} r^0 \right].$$

Compare this with (48). We see that we will obtain the desired result if we set $\operatorname{div} V^0 = -\frac{\alpha_1}{\gamma_1} r^0$ and take into account that

$$\operatorname{Re}\left( i\frac{\gamma_1}{\alpha_1} \right) = -\frac{\mu_2}{\alpha_1}.$$

Then we obtain in (58) the exponential decreasing $o(\frac{1}{t}) = O(e^{-\delta t})$.

## Auxiliary propositions

In this section we will formulate, with an explanation of the proofs, some technical results, which are necessary for a verification of the separation dynamics under the projection in the phase space of the conservative variables.

**The Liapunov equation and the corrector construction.** Note that the Liapunov matrix equation

$$-M_{11}Q_{12} + Q_{12}M_{22} - M_{12} = 0 \tag{59}$$

is a partial case of (18), when the quadratic part is strictly equal to zero. Then we have the following result:

**Theorem 3.** *Let* $\det(M_{11}) \neq 0$, $\det(M_{22}) \neq 0$ *such that the matrix*

$$M = \begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix}$$

*has no eigenvalue* $\lambda$ *for which, after subdivision into blocks of corresponding size, the eigenvector to* $\lambda$ *has the form* $v_0 = \begin{pmatrix} v_{0,1} \\ 0 \end{pmatrix}$, *and a corresponding adjoint vector is* $v_1 = \begin{pmatrix} v_{1,1} \\ v_{1,2} \end{pmatrix}$, *where* $v_{1,2} \neq 0$. *Then there exists a solution* $Q_{12}$ *of the equation* (59), *corresponding to the matrixes* $M_{11}$, $M_{12}$, $M_{22}$.

Consider the equation (59) as a partial case of a matrix equation. Then the matrix $\Lambda$ corresponding to this equation is obtained from the matrix $M$ by a permutation of lines and columns:

$$\Lambda = \begin{pmatrix} M_{22} & 0 \\ M_{12} & M_{11} \end{pmatrix}.$$

Since $\det(M_{11}) \neq 0$, $\det(M_{22}) \neq 0$, then $det(\Lambda) \neq 0$. The solvability of the equation (59) follows from Theorem 2, if the system of vectors is reduced from the Jordan basis to $\Lambda$ by the receding of vectors $\begin{pmatrix} 0 \\ v \end{pmatrix}$, determining the basis of the proper subspace for the matrix $\Lambda$ (see the proof of this theorem considered below and in [27]). The answer to the question when the matrix $\Lambda$ can be reduced to the block-diagonal form, is given by the following result.

**Theorem 4.** *Let the matrix $\Lambda$ be invertible such that there exist the basis $v_1, \dots, v_m$ of the proper subspace $V$ and $\mathrm{Lin}\{v_1, \dots, v_m, e_{m+1}, \dots, e_n\} = \mathbb{R}^n$ and such that $V$ can not be expanded up to a proper subspace to the matrix $\Lambda$ of $m+1$ dimension by an addition to the basis $v_1, \dots, v_m$ of an adjoint vector of the matrix $\Lambda$. Then there exist matrices $P_{21}$, $Q_{12}$ such that*

$$\begin{pmatrix} E & -Q_{12} \\ 0 & E \end{pmatrix}\begin{pmatrix} E & 0 \\ -P_{21} & E \end{pmatrix}\Lambda\begin{pmatrix} E & 0 \\ P_{21} & E \end{pmatrix}\begin{pmatrix} E & Q_{12} \\ 0 & E \end{pmatrix} = \begin{pmatrix} M_{11} & 0 \\ 0 & M_{22} \end{pmatrix}.$$

**The condition of the stiff crack and the existence of an attracting invariant manifold.** Let the matrix $\Lambda$ satisfy the conditions of Theorem 4. Reduce our problem to the Fourier image and make the transformation $\widetilde{U} = S^{-1}u$. Then the solution to the Cauchy problem (9) with the initial data

$$U|_{t=0} = \begin{pmatrix} \mathcal{U}_0 \\ \mathcal{V}_0 \end{pmatrix}$$

can be rewritten in the form

$$U = e^{-Mt}\begin{pmatrix} \mathcal{U}_0 \\ \mathcal{V}_0 \end{pmatrix},$$

where $M = S^{-1}\Lambda S$. Due to Theorem 4, the matrix $M$ has the form

$$M = \begin{pmatrix} E & Q_{12} \\ 0 & E \end{pmatrix}\begin{pmatrix} M_{11} & 0 \\ 0 & M_{22} \end{pmatrix}\begin{pmatrix} E & -Q_{12} \\ 0 & E \end{pmatrix},$$

whence we obtain

$$\begin{aligned} U &= \begin{pmatrix} E & Q_{12} \\ 0 & E \end{pmatrix} exp\left(-\begin{pmatrix} M_{11} & 0 \\ 0 & M_{22} \end{pmatrix}t\right)\begin{pmatrix} E & -Q_{12} \\ 0 & E \end{pmatrix}\begin{pmatrix} \mathcal{U}_0 \\ \mathcal{V}_0 \end{pmatrix} \\ &= \begin{pmatrix} e^{-M_{11}t}\mathcal{U}_0 \\ 0 \end{pmatrix} + \begin{pmatrix} -e^{-M_{11}t}Q_{12}\mathcal{V}_0 \\ 0 \end{pmatrix} + \begin{pmatrix} Q_{12}e^{-M_{22}t}\mathcal{V}_0 \\ e^{-M_{22}t}\mathcal{V}_0 \end{pmatrix} \\ &= e^{-Mt}\begin{pmatrix} \mathcal{U}_0 \\ 0 \end{pmatrix} + e^{-Mt}\begin{pmatrix} -Q_{12}\mathcal{V}_0 \\ 0 \end{pmatrix} + \begin{pmatrix} Q_{12}e^{-M_{22}t}\mathcal{V}_0 \\ e^{-M_{22}t}\mathcal{V}_0 \end{pmatrix}. \end{aligned}$$

Set

$$U_{\mathrm{Ch}} = e^{-Mt}\begin{pmatrix} \mathcal{U}_0 \\ 0 \end{pmatrix}, \quad U_{\mathrm{Cor}} = e^{-Mt}\begin{pmatrix} -Q_{12}\mathcal{V}_0 \\ 0 \end{pmatrix},$$

$$U_H = \begin{pmatrix} Q_{12}e^{-M_{22}t}\mathcal{V}_0 \\ e^{-M_{22}t}\mathcal{V}_0 \end{pmatrix}.$$

Whence the solution of the Cauchy problem is represented as the sum of three terms:

$$U = U_{\mathrm{Ch}} + U_{\mathrm{Cor}} + U_H, \tag{60}$$

and also each of these terms is a solution of the equation (9) with some initial data. The first term corresponds to the projection in the phase space consolidated variables. The second is a correction term, describing the influence of the initial data for the nonequilibrium variables, and the third term is the unessential part. Look for the conditions under which the following estimates are fulfilled:

$$||U_H|| = o(||U_{\mathrm{Ch}}||),\ t \to \infty,$$

where $||f||$ is $L_2-$ norm of $f$. For this use the auxiliary statement (see [27]):

**Lemma 3.** *Let the matrix $\Lambda$ depend on the parameter $\xi$ polynomially such that there is a number $k_0 > 0$, that $\forall \xi : |\xi| > k_0$ all eigenvalues of the matrix $\Lambda$ are algebraic of multiplicity 1 and that the following estimates are fulfilled:*

$$|\lambda(\xi)| \leq\ C_1(1+|\xi|)^{d_1},$$

*where $C_1$, $d_1$ are constant. Let the vector $v$ be an eigenvector for $\Lambda$. Then for $|\xi| > k_0$ the following inequality is true:*

$$\frac{\max\{|e_i^T v|\}}{\min\{|e_i^T v| \neq 0\}} \leq\ C_2(1+|\xi|)^{d_2}, \tag{61}$$

*where $C_2$, $d_2$ are some constants.*

**Lemma 4.** *Let the matrix $\Lambda$, depending on the parameter $\xi$, be determined for all $\xi \in \mathbb{R}$ and satisfy the conditions of Theorem 3 for all $\xi \in \Xi$ also, where the set $\Xi = \mathbb{R}\backslash\Xi_-$ and the set $\Xi_-$ consist of a finite number of points. Then $P_{21}$, $Q_{12}$ are determined on the set $\Xi$.*

*Moreover, let the matrix $P_{21}(\xi)$, $Q_{12}(\xi)$ be continued on the set $\Xi_-$ and in addition the matrix $\Lambda$ depend on $\xi$ polynomially such that there exists a number $k_0 > 0$, that $\forall \xi : |\xi| > k_0$ all eigenvalues of the matrix $\Lambda$ are algebraic of multiplicity 1 and that the estimate*

$$|\lambda(\xi)| \leq\ C_1(1+|\xi|)^{d_1}$$

*is fulfilled, where $C_1$, $d_1$ are some constants. Then there exists $N \in \mathbb{N}$ such that for all $\xi \in \mathbb{R}$ the following estimate is true:*

$$|P_{21}| \leq\ K_1(1+|\xi|)^{N},$$

$$|Q_{12}| \leq\ K_2(1+|\xi|)^{5N},$$

*where $K_1$, $K_2$ are some constants and $|A|$ is the matrix norm of $A$ in $L_\infty$.*

**Dessination 1.** *Denote the smallest $N$ from all $N \in \mathbb{N}$, satisfying Lemma 4, by $N_\Lambda$.*

Except the estimates cited above, we use the two-sides estimate for $|e^{-Mt}v|$, where $|.|$ is the $L_\infty(\mathbb{R})$-norm. For shortening of the formulations, we introduce some designations.

**Dessination 2.** *Let $M$ be a quadratic matrix, irreducibly depending on the parameter $\xi$ such that $\lambda_j$, $j = 1, \ldots, s$ are its eigenvalues. Denote by $d_j$ the maximum size of the Jordan box, corresponding to the eigenvalue $\lambda_j$. Let in addition $\lambda_j$ be put in* *order of the increase of real parts, such that the minimum from these is $l(M)$, and the maximum is $L(M)$, i.e.:*

$$l(M) = \operatorname{Re}\lambda_1 \le \operatorname{Re}\lambda_2 \le \cdots \le \operatorname{Re}\lambda_s = L(M).$$

*Also write $d(M) = d_1$.*

Now we can formulate the lemma of the two-sides estimate.

**Lemma 5.** *Let $M$ be a quadratic matrix, continuously depending on the parameter $\xi$. Then for all $\varepsilon > 0$ there exists $T_0 > 0$ such that $\forall t > T_0$ the following estimate is fulfilled:*

$$e^{-L(M)t}|v| \le |e^{-Mt}v| \le \frac{1+\varepsilon}{(d(M)-1)!}|M|^{d(M)-1}e^{-l(M)t}t^{d(M)-1}|v|, \tag{62}$$

*where $|A|$ is the matrix norm of $A$ in $L_\infty(\mathbb{R})$.*

The proof of this lemma can also be found in [27]. Next define:

**Dessination 3.** *Let $\Gamma(\xi)$ be a finite set of continuous functions $\gamma_1(\xi), \dots, \gamma_s(\xi)$ in $\xi$. Designate by $l(\xi, \Gamma(\xi)) = \inf_s\{\gamma_s(\xi) \in \Gamma(\xi)\}$, $l_0(\Gamma) = \inf_\xi l(\xi, \Gamma(\xi))$, $L(\xi, \Gamma(\xi)) = \sup_s\{\gamma_s(\xi) \in \Gamma(\xi)\}$, $L_0(\Gamma) = \sup_\xi L(\xi, \Gamma(\xi)))$.*

We introduce the so-called condition of the stiff crack.

**Condition 1 (Condition of stiff crack).** *We will say that for a pair of sets $\Gamma_1(\xi)$, $\Gamma_2(\xi)$ the stiff crack condition is true, if*

$$\exists\gamma > 0 : \; l_0(\Gamma_2) - L_0(\Gamma_1) \ge \gamma. \tag{63}$$

Next we can formulate and prove the existence of an attracting manifold.

**Theorem 5 ($L_2$-Chapman–Enskog correctness).** *Let the matrix $\Lambda$, determined by the problem* (9)*, satisfy Condition* 4*. Let in addition, $\Gamma_1$ be the set of all eigenvalues of the matrix $\Lambda$, corresponding to the proper subspace $V$ which yields the dynamic separation. Let $\Gamma_2$ be all other eigenvalues of $\Lambda$ such that for $\Gamma_1$, $\Gamma_2$ the stiff crack Condition* 1 *is fulfilled. Then, if the Fourier images $\mathcal{V}_0$ of the initial data for nonequilibrium variables satisfy the condition*

$$(1+|\xi|)^{5N_\Lambda}|M_{22}|^{d(M_{22})-1}\mathcal{V}_0 \in L_2(\mathbb{R}),$$

*i.e., the initial nonequilibrium data are sufficiently smooth, and the initial data of conservative variables are not equal to zero* (*it means that $||\mathcal{U}_0|| \ne 0$*)*, then there exists $T_0 > 0$ such that for the solution expansion* (60) *constructed above, the following estimate is true:*

$$\frac{||U_H||(t)}{||U_{\mathrm{Ch}}||(t)} \le Ke^{-\delta t}, \; t > T_0, \tag{64}$$

*where $K$, $\delta > 0$ are some constants. Here $K$ depends on $||\mathcal{U}_0||$, $||\mathcal{V}_0||$ and $\delta$ is determined by $\gamma$ and the properties of the matrix $M$.*

We will refer to the estimate obtained here as $L_2$-Chapman–Enskog correctness with respect to the projection in the phase space of the consolidated variables.

Concretely, we have

$$||U_H(t)|| = \left(\int_{\mathbb{R}} \left| \begin{pmatrix} Q_{12} e^{-M_{22}t} \mathcal{V}_0 \\ e^{-M_{22}t} \mathcal{V}_0 \end{pmatrix} \right|^2 d\xi \right)^{\frac{1}{2}} \leq \left(\int_{\mathbb{R}} |1 + |Q_{12}|^2 || e^{-M_{22}t} \mathcal{V}_0|^2 d\xi \right)^{\frac{1}{2}}.$$

Whence, using the lemma obtained above we obtain:

$$||U_H(t)|| \leq \Big( \int_{\mathbb{R}} (1 + K_2^2 (1 + |\xi|)^{10N_\Lambda}) (\frac{1+\varepsilon}{(d(M_{22}) - 1)!})^2$$

$$\times |M_{22}|^{2d(M_{22})-2} e^{-2l(M_{22})t} t^{2d(M_{22})-2} |\mathcal{V}_0|^2 d\xi \Big)^{\frac{1}{2}}.$$

Next, from the condition (63) it follows that

$$e^{-l(M_{22})t} \leq e^{-l_0(\Gamma_2)t} \leq e^{-\gamma t} e^{-L_0(\Gamma_1)t};$$

$$e^{-L(M_{11})t} \geq e^{-L_0(\Gamma_1)t}.$$

In addition, from Lemma 5 we have

$$||U_{\mathrm{Ch}}(t)|| \geq \left( \int_{\mathbb{R}} e^{-2L(M_{11})t} |\mathcal{U}_0|^2 d\xi \right)^{\frac{1}{2}}.$$

Combining the last lammas we obtain

$$\left( \frac{||U_H||(t)}{||U_{\mathrm{Ch}}||(t)} \right)^2$$

$$\leq \frac{e^{-2\gamma t} t^{2d(M_{22})-2} \int_{\mathbb{R}} e^{-2L_0(\Gamma_1)t} (1 + K_2^2 (1 + |\xi|)^{10N_\Lambda}) \left( \frac{1+\varepsilon}{(d(M_{22})-1)!} \right)^2 |\mathcal{V}_0|^2 d\xi}{\int_{\mathbb{R}} e^{-2L_0(\Gamma_1)t} |\mathcal{U}_0|^2 d\xi}.$$

Whence, since $L_0(\Gamma_1)$ does not depend on $\xi$, the demonstrated estimate (64) follows.

**Observation 3.** *The results above allow us to finish the proof of $L_2$-Chapman–Enskog correctness of the Cauchy problem for the system* (30) *with respect to the projection in the phase space of the variables $U$. I.e., the statement about the attracting manifold* (*the unessentialness of the variable $\varrho$*) *for the system* (30) *is true since in this case the condition of the stiff crack is fulfilled* (*Condition* 1).

*The conditions on the relaxation operator $\gamma_1$ in* (30) *can be weakened, if we assume that $\mu_2 = \mu_2(|\xi|^2)$, where $\mu_2(0) = 0$ and $\mu_2(|\xi|^2)$ is a smooth, stabilizing-at-infinity function, like the so-called kink function, such that we could apply to the Fourier operator the $L_p - L_q$ estimates* (*see, for example,* [22, 23]). *More essential will be the opening of the exterior force in* (30) *by a construction of conservation laws with dissipation determining the magnetic and electric currents.*

**Condition of a degenerate crack.** In this section we study the example of a violation of the stiff crack condition, when the expression in Condition 1 can be equal to zero: $l_0(\Gamma_2) - L_0(\Gamma_1) = 0$, i.e., $l(\xi, \Gamma(\xi)) = L(\xi, \Gamma(\xi)))$ for some $|\xi| > 0$ and

$$l(\xi, \Gamma(\xi)) = L(\xi, \Gamma(\xi))) \geq \ 0, \ \forall \xi \in R^n.$$

This condition will be called the **condition of the degenerate crack**. Consider the two-dimensional isothermal 9-moment Grad system of the Boltzmann kinetic equation

$$\begin{gathered}(\partial_t + u\cdot\partial_x)\varrho + \varrho \operatorname{div}_x \, u = 0, \\ (\partial_t + u\cdot\partial_x)u_i + \frac{1}{\varrho}\partial_{x_i}p + \frac{1}{\varrho}\partial_{x_j}\sigma_{ij} = 0, \\ (\partial_t + u\cdot\partial_x)p + \frac{5}{3}p \operatorname{div}_x \, u + \frac{2}{3}\sigma_{ij}(\widetilde{\nabla} u)_{ij} = 0, \\ (\partial_t + u\cdot\partial_x)\sigma_{ij} + \frac{4}{3}p(\widetilde{\nabla} u)_{ij} + \frac{7}{3}\sigma_{ij}\operatorname{div} \, u + \frac{\sqrt{p\varrho}}{\eta(p)}\sigma_{ij} = 0, \quad i,j = 1,2,\end{gathered} \tag{65}$$

We will assume that the particle mass $m = 1$. Then pressure is $p = k_B\varrho T)$, the sound velocity $c^2 = k_B T$, the viscosity coefficient $\mu(T) = \eta(T)T$, $\eta(T)$ is determined by the choice of a model of the interaction particles ($\eta = \text{const}$ for the Maxwell molecule, $\eta \approx \ \sqrt{T}$ in the case of the hard sphere). Here $\widetilde{\nabla} u = \nabla_x u + \nabla_x u^\top - \operatorname{div}_x u \, I$ is a traceless deformation tensor, $I$ is a unique matrix, the matrix $\nabla_x u = (\partial_{x_i} u_j)$. Since the tensor is traceless it follows that

$$\sigma_{11} + \sigma_{22} = 0, \tag{66}$$

i.e., it is sufficient to consider independent variables

$$\varrho, \ u = (u_1, u_2), \ T, \ \sigma_{11}, \ \sigma_{12}.$$

Then the last equation for $\sigma_{33}$ follows from another one and it can be discarded, if we substitute $\sigma_{33}$ in the remaining equations in (66). The linearization of this system in the neighborhood of an equilibrium state $\varrho_e, \ u_e = 0, \ p_e, \ \sigma_e = 0$ is

$$\begin{aligned}\partial_t \varrho + \operatorname{div}_x u &= 0, \\ \partial_t p + \frac{5}{3}\operatorname{div}_x u &= 0, \\ \partial_t u_i + \partial_{x_i} p + \partial_{x_j}\sigma_{ij} &= 0, \\ \partial_t \sigma_{ij} + \frac{4}{3}(\widetilde{\nabla} u)_{ij} + \frac{1}{\mu(p_e)}\sigma_{ij} &= 0,\end{aligned} \tag{67}$$

where $p_e = c_e^2$. In the isothermal case the density equation is separated in the system (67), such that below we can study this system only. Our goal is the proof of the following statement:

**Proposition 4.** *The Cauchy problem for the linearized system* (65) *is $L_2$-Chapman–Enskog correct with respect to the projection in the phase space of conservative variable $U = (\varrho, u, p)$ with an additional restriction on the initial data for nonequilibrium variables in the following form: the supports of the Fourier image of nonequilibrium variables $(\sigma_{11}, \sigma_{12})$ do not intersect the critical set $M_{cr} = \{\xi \in R^2;\ |\xi|^2 = 1/2\mu(p_e)$ and $|\xi| = 0\}$.*

**The spectrum properties of the resolvent matrix $\Lambda(\xi)$.** In the considered case the resolvent matrix $\Lambda$ has the form:

$$\Lambda = \begin{pmatrix} 0 & \frac{5}{3}i\xi_1 & \frac{5}{3}i\xi_2 & 0 & 0 \\ i\xi_1 & 0 & 0 & i\xi_1 & i\xi_2 \\ i\xi_2 & 0 & 0 & -i\xi_2 & i\xi_1 \\ 0 & \frac{4}{3}i\xi_1 & -\frac{4}{3}i\xi_2 & \Pi_0 & 0 \\ 0 & \frac{4}{3}i\xi_2 & \frac{4}{3}i\xi_1 & 0 & \Pi_0 \end{pmatrix},$$

$\Pi_0 = 1/\mu(p_e)$, and the Fourier image of the system 1D13-lin) is

$$\partial_t \widetilde{U}(t,\xi) + \Lambda(\xi)\widetilde{U} = 0$$

where $\widetilde{U}(t,\xi)$ is the Fourier transformation of vector-function $U = (u, p, \sigma_{11}, \sigma_{12})$. It is not difficult to obtain the factorization for the characteristic polynomial of the resolvent matrix $\Lambda$ as the following:

$$\begin{aligned} \det(\Lambda - \lambda E) &= \lambda^5 - 2\Pi_0\lambda^4 + \left(\frac{13}{3}|\xi|^2 + \Pi_0^2\right)\lambda^3 - 6|\xi|^2\Pi_0\lambda^2 \\ &\quad + |\xi|^2\left(4|\xi|^2 + \frac{5}{3}\Pi_0^2\right)\lambda - \frac{20}{9}|\xi|^4\Pi_0 \\ &= \left(\lambda^3 - \Pi_0\lambda^2 + 3c_0^2|\xi|^2\lambda - \frac{5}{3}c_0^2|\xi|^2\Pi_0\right)\left(\lambda^2 - \Pi_0\lambda + \frac{4}{3}c_0^2|\xi|^2\right). \end{aligned} \tag{68}$$

Set $\lambda = i\tau$, $\zeta^2 = \frac{1}{3}c_0^2|\xi|^2$ and multiply the characteristic polynomial by $i$. Then we obtain the dispersion equation of (67):

$$\begin{aligned} P(\tau,\zeta) &= \tau(\tau^4 - 13\zeta^2\tau^2 + 36\zeta^4) + 2i\Pi_0(\tau^4 - 9\zeta^2\tau^2 + 10\zeta^4) - \Pi_0^2\tau(\tau^2 - 5\zeta^2) \\ &= P_0(\tau,\zeta) - i\Pi_0 P_1(\tau,\zeta) - \Pi_0^2 P_2(\tau,\zeta), \end{aligned} \tag{69}$$

$$P_0 = \tau(\tau^4 - 13\zeta^2\tau^2 + 36\zeta^4) = 0,\ \tau_{1,2}^2 = 9\zeta^2,\ \tau_{3,4}^2 = 4\zeta^2,$$

$$P_1 = \tau^4 - 9\zeta^2\tau^2 + 10\zeta^4 = 0,\ \tau_{5,6}^2 = \frac{1}{2}\zeta^2(9 + \sqrt{41}),\ \tau_{7,8}^2 = \frac{1}{2}\zeta^2\left(9 - \sqrt{41}\right),$$

$$P_2 = \tau(\tau^2 - 5\zeta^2) = 0,\ \tau_{9,10}^2 = \pm\sqrt{5}\,\zeta^2.$$

The polynomials, corresponding to different degrees of $\Pi_0$, are strictly hyperbolic and the roots $\tau_{k,j}$ of the polynomials to neighboring degrees of $\Pi_0$ separate each other. Hence (see [2]) the polynomial $P(\tau,\kappa)$ is stable. Study the roots of the characteristic polynomial. Take the following designations: the roots $\lambda(\kappa)$ are called wave-roots, if $\lambda(0) = 0,\ \lambda'(0) \neq 0$; as the diffusion root, if $\lambda(0) = 0,\ \lambda'(0) = 0,\ \lambda''(0) \neq 0$; and as boundary-layer root, if $\lambda(0) \neq 0$. Set $\kappa^2 = c_0^2|\xi|^2$.

**Lemma 6.** *For sufficiently large* $\Pi_0 > \frac{9}{52}\left(13+\sqrt{13}\right)$, *simple complex conjugate wave roots* $\lambda_j^w$, $j=1,2$, *of the cubic factor* $P_3$ *satisfy the condition*

$$\operatorname{Re}\lambda_j^w(\kappa^2) < \frac{2}{9}\Pi_0,\ j=1,2,\ \forall\kappa^2 \geq 0;$$

*the simple real boundary-layer root* $\lambda_3^b$, $\lambda_3^b(0) = \Pi_0$, *of the cubic factor* $P_3$ *lies on the half-interval* $(5\Pi_0/9, \Pi_0]$, $\lambda_3^b(\kappa^2) \to 5\Pi_0/9$, $\kappa^2 \to \infty$. *The diffusion and boundary-layer roots of the quadratic factor* $P_2$ *satisfy the condition*

$$\operatorname{Re}\lambda_2^b(\kappa^2) \geq \frac{1}{2}\Pi_0,\ \operatorname{Re}\lambda_d(\kappa^2) \leq \frac{1}{2}\Pi_0,\ \forall\kappa^2 \geq 0;$$

$\lambda_2^b(\kappa^2), \lambda_d(\kappa^2) \to \frac{1}{2}\Pi_0$, $\kappa^2 \to \frac{3\Pi_0^2}{4}$, $\lambda_2^b(0) = \Pi_0$.

Concretely, we have:

1. Factors $P_3, P_2$ of the characteristic polynomial have no common root when $\kappa \neq 0$. There are $P_3 = \lambda\, P_2 + \frac{5}{3}\kappa^2(\lambda - \Pi_0)$, where $P|_{\lambda=\Pi_0} = (\frac{4}{3}\kappa^2)^2 \neq 0,\ \forall\xi \neq 0$.

2. The cubic factor has no three-multiple for $\kappa \geq 0$, i.e., for all $\kappa \geq 0$ there are two different roots. For three-multiple root $a$ we have

$$3a = \Pi_0,\ 3a^2 = 3\kappa^2,\ a^3 = \frac{5}{3}\kappa^2\Pi_0.$$

Obviously, this system is not solved for all $\kappa^2 \geq 0$. So that for all $|\xi| \geq 0$ the polynomial $P_3$ has two different roots.

For a two-multiple root one obtains

$$P_3 = \left(\frac{1}{3}\lambda - \frac{1}{9}\pi_0\right) P_3' + 2\left(\kappa^2 - \frac{1}{9}\Pi_0^2\right)\lambda - \frac{4}{3}\kappa^2\Pi_0.$$

Hence, for a multiple real root we have

$$P_3' = 0,\ \left(\kappa^2 - \frac{1}{9}\Pi_0^2\right)\lambda - \frac{2}{3}\kappa^2\Pi_0 = 0.$$

This system is not solved, if $\kappa^2 - \frac{1}{9}\Pi_0^2 \leq 0$. When $\kappa^2 - \frac{1}{9}\Pi_0^2 >\ 0$ we have

$$\left(\kappa^2 - \frac{1}{9}\Pi_0^2\right)^2 P_3' = \frac{4}{27}\kappa^2\Pi_0^2 + 3\kappa^2\left(\kappa^2 - \frac{1}{9}\Pi_0^2\right)^2 \neq 0$$

for all $\kappa^2 > 0$. Whence we obtain that complex conjugate wave roots $\lambda_1^w$, $\lambda_2^w$, $\lambda_j^w(0) = 0$, $(\lambda_j^w)'(0) \neq 0$, of the cubic factor $P_3$ are simple. At the same time, from the representation

$$(\lambda - \Pi_0)\lambda^2 = -3\kappa^2(\lambda - \frac{5}{9}\Pi_0)$$

follows the existence of the smooth simple real boundary-layer root $\lambda_3^b(\kappa^2)$, such that $\lambda_3^b(0) = \Pi_0$ $\lambda_3^b(\kappa^2) \to 5/9$, $\kappa^2 \to \infty$.

3. The boundary-layer root $\lambda_2^b$, $\lambda_2^b(0) = \Pi_0 \neq 0$, and the diffusion root $\lambda_d$, $\lambda_d(|\xi|^2) = O(|\xi|^2)$ of the quadratic factor $P_2$, for small $|\xi| \ll 1$, are real up

to $\kappa^2 = \frac{3\Pi_0^2}{4}$ and are complex conjugate when $\kappa^2 > \frac{3\Pi_0^2}{4}$ with the real part $\operatorname{Re}\lambda_2^b = \operatorname{Re}\lambda_d = \frac{1}{2}\Pi_0$.

4. Now show that

$$\operatorname{Re}\lambda_j^w(\kappa^2) < \frac{1}{2}\Pi_0 \ \forall\kappa^2 \geq 0.$$

At first, verify the root asymptotics for large $\kappa \gg 1$. We have

$$\lambda^w = \pm i\sqrt{3}\kappa + \frac{2}{9}\Pi_0 + O\left(\frac{1}{\kappa}\right).$$

At the same time, due to the difference of roots for factors $P_3$ and $P_2$ and the fact that $\operatorname{Im}\lambda_2^b = \lambda_d = \Pi_0/2$ for $\kappa^2 \geq \frac{3\Pi_0^2}{4}$, the phase picture on complex plane $\lambda$ shows that graphics of the wave roots $\lambda_j^w$ are situated on the left from the straight line $\lambda = \Pi_0/2$ or intersect this line on the interval $0 < \kappa^2 < \frac{3\Pi_0^2}{4}$ and after this come back on the straight line $|\xi|^2 = \frac{2}{9}\Pi_0$.

Next, observe that if $a+ib$ is a root of the cubic factor, the equations for real and imaginary parts are

$$b^2 = 3a\left(a - \frac{2}{3}\Pi_0\right) + 3\kappa^2$$

and

$$p(a) = a(13a^2 - 13\Pi_0 a + 3\Pi_0^2) + 6\kappa^2\left(a - \frac{2}{9}\Pi_0\right) = 0.$$

Whence, it follows that $p(a) > 0,\ \forall a \geq \frac{2}{9}\Pi_0$, if

$$\Pi_0 > \frac{9}{52}\left(13 + \sqrt{13}\right).$$

By this condition the wave root trajectories are situated on the left from the vertical $\lambda = \frac{2}{9}\Pi_0$ and $\operatorname{Re}\lambda_j^w(\kappa^2) \to \frac{2}{9}\Pi_0,\ \kappa^2 \to \infty$.

Therefore, for construction of the Chapman–Enskog projection in the phase space of the variables $u$, $p$ we can choose three eigenvectors of the resolvent matrix, corresponding to two wave roots $\lambda_j^w,\ j = 1,2$, of the cubic factor $P_3$ and the diffusion root $\lambda_d$ of the factor $P_2$ if

$$\kappa^2 \neq \Pi_0/2,\ 0.$$

For $\kappa^2 \neq \Pi_0/2$ we can separate the roots of the characteristic polynomial (68) on two groups

$$\Gamma_1 = \{\lambda_j^w,\ j = 1,2,\ \lambda_d\},\ \Gamma_1 = \{\lambda_3^b, \lambda_2^b\},$$

such that

$$\max_{\lambda(\kappa^2)\in\Gamma_1} \operatorname{Re}\lambda(\kappa^2) < \min_{\lambda(\kappa^2)\in\Gamma_2} \operatorname{Re}\lambda(\kappa^2),\ \forall\kappa^2 \neq \Pi_0/2,\ \kappa \neq 0. \tag{70}$$

Whence follows the existence of the special proper subspace. For such values of $|\xi|$ the resolvent matrix $\Lambda$ has the proper subspace $V$ of three dimensions such that $Lin\{V, e_4, e_5\} = \mathbb{R}^5$. The following statement is true.

**Lemma 7.** *Let $V = \mathrm{Lin}\{v_1,\ v_2,\ v_3\}$, where $v_1,\ v_2,\ v_3$ are eigenvectors corresponding to the roots of the characteristic polynomial* (68) *from the first group $G_1$ when $\kappa^2 \neq \Pi_0/2$. Then*

$$\mathrm{Lin}\{V,\ e_4,\ e_5\} = \mathbb{R}^5. \tag{71}$$

Hence, in this case for any small $\varepsilon > 0$ on the set of the regular values for $|\xi|$:

$$M_\varepsilon = \{\xi \in R^2;\ \varepsilon < \ |\xi|^2 < \ \Pi_0/2 - \varepsilon\} \cup \{\xi \in R^2;\ \Pi_0/2 + \varepsilon < \ |\xi|^2\},$$

thus the stiff crack condition (1) is fulfilled. So we can use the results of Theorem 5 with only one restriction, that the supports of the Fourier image of nonequilibrium variables $(\sigma_{11}, \sigma_{12})$ do not intersect the critical set $M_{cr} = \{\xi \in R^2;\ |\xi|^2 = 1/2\mu(p_e)\} \cup \{\xi \in R^2;\ |\xi| = 0\}$. The proof of Proposition 4 is complete.

**Observation 4.** 1. *We can not obtain the result of Lemma 7 when $\kappa^2 = \Pi_0/2$, since in this case for a two-multiple root of $P_2$ $(\lambda^d|_{\kappa^2=\Pi_0/2} = \lambda_2^b|_{\kappa^2=\Pi_0/2})$ we have the Jordan box. Whence, due to the main result about the solvability of the matrix equation, we can not separate the roots $\lambda^d, \lambda_2^b$ into two different groups to derive the proper subspace $V$ determining the projection in the phase space of the variables $(\varrho, u, p)$.*
2. *For the resolvent matrix $\Lambda$ there exists the proper subspace $V$ of dimension 1 such that $Lin\{V,\ e_2,\ e_3,\ e_4,\ e_5\} = \mathbb{R}^5$ and this subspace $V$ depends on the parameter $\xi$ smoothly for all $|\xi| \neq 0$. Obviously, the subspace $V$ is determined by separating the roots of the characteristic polynomial* (68) *into two groups*

$$\Gamma_1 = \{\lambda_j^w,\ j = 1, 2,\ \lambda_d, \lambda_2^b\},\ \Gamma_1 = \{\lambda_3^b\},$$

*such that*

$$\max_{\lambda(\kappa^2)\in\Gamma_1} \mathrm{Re}\,\lambda(\kappa^2) < \min_{\lambda(\kappa^2)\in\Gamma_2} \mathrm{Re}\,\lambda(\kappa^2),\ \forall\kappa \neq 0. \tag{72}$$

*In this case the full Jordan basis of the multiple root, when $|\xi|^2 = \Pi_0/2$, belongs to the proper subspace $V$ and the unessential variable is $\sigma_{12}$. Whence, we can use again the results of Theorem 5 with the restriction that the supports of the Fourier image of nonequilibrium variables $\sigma_{12}$ does not intersect the critical point $M_{cr}^{(1)} = \{\xi = 0\}$. To annulate this restriction we need to consider the asymptotics of the roots $\lambda_2^b,\ \lambda_3^b$ in a neighborhood of $\xi = 0$. We have*

$$\lambda_2^b = \Pi_0 - \frac{2}{3}\kappa^2 + O(\kappa^4),\ |\kappa| \ll 1,$$
$$\lambda_3^b = \Pi_0 - \frac{4}{3}\frac{1}{\pi_0}\kappa^2 + O(\kappa^4),\ |\kappa| \ll 1,$$

*so that there is*

$$\lambda_3^b - \lambda_2^b = \frac{2}{3}\left(1 - 2\frac{1}{\Pi_0}\right)\kappa^2 + O(\kappa^4),\ |\kappa| \ll 1.$$

*We can assume that*

$$1 - 2\frac{1}{\Pi_0} > \ 0$$

*which is true for large* $\Pi_0 > 2$*. Hence, in an* $\varepsilon$*-neighborhood* $\mathcal{O}_\varepsilon(M_{cr}^{(1)})$ *of the point* $M_{cr}^{(1)}$ *we can use a modification Theorem* 5 *in the case of a degenerate crack, taking into account* $L_p \to L_q$ *estimates in* [23, 27].

**Conclusion.** The above analysis of linear and linearized problems allows us to study the specific character of the Cauchy problem with insufficient information about initial data and conformation to the Chapman–Enskog conjecture. The study shows the meaning of this subject from the physical and mathematical points of view. Our next publication will be devoted to linear analysis of the mixed problem with insufficient information about initial-boundary data.

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# On the Stability of Non-symmetric Equilibrium Figures of Rotating Self-gravitating Liquid not Subjected to Capillary Forces

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*To the memory of Professor A.V. Kazhikhov*

**Abstract.** The paper contains a justification of the principle of minimum of potential energy in the problem of stability of a rotating viscous incompressible self-gravitating liquid bounded only by a free surface. It is assumed that the domain occupied by a rotating liquid that is referred to as an equilibrium figure is not symmetric with respect to the axis of rotation. The surface tension is not taken into account. The proof of stability is based on analysis of the evolution free boundary problem for perturbations of the velocity and pressure.

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**Keywords.** free boundary problems, equilibrium figures, stability.

## 1. Introduction

In the present article we continue the analysis of the stability of an isolated mass of uniformly rotating viscous incompressible self-gravitating liquid initiated in [1]. As in [1], we do not take into account capillary forces on the free boundary. We recall that the velocity and the pressure of a liquid rotating as a rigid body about the $x_3$-axis is given by

$$\boldsymbol{V}(x) = \omega(\boldsymbol{e}_3 \times \boldsymbol{x}) = \omega(-x_2, x_1, 0), \qquad P(x) = \frac{\omega^2}{2}|x'|^2 + p_0 \tag{1.1}$$

where $x' = (x_1, x_2, 0)$, $p_0 = \text{const}$, $\boldsymbol{e}_3$ is a unit vector directed along the $x_3$-axis and $\omega$ is the angular velocity of rotation. The domain $\mathcal{F}$ occupied by the liquid, so called equilibrium figure, is defined by the equation

$$\frac{\omega^2}{2}|x'|^2 + \kappa\mathcal{U}(x) + p_0 = 0, \quad x \in \mathcal{G} = \partial\mathcal{F}, \tag{1.2}$$

where

$$\mathcal{U}(x) = \int_{\mathcal{F}} \frac{dz}{|x-z|}$$

is a gravitational potential of the domain $\mathcal{F}$ (the density of the liquid equals 1).

We consider the functions (1.1) given in $\mathcal{F}$ as a solution of a free boundary problem governing the evolution of an isolated liquid mass bounded only by a free surface. This problem consists of determination of a bounded domain $\Omega_t \subset \mathbb{R}^3$, $t > 0$, as well as of the vector field of velocities $\boldsymbol{v}(x,t) = (v_1, v_2, v_3)$ and the pressure function $p(x,t)$, $x \in \Omega_t$, $t > 0$, satisfying the equations

$$\begin{gathered} \boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nu\nabla^2\boldsymbol{v} + \nabla p = 0, \\ \nabla\cdot\boldsymbol{v} = 0, \qquad x \in \Omega_t, \;\; t > 0, \\ T(\boldsymbol{v},p)\boldsymbol{n} = \kappa U(x,t)\boldsymbol{n}, \quad V_n = \boldsymbol{v}\cdot\boldsymbol{n}, \qquad x \in \Gamma_t \equiv \partial\Omega_t, \\ \boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \qquad x \in \Omega_0, \end{gathered} \tag{1.3}$$

where $\nu, \kappa = \mathrm{const} > 0$,

$$U(x,t) = \int_{\Omega_t} \frac{dz}{|x-z|}$$

is the Newtonian potential depending on an unknown domain $\Omega_t$, $T(\boldsymbol{v},p) = -pI + \nu S(\boldsymbol{v})$ is the stress tensor, $S(\boldsymbol{v}) = \left(\frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial v_j}\right)_{j,k=1,2,3}$ is the doubled rate-of-strain tensor, $\boldsymbol{n}$ is the exterior normal to $\Gamma_t$, and $V_n$ is the velocity of evolution of $\Gamma_t$ in the normal direction. The domain $\Omega_0$ is given.

We assume that the equilibrium figure $\mathcal{F}$ is a given bounded domain. If it is axially symmetric with respect to the $x_3$-axis (as the Maclaurin ellipsoids), then the functions (1.1) given in the domain $\mathcal{F}$ represent a stationary solution of (1.3). If $\mathcal{F}$ does not possess the symmetry property (as the Jacobi ellipsoids, pear-formed figures of Poincaré etc., see [2–5]), then there exists a one-parameter family of the equilibrium figures, $\mathcal{F}_\theta$, obtained by rotation of the angle $\theta$ about the $x_3$-axis of one of them, $\mathcal{F}_0$. We assume that $\theta \in \mathbb{R}$ and $\mathcal{F}_\theta = \mathcal{F}_{\theta+2\pi}$. In this case the functions (1.1) defined in the variable domain $\mathcal{F}_{\omega t+\varphi}$ represent a periodic solution of (1.3).

We observe that in the case of non-symmetric $\mathcal{F}$ the function $h(y) = \boldsymbol{N}(y)\cdot(\boldsymbol{e}_3 \times y)|_{\mathcal{G}}$, where $\boldsymbol{N}(y)$ is the exterior normal to $\mathcal{G} = \partial\mathcal{F}$ and $\boldsymbol{e}_3$ is a unit vector directed along the $x_3$-axis, is different from identical zero, whereas for axially symmetric $\mathcal{F}$ this function vanishes.

We are interested in the problem of stability of these solutions, that is closely related to the well-known problem of stability of equilibrium figures. According to the classical theory, the figure is stable, if the quadratic form

$$\delta^2\mathcal{R}[\rho] = \int_{\mathcal{G}} b(x)\rho^2(x)dS + \frac{\omega^2}{\int_{\mathcal{F}}|z'|^2dz}\Big(\int_{\mathcal{G}}|y'|^2\rho(y)dS\Big)^2 - \kappa\int_{\mathcal{G}}\int_{\mathcal{G}}\frac{\rho(y)\rho(z)}{|y-z|}dS_ydS_z \tag{1.4}$$

where

$$b(x) = -\omega^2\boldsymbol{x}'\cdot\boldsymbol{N}(x) - \kappa\frac{\partial\mathcal{U}(x)}{\partial N} \geq b_0 > 0, \tag{1.5}$$

is positive definite, i.e.,

$$c_1\|\rho\|^2_{L_2(\mathcal{F})} \leq \delta^2R[\rho] \leq c_2\|\rho\|^2_{L_2(\mathcal{F})} \tag{1.6}$$

for arbitrary function $\rho(x)$ given on $\mathcal{G}$ and satisfying the conditions

$$\int_{\mathcal{G}}\rho(x)dS = 0, \quad \int_{\mathcal{G}}\rho(x)x_idS = 0, \quad i = 1,2,3, \tag{1.7}$$

$$\int_{\mathcal{G}}\rho(x)h(x)dS_x = 0, \tag{1.8}$$

and unstable, if this form can take negative values. We give the justification of the first statement by the analysis of the evolution free boundary problem for the perturbations $\boldsymbol{w}(x,t) = \boldsymbol{v} - \boldsymbol{V}$, $s = p - P$ of the velocity and pressure. This problem consists of determination of a bounded domain in $\mathbb{R}^3$ (denoted also by $\Omega_t$) with the boundary $\Gamma_t$, $t > 0$, as well as of the functions $\boldsymbol{w}(x,t)$ and $s(x,t)$, satisfying the relations

$$\begin{gathered}
\boldsymbol{w}_t + (\boldsymbol{w}\cdot\nabla)\boldsymbol{w} + 2\omega(\boldsymbol{e}_3\times\boldsymbol{w}) - \nu\nabla^2\boldsymbol{w} + \nabla s = 0,\\
\nabla\cdot\boldsymbol{w} = 0, \qquad x\in\Omega_t, \quad t>0,\\
T(\boldsymbol{w},s)\boldsymbol{n} = \Big(\frac{\omega^2}{2}|x'|^2 + \kappa U(x,t) + p_0\Big)\boldsymbol{n},\\
V_n = \boldsymbol{w}\cdot\boldsymbol{n}, \quad x\in\Gamma_t,\\
\boldsymbol{w}(x,0) = \boldsymbol{w}_0(x), \quad x\in\Omega_0.
\end{gathered} \tag{1.9}$$

The vector field $\boldsymbol{w}_0 = \boldsymbol{v}_0 - \boldsymbol{V}$ should satisfy the orthogonality conditions

$$\begin{gathered}
\int_{\Omega_0}\boldsymbol{w}_0(x)dx = 0,\\
\int_{\Omega_0}\boldsymbol{w}_0(x)\cdot\boldsymbol{\eta}_i(x)dx + \omega\int_{\Omega_0}\boldsymbol{\eta}_3(x)\cdot\boldsymbol{\eta}_i(x)dx = \omega\int_{\mathcal{F}}\boldsymbol{\eta}_3(x)\cdot\boldsymbol{\eta}_i(x)dx,
\end{gathered} \tag{1.10}$$

and it is easily verified that they hold at any moment of time $t \geq 0$:

$$\int_{\Omega_t}\boldsymbol{w}(x,t)dx = 0, \tag{1.11}$$

$$\int_{\Omega_t}\boldsymbol{w}(x,t)\cdot\boldsymbol{\eta}_i(x)dx + \omega\int_{\Omega_t}\boldsymbol{\eta}_3(x)\cdot\boldsymbol{\eta}_i(x)dx = \omega\int_{\mathcal{F}}\boldsymbol{\eta}_3(x)\cdot\boldsymbol{\eta}_i(x)dx,$$

$i = 1, 2, 3$. In addition, we have

$$|\Omega_t| = |\mathcal{F}|, \tag{1.12}$$
$$\int_{\Omega_t} x_i dx = 0, \quad i = 1, 2, 3.$$

We find it convenient to pass to the Lagrangian coordinates $\xi \in \Omega_0$ connected with the Eulerian coordinates $x \in \Omega_t$ by

$$x = \xi + \int_0^t \boldsymbol{u}(\xi, \tau) d\tau \equiv X(\xi, t), \tag{1.13}$$

where $\boldsymbol{u}(\xi, t) = \boldsymbol{w}(X(\xi, t), t)$. Under this transformation (1.9) is converted to

$$\begin{gathered}
\boldsymbol{u}_t + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{u}) - \nu \nabla_u^2 \boldsymbol{u} + \nabla_u q = 0, \\
\nabla_u \cdot \boldsymbol{u}(\xi, t) = 0, \quad \xi \in \Omega_0, \ \ t > 0, \\
T_u(\boldsymbol{u}, q)\boldsymbol{n} = \Big(\kappa U(X, t) + \frac{\omega^2}{2} |X'^2(\xi, t)|^2 + p_0\Big)\boldsymbol{n}, \quad \xi \in \Gamma_0, \\
\boldsymbol{u}(\xi, 0) = \boldsymbol{w}_0(\xi), \qquad \xi \in \Omega_0,
\end{gathered} \tag{1.14}$$

where $q(\xi, t) = s(X(\xi, t), t)$, and $\nabla_u$, $T_u$ are the transformed gradient and the stress tensor, respectively. Since the Jacobian of the transformation (1.13) equals 1, we have $\nabla_u = A\nabla_\xi$, $T_u(\boldsymbol{u}, q) = -qI + \nu S_u(\boldsymbol{u})$, where $S_u(\boldsymbol{u}) = A\nabla_u \boldsymbol{u} + (A\nabla_u \boldsymbol{u})^T$ is the transformed doubled rate-of-strain tensor, and $A(\xi, t) = (A_{ij})_{i,j=1,2,3}$ is the co-factors matrix corresponding to the transformation (1.13). Finally, $U(X, t) = \int_{\Omega_0} |X(\xi, t) - X(\eta, t)|^{-1} d\eta$ and $\boldsymbol{n}(x)$ is the exterior normal to the surface $\Gamma_t = X\Gamma_0$ connected with the normal $\boldsymbol{n}_0(\xi)$ to $\Gamma_0$ by

$$\boldsymbol{n}(X(\xi, t)) = \frac{A(\xi, t)\boldsymbol{n}_0(\xi)}{|A(\xi, t)\boldsymbol{n}_0(\xi)|}. \tag{1.15}$$

The problem (1.14) is studied in the weighted anisotropic Sobolev–Slobodetskii spaces introduced by Y. Hataya [6]. Let $Q_T = \Omega_0 \times (0, T)$ and let $W_2^{l,l/2}(Q_T)$, $l \geq 1$, be a standard anisotropic Sobolev–Slobodetskii space. The weighted space $\widetilde{W}_2^{l,l/2}(Q_T)$ is defined as the set of functions (or vector fields) $u(\xi, t)$, such that $u \in W_2^{l,l/2}(Q_T)$, $tu \in W_2^{l-1,l/2-1/2}(Q_T)$ (the weight improves the behavior of $u$ for large $t$), and supplied with the norm

$$\|u\|_{\widetilde{W}_2^{l,l/2}(Q_T)} = \|u\|_{W_2^{l,l/2}(Q_T)} + \|tu\|_{W_2^{l-1,l/2-1/2}(Q_T)}.$$

We also set

$$\|u\|_{\widetilde{W}_2^{l,0}(Q_T)} = \|u\|_{W_2^{l,0}(Q_T)} + \|tu\|_{W_2^{l-1,0}(Q_T)},$$
$$\|u\|_{\widetilde{W}_2^{0,l/2}(Q_T)} = \|u\|_{W_2^{0,l/2}(Q_T)} + \|tu\|_{W_2^{0,l/2-1/2}(Q_T)}.$$

The weighted spaces of functions given on smooth manifolds, in particular, on $G_T = \Gamma_0 \times (0, T)$, are defined in a similar way.

The main result of the paper is as follows.

**Theorem 1.1.** *Assume the following:*

1. $\boldsymbol{w}_0 \in W_2^{l+1}(\Omega_0)$, $l \in (1, 3/2)$, *satisfies the orthogonality conditions* (1.10) *and the compatibility conditions*
$$\nabla \cdot \boldsymbol{w}_0 = 0, \quad \Pi_0 S(\boldsymbol{w}_0)\boldsymbol{n}_0|_{\Gamma_0} = 0, \tag{1.16}$$
*where* $\Pi_0 \boldsymbol{f} = \boldsymbol{f} - \boldsymbol{n}_0(\boldsymbol{f} \cdot \boldsymbol{n}_0)$ *is the projection on the tangent plane to* $\Gamma_0$.
2. *The domain* $\Omega_0$ *satisfies* (1.12), *the surface* $\Gamma_0 = \partial\Omega_0$ *is given by the equation*
$$x = y + \boldsymbol{N}_0(y)\rho_0(y), \quad y \in \mathcal{G}, \tag{1.17}$$
*where* $\boldsymbol{N}_0$ *is the unit normal to* $\mathcal{G}_0$, *and* $\rho_0(y) \in W_2^{l+3/2}(\mathcal{G})$ *satisfies the condition*
$$\int_{\Gamma_0} \rho_0(\bar{\xi})\boldsymbol{N}(\bar{\xi}) \cdot (\boldsymbol{e}_3 \times \bar{\xi}) dS_\xi = 0, \tag{1.18}$$
$\bar{\xi}$ *being the closest point of* $\mathcal{G}_0$ *to* $\xi$.
3. *The following smallness condition holds:*
$$\|\boldsymbol{w}_0\|_{W_2^{l+1}(\Omega_0)} + \|\rho_0\|_{W_2^{l+3/2}(\mathcal{G})} \le \epsilon \ll 1. \tag{1.19}$$
4. *The quadratic form* (1.4) *satisfies the condition* (1.6), *where* $\mathcal{G}$ *is an arbitrary* $\mathcal{G}_\theta$.

*Then the problem* (1.14) *has a unique solution*
$$\boldsymbol{u} \in \widetilde{W}_2^{2+l,1+l/2}(Q_\infty), \quad \nabla s \in \widetilde{W}_2^{l,l/2}(Q_\infty)$$
*such that* $s|_{\xi\in\Gamma_0} \in \widetilde{W}_2^{1/2+l,1/4+l/2}(G_\infty)$, *and*
$$\begin{aligned}\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,1+l/2}(Q_\infty)} &+ \|\nabla s\|_{\widetilde{W}_2^{l,l/2}(Q_\infty)} + \|s\|_{\widetilde{W}_2^{1/2+l,1/4+l/2}(G_\infty)} \\ &\le c\Big(\|\boldsymbol{w}_0\|_{W_2^{l+1}(\Omega_0)} + \|\rho_0\|_{W_2^{l+1}(\mathcal{G})}\Big).\end{aligned} \tag{1.20}$$

*The surface* $\Gamma_t$ *is given by the equation*
$$x = z + \boldsymbol{N}_{\theta(t)}(z)\widehat{\rho}(z,t), \quad z \in \mathcal{G}_{\theta(t)}, \tag{1.21}$$
*where* $\boldsymbol{N}_\theta$ *is a unit exterior normal to* $\mathcal{G}_\theta$. *The derivative of* $\theta(t)$ *satisfies the inequality*
$$|\theta'(t)| \le c\int_{\Gamma_0} |\boldsymbol{u}(\xi,t)| dS_\xi, \tag{1.22}$$
*whereas*
$$\theta(t) = \int_0^t \theta'(\tau)d\tau \to \theta_\infty \tag{1.23}$$
*as* $t \to \infty$. *The function*
$$r(\xi,t) \equiv \widehat{\rho}(z,t), \tag{1.24}$$
*where* $z$ *is the closest point of* $\mathcal{G}_{\theta(t)}$ *to* $X(\xi,t) \in \Gamma_t$, *satisfies the condition*
$$\int_{\Gamma_0} r(\xi,t)h_{\theta(t)}(z)dS_\xi = \int_{\Gamma_0} \widehat{\rho}(z,t)h_{\theta(t)}(z)dS_\xi = 0 \tag{1.25}$$

*and the inequality*

$$\|r\|_{\widetilde{W}_2^{l+1/2,0}(G_\infty)} + \sup_{t>0} \|r(\cdot,t)\|_{W_2^{l+1}(\Gamma_0)} + \sup_{t>0} t\|r(\cdot,t)\|_{W_2^{l}(\Gamma_0)}$$
$$\leq c\Big(\|\boldsymbol{w}_0\|_{W_2^{l+1}(\Omega_0)} + \|\rho_0\|_{W_2^{l+1}(\mathcal{G})}\Big). \tag{1.26}$$

Thus, $\boldsymbol{w}, s \to 0$ and $\Omega_t \to \mathcal{F}_{\theta_\infty}$ as $t \to \infty$, which means the stability of the regime (1.1) of rigid rotation.

The conditions (1.8), (1.18) are trivial in the case of axially symmetric $\mathcal{F}$ (since $h(y) = 0$). In the general case we have (1.25); as we shall see, it may be regarded as the approximate condition (1.8) for $\widehat{\rho}(z,t)$, $z \in \mathcal{G}_{\theta(t)}$.

The quadratic form (1.4) is the second variation of the energy functional

$$\mathcal{R} = \frac{\beta^2}{2\int_\Omega |x'|^2 dx} - \frac{\kappa}{2}\int_\Omega \int_\Omega \frac{dxdy}{|x-y|} - p_0|\Omega| \tag{1.27}$$

where $\beta = \omega \int_{\mathcal{F}} |x'|^2 dx$ is the magnitude of the total angular momentum of the rotating liquid and $\Omega$ is the domain in $\mathbb{R}^3$ close to $\mathcal{F}$ and having the same volume and the position of the barycenter as $\mathcal{F}$. If the boundary of $\Omega$ is given by the equation $x = y + \boldsymbol{N}(y)\rho(y,t)$, $y \in \mathcal{F}$, then the above-mentioned properties of $\Omega$ can be expressed in terms of $\rho$ as follows:

$$\int_{\mathcal{G}} \varphi(y,\rho)dS = 0, \quad \int_{\mathcal{G}} \psi_i(y,\rho)dS = 0, \quad i = 1,2,3, \tag{1.28}$$

where

$$\varphi(y,\rho) = \rho - \frac{\rho^2}{2}\mathcal{H}(y) + \frac{\rho^3}{3}\mathcal{K}(y),$$
$$\psi_i(y,\rho) = \varphi(y,\rho)y_i + N_i(y)\Big(\frac{\rho^2}{2} - \frac{\rho^3}{3}\mathcal{H}(y) + \frac{\rho^4}{4}\mathcal{K}(y)\Big), \tag{1.29}$$

$\mathcal{H}(y)$ and $\mathcal{K}(y)$ are the doubled mean curvature and the Gaussian curvature of $\mathcal{G}$, respectively. In particular, these conditions are satisfied by $\widehat{\rho}$. Direct calculation shows that the first variation of $\mathcal{R}$ (considered as a functional defined on the set of small $\rho$ satisfying (1.28)) vanishes in view of (1.2) and the second variation coincides with the form (1.4); moreover, if the form (1.4) is positive definite for arbitrary $\rho$ satisfying (1.7), (1.8), then the difference $\mathcal{R} - \mathcal{R}_0$ where $\mathcal{R}_0 = \mathcal{R}|_{\rho=0}$ is equivalent to $\|\rho\|^2_{L_2(\mathcal{G})}$ for small $\rho$ satisfying (1.28), (1.25).

It should be observed that $\delta^2\mathcal{R}[h] = 0$.

When the surface tension is taken into account, then the extra term $\sigma H$ appears in the boundary conditions, where $\sigma$ is a positive constant coefficient of the surface tension and $H$ is the doubled mean curvature of $\Gamma_t$. This term is a strong regularizer of the problem, moreover, it guarantees the exponential decay of the solution of (1.9), as $t \to \infty$. The problem of stability of the rotating capillary viscous incompressible self-gravitating liquid is treated in a series of papers of the author, partly in collaboration with Professor M. Padula. In particular, the analogue of Theorem 1.1 for non-symmetric equilibrium figures is proved in [7].

As it has been pointed out, our main attention is given to the case of non-symmetric $\mathcal{F}$. Section 2 is devoted to the construction of $\theta(t)$ and to the proof of (1.22). In Section 3 the general scheme of the proof of Theorem 1.1 is presented and the necessary transformations of the problem (1.14) are carried out. In Section 4 the main estimate of $\theta'(t)$ is obtained, as well as some important auxiliary inequalities, whose proof requires additional calculations in the case of non-symmetric $\mathcal{F}$. Finally, in Section 5 the "generalized energy" is estimated, which furnishes uniform bounds for some weak norms of the solution of the problem (1.14). In the case of symmetric $\mathcal{F}$ these bounds are obtained in [8].

## 2. On the construction of $\theta(t)$

This section is devoted to the construction of the function $\theta(t)$. At first we introduce some notations (some of them are introduced above).

By $\mathcal{F}_\theta$ we mean the family of equilibrium figures obtained by rotation of the angle $\theta$ of one of them, $\mathcal{F}_0$, about the $x_3$-axis, $\mathcal{G}_\theta$ is the boundary of $\mathcal{F}_\theta$, $\boldsymbol{N}_\theta$ is the exterior normal to $\mathcal{G}_\theta$.

We set

$$R_\theta(x) = \pm \mathrm{dist}\,(x, \mathcal{G}_\theta), \tag{2.1}$$

with the signs "+" and "−" corresponding to the cases $x \in \mathbb{R}^3 \setminus \mathcal{F}_\theta$ and $x \in \mathcal{F}_\theta$, respectively. The function $R_\theta$ is smooth in a certain neighborhood ($\delta_1$-neighborhood) of $\mathcal{G}_\theta$ and it possesses the property

$$\nabla R_\theta(x) = \boldsymbol{N}_\theta(\bar{x}^\theta), \tag{2.2}$$

where $\bar{x}^\theta$ is the closest point of $\mathcal{G}_\theta$ to $x$. We have $x = \bar{x}^\theta + \boldsymbol{N}(\bar{x}^\theta)R_\theta(x)$, i.e.,

$$\bar{x}^\theta = x - R_\theta(x)\nabla R_\theta(x) \equiv \mathfrak{R}_\theta(x). \tag{2.3}$$

The function $\mathfrak{R}_\theta$ is also smooth in the $\delta_1$-neighborhood of $\mathcal{G}_\theta$. In the case $\theta = 0$ the index 0 is sometimes omitted, in particular, $R_0(x) = R(x)$.

It is easily seen that $R(y) = R_\theta(\mathcal{Z}(\theta)y)$, i.e., $R_\theta(z) = R(\mathcal{Z}(-\theta)z)$, and $\mathcal{Z}(\theta)\boldsymbol{N}_0(y) = \boldsymbol{N}_\theta(z)$. It is also easily verified that $h_\theta(\mathcal{Z}(\theta)y) = h_0(y)$, $y \in \mathcal{G}_0$, and that $b_\theta(z) = b_0(y)$, where $b_\theta(z) = -\omega^2 z' \cdot \boldsymbol{N}_\theta(z) - \kappa \frac{\partial \mathcal{U}_\theta(z)}{\partial N_\theta}$, $\mathcal{U}_\theta(z) = \int_{\mathcal{F}_\theta} \frac{d\zeta}{|z-\zeta|}$. It follows that the quadratic form (1.4) is invariant under the rotation about the $x_3$-axis.

Let us consider the family of surfaces $\Gamma_t$ given by the equation (1.13) with $\xi \in \Gamma_0$. In the case of small $\rho_0$ and $\boldsymbol{u}$ these surfaces are close to a certain $\mathcal{G}$ (say, $\mathcal{G}_0$) – see [1], Proposition 4.5. We want to construct the function $\theta(t)$ such that $\Gamma_t$ can be given by (1.21) with $\widehat{\rho}$ satisfying the condition similar to (1.8).

Let $\Gamma_{t,\lambda}$ be a surface obtained by rotation of $\Gamma_t$ through the angle $\lambda$ about the $x_3$-axis: $\Gamma_{t,\lambda} = \mathcal{Z}(\lambda)\Gamma_t$, where

$$\mathcal{Z}(\lambda) = \begin{pmatrix} \cos\lambda & -\sin\lambda & 0 \\ \sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For small $\lambda$, $\Gamma_{t,\lambda}$ is also located in a certain small neighborhood of $\mathcal{G}_0$, and can be defined by the equation

$$x = y + \boldsymbol{N}_0(y)\widetilde{\rho}(y,t,\lambda), \quad y \in \mathcal{G}_0. \tag{2.4}$$

It follows that

$$\widetilde{\rho}(y,t,\lambda) = R(\mathcal{Z}(\lambda)X(\xi,t))$$

and $y = \overline{\mathcal{Z}(\lambda)X}$.

We look for the function $\lambda(t)$ such that

$$\int_{\Gamma_0} R(\mathcal{Z}(\lambda(t))X(\xi,t))h_0(\overline{\mathcal{Z}X})dS_\xi = 0, \tag{2.5}$$

which is equivalent to (1.25) with $r(\xi,t) = R(\mathcal{Z}(\lambda(t))X(\xi,t))$, $\theta(t) = -\lambda(t)$. Moreover, by Proposition 4.2 in [1], (2.5) can be written in the form

$$\int_{\mathcal{G}_0} \widetilde{\rho}(y,t,\lambda(t))h_0(y)\Psi^{-1}dS = 0, \tag{2.6}$$

where

$$\Psi = \frac{|A(\xi,t)\boldsymbol{n}_0(\xi)|}{|\widehat{\mathcal{L}}^T(y,\widetilde{\rho})\boldsymbol{N}_0(y)|}, \quad y = \overline{\mathcal{Z}(\lambda(t))X(\xi,t)}. \tag{2.7}$$

By $\widehat{\mathcal{L}}^T(y,\widetilde{\rho})$ we mean the co-factors matrix of the matrix of Jacobi of the transformation (2.4), and the sign "$T$" means transposition. If $\rho_0$ and $\boldsymbol{u}$ are small, then $\Psi^{-1}$ is close to 1.

In the paper [7] where the stability of the rotating capillary liquid was analyzed, we were looking for $\lambda = \lambda(t)$ such that

$$\int_{\mathcal{G}_0} \widetilde{\rho}(y,t,\lambda(t))h_0(y)dS = 0,$$

but when the surface tension is neglected, then the equation (2.5) is more convenient for technical reasons.

Let us compute the partial derivative of the function

$$f(t,\lambda) = \int_{\Gamma_0} R(\mathcal{Z}(\lambda)X(\xi,t))h_0(\overline{\mathcal{Z}X})dS_\xi \tag{2.8}$$

with respect to $\lambda$. Since

$$\begin{aligned}\frac{\partial R(\mathcal{Z}(\lambda)X(\xi,t))}{\partial\lambda} &= \boldsymbol{N}_0(\overline{\mathcal{Z}X})\cdot\mathcal{Z}'(\lambda)X = \boldsymbol{N}_0(\overline{\mathcal{Z}X})\cdot\mathcal{Z}(\boldsymbol{e}_3\times X)\\ &= \boldsymbol{N}_\theta(\bar{X}^\theta)\cdot(\boldsymbol{e}_3\times\bar{X}^\theta) = h_\theta(\bar{X}^\theta) = h_0(\overline{\mathcal{Z}(\lambda)X}),\end{aligned}$$

we have

$$f_\lambda(t,\lambda) = \int_{\mathcal{G}_0} h_0^2(y)\Psi^{-1}dS_y + \int_{\Gamma_0} R(\mathcal{Z}X)\nabla h_0(\overline{\mathcal{Z}X})\cdot\Big(\nabla\mathfrak{R}(\mathcal{Z}(\lambda)X)\mathcal{Z}(\boldsymbol{e}_3\times X)\Big)dS_\xi. \tag{2.9}$$

If $\boldsymbol{u} \in \widetilde{W}_2^{2+l,1+l/2}(Q_T)$ is small, then, by Proposition 5.4 in [9], $X(\xi,t)$ is bounded by a constant independent of $t$ and $|\Psi^{-1}| \geq k > 0$. This implies

$$f_\lambda(t,\lambda) \geq k \int_{\mathcal{G}_0} h_0^2(y)dS - c_0\delta_1 \geq \frac{k}{2}\int_{\mathcal{G}_0} h_0^2(y)dS, \tag{2.10}$$

provided $c_0\delta_1 \leq \frac{k}{2}\int_{\mathcal{G}_0} h_0^2(y)dS$. For $\lambda = 0$ we have

$$f(t,0) = \int_{\Gamma_0} R(X)h_0(\bar{X})dS_\xi,$$

hence in the interval

$$|\lambda| \leq 2k^{-1}|f(t,0)|\Big(\int_{\mathcal{G}_0} h_0^2(y)dS\Big)^{-1} = 2k^{-1}\Big|\int_{\Gamma_0} R(X)h_0(\bar{X})dS\Big|\Big(\int_{\mathcal{G}_0} h_0^2(y)dS\Big)^{-1}$$

there exists the number $\lambda(t)$ that is sought.

We set $\widetilde{\rho}(y,t) = \widetilde{\rho}(y,t,\lambda(t))$, $y \in \mathcal{G}_0$, $\theta(t) = -\lambda(t)$ and

$$\widehat{\rho}(z,t) = \widetilde{\rho}(\mathcal{Z}(\lambda(t))z,t), \quad z \in \mathcal{G}_{\theta(t)}. \tag{2.11}$$

It is clear that the equation (2.4) for the surface $\mathcal{Z}(\lambda)\Gamma_t$ is equivalent to the equation (1.21) for $\Gamma_t$. The condition (1.25) is equivalent to (2.5) and to

$$\int_{\mathcal{G}_{\theta(t)}} \widehat{\rho}(z,t)h_{\theta(t)}(z)\Psi_{\theta(t)}^{-1}dS = 0, \tag{2.12}$$

where

$$\Psi_\theta = \frac{|A(\xi,t)\boldsymbol{n}_0(\xi)|}{|\widehat{\mathcal{L}}^T(z,\widehat{\rho})\boldsymbol{N}_\theta(z)|}, \quad z = \bar{X}^\theta(\xi,t),$$

and $\widehat{\mathcal{L}}^T(z,\widehat{\rho})$ is a co-factors matrix corresponding to the transformation (1.21). It can be verified that $\Psi_\theta = \Psi$.

In particular, if $\Gamma_0$ is sufficiently close to a certain $\mathcal{G}'$, then there exists such $\theta_0$ that $\Gamma_0$ is representable in the form (1.17) with $y \in \mathcal{Z}(\theta_0)\mathcal{G}' \equiv \mathcal{G}_0$ and with $\rho_0$ satisfying (1.18). This defines the choice of $\mathcal{G}_0$; we also have $\lambda(0) = 0$.

By the implicit function theorem, $\lambda(t)$ possesses the derivative

$$\lambda'(t) = -\frac{f_t(t,\lambda)}{f_\lambda(t,\lambda)}\Big|_{\lambda=\lambda(t)}, \tag{2.13}$$

where

$$\begin{aligned} f_t(t,\lambda) = &\int_{\Gamma_0} \boldsymbol{N}_0(\overline{\mathcal{Z}X})\cdot\mathcal{Z}(\lambda)\boldsymbol{u}(\xi,t)h_0(\overline{\mathcal{Z}X})dS \\ &+ \int_{\Gamma_0} R(\mathcal{Z}X)\nabla h_0(\overline{\mathcal{Z}X})\cdot\nabla\mathfrak{R}(\mathcal{Z}X)\mathcal{Z}(\lambda)\boldsymbol{u}(\xi,t)dS, \end{aligned} \tag{2.14}$$

and $f_\lambda$ is defined in (2.9). It is easily seen that

$$|\lambda'(t)| \leq \frac{|f_t(t,\lambda)|}{|f_\lambda(t,\lambda)|}\Big|_{\lambda=\lambda(t)} \leq c\int_{\Gamma_0} |\boldsymbol{u}(\xi,t)|dS_\xi, \tag{2.15}$$

hence for $\boldsymbol{u} \in \widetilde{W}_2^{l+2,1+l/2}(Q_\infty)$

$$\lambda(t) = \int_0^t \lambda'(\tau)d\tau \to \lambda_\infty, \quad \text{as} \quad t \to \infty. \tag{2.16}$$

Thus we have proved the following proposition.

**Proposition 2.1**. *If $\Gamma_t$ is defined by* (1.13) *with $\xi \in \Gamma_0$ and the norms $\|\rho_0\|_{W_2^{l+3/2}(\Gamma_0)}$ and $\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,1+l/2}(Q_T)}$ are sufficiently small, then there exists a function $\lambda(t)$ satisfying* (2.15), (2.16) *such that $\Gamma_t$ can be given by* (1.21), *and $\widehat{\rho}$ satisfies* (2.12) *with $\theta(t) = -\lambda(t)$.*

Moreover, the following proposition holds.

**Proposition 2.2.** *If $\boldsymbol{u} \in \widetilde{W}_2^{2+l,1+l/2}(Q_T)$, then*

$$\|\lambda'\|_{\widetilde{W}_2^{l/2+3/4}(0,T)} + \sup_{t<T}|\lambda(t)| \leq c\|\boldsymbol{u}\|_{\widetilde{W}_2^{0,l/2+3/4}(G_T)} \leq c\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,1+l/2}(G_T)} \tag{2.17}$$

*with the constant independent of $T \leq \infty$.*

We observe in conclusion that $\lambda'(t)$ can be represented in the form

$$\lambda'(t) = -\frac{\int_{\Gamma_0} \boldsymbol{N}_0(\bar{\xi}) \cdot \boldsymbol{u}(\xi,t)h_0(\bar{\xi})dS_\xi}{\int_{\Gamma_0} h_0^2(\bar{\xi})dS_\xi} + m(t), \tag{2.18}$$

where

$$m(t) = \frac{\int_{\Gamma_0} \boldsymbol{N}_0(\bar{\xi}) \cdot \boldsymbol{u}(\xi,t)h_0(\bar{\xi})dS_\xi - f_t(t,\lambda(t))}{f_\lambda(t,\lambda(t))}$$
$$+ \int_{\Gamma_0} \boldsymbol{N}_0(\bar{\xi}) \cdot \boldsymbol{u}(\xi,t)h_0(\bar{\xi})dS_\xi\Big(\frac{1}{\int_{\Gamma_0} h_0^2(\bar{\xi})dS_\xi} - \frac{1}{f_\lambda(t,\lambda(t))}\Big). \tag{2.19}$$

The first term in (2.18) is a linear part of $\lambda'(t)$ with respect to $\boldsymbol{u}$ and $m(t)$ is a nonlinear remainder. The estimate of $m(t)$ and the proof of Proposition 2.2 is given below in Section 4.

## 3. Scheme of the proof of Theorem 1.1

As the first step, we reproduce (with necessary modifications) the transformation of the problem (1.14) made in [1] in the symmetric case. We introduce the projection $\Pi\boldsymbol{f} = \boldsymbol{f} - \boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{f})$ and write the boundary condition $T_u(\boldsymbol{u},q)\boldsymbol{n} = M\boldsymbol{n}$, where $M = \frac{\omega^2}{2}|x'|^2 + \kappa U(x,t) + p_0$, in an equivalent way as follows:

$$\Pi_0\Pi S_u(\boldsymbol{u})\boldsymbol{n} = 0, \quad -q + \nu\boldsymbol{n} \cdot S_u(\boldsymbol{u})\boldsymbol{n} = M.$$

Next, we make use of (1.2) and write $M$ in the form

$$M = \frac{\omega^2}{2}|X'|^2 + \kappa U(X,t) + p_0 = \frac{\omega^2}{2}(|x'|^2 - |z'|^2) + \kappa(U(x,t) - \mathcal{U}(z)),$$

where $x = X(\xi,t) \in \Gamma_t$ and $z = \bar{X}^\theta \in \mathcal{G}_{\theta(t)}$. Let $y = \mathcal{Z}(\lambda(t))z$. As in [1], we have

$$M = -B_0(z)\widehat{\rho}(z,t) + \frac{\omega^2}{2}|\boldsymbol{N}'_\theta(z)|^2\widehat{\rho}^2(z,t) + \kappa\int_0^1 (1-s)\frac{\partial^2 U_s}{\partial s^2}ds, \tag{3.1}$$

where

$$B_0(z)\widehat{\rho}(z,t) = b(z)\widehat{\rho}(z,t) - \kappa\int_{\mathcal{G}_{\theta(t)}} \frac{\widehat{\rho}(\zeta,t)dS}{|z-\zeta|} = b(y)\widetilde{\rho}(y,t) - \kappa\int_{\mathcal{G}_0}\frac{\widetilde{\rho}(\eta,t)dS}{|y-\eta|}, \tag{3.2}$$

$$U_s(z,t) = \int_{\mathcal{F}_\theta} \frac{L_s(\zeta,t)d\zeta}{|\boldsymbol{e}_{s\widehat{\rho}}(z) - \boldsymbol{e}_{s\widehat{\rho}}(\zeta)|}, \tag{3.3}$$

$$\boldsymbol{e}_{s\widehat{\rho}}(z) = z + \boldsymbol{N}^*_\theta(z)\widehat{\rho}^*(z,t), \tag{3.4}$$

$\boldsymbol{N}^*$ and $\widehat{\rho}^*$ are extensions of $\boldsymbol{N}_\theta$ and $\widehat{\rho}$ from $\mathcal{G}_\theta$ in $\mathcal{F}_\theta$, and $L_s(z,t)$ is the Jacobian of the transformation (3.4). When we pass in (3.2) to the variables $\xi \in \Gamma_0$, according to the formula $y = \overline{\mathcal{Z}(\lambda(t))X(\xi,t)}$, we obtain

$$\begin{aligned} B_0\widehat{\rho} &= b(\bar{X}^\theta)r(\xi,t) - \kappa\int_{\Gamma_0}\frac{r(\eta,t)\Psi(\eta,t)dS}{|\bar{X}^\theta(\xi,t) - \bar{X}^\theta(\eta,t)|} \\ &= b(\overline{\mathcal{Z}(\lambda(t))X})r - \kappa\int_{\Gamma_0}\frac{r(\eta,t)\Psi(\eta,t)dS}{|\overline{\mathcal{Z}X}(\xi,t) - \overline{\mathcal{Z}X}(\eta,t)|}, \end{aligned}$$

where $r$ is the function (1.24), i.e.,

$$r(\xi,t) = R(\mathcal{Z}(\lambda(t))X(\xi,t)) = \widetilde{\rho}(\overline{\mathcal{Z}X},t) = \widehat{\rho}(\bar{X}^\theta,t).$$

It follows that

$$B_0\widehat{\rho} = B'_0(\xi)r + B_1(r,\boldsymbol{u}),$$

where

$$\begin{aligned} B'_0(\xi)r &= b(\bar{\xi})r(\xi,t) - \kappa\int_{\Gamma_0}\frac{r(\eta,t)dS}{|\bar{\xi}-\bar{\eta}|}, \\ B_1(r,\boldsymbol{u}) &= (b(\overline{\mathcal{Z}(\lambda(t))X}) - b(\bar{\xi}))r(\xi,t) \\ &\quad - \kappa\int_{\Gamma_0}\frac{r(\eta,t)\Psi(\eta,t)dS}{|\overline{\mathcal{Z}X}(\xi,t) - \overline{\mathcal{Z}X}(\eta,t)|} + \kappa\int_{\Gamma_0}\frac{r(\eta,t)dS}{|\bar{\xi}-\bar{\eta}|}, \\ M &= -B'_0 r + B_1(r,\boldsymbol{u}) + \frac{\omega^2}{2}|\boldsymbol{N}'_0(y)|^2\widetilde{\rho}^2(y,t) + \kappa\int_0^1(1-s)\frac{\partial^2 U_s}{\partial s^2}ds. \end{aligned} \tag{3.5}$$

Next, we make one more modification of the problem (1.14) by inserting the function $r$ into it. We note that $r(\xi,0) = R(\xi) = \rho_0(\bar{\xi})$ and

$$\begin{aligned} r_t(\xi,t) &= \boldsymbol{N}_0(\overline{\mathcal{Z}(\lambda(t))X})\cdot\mathcal{Z}(\lambda(t))\Big(\boldsymbol{u}(\xi,t) + \lambda'(t)(\boldsymbol{e}_3\times X(\xi,t))\Big) \\ &= \boldsymbol{N}_0(\overline{\mathcal{Z}X})\cdot\mathcal{Z}\boldsymbol{u} + h_0(\overline{\mathcal{Z}X})\lambda'(t), \end{aligned} \tag{3.6}$$

because

$$\begin{aligned} \boldsymbol{N}_0(\overline{\mathcal{Z}X})\cdot\mathcal{Z}(\boldsymbol{e}_3\times X) &= \boldsymbol{N}_\theta(\bar{X}^\theta)\cdot(\boldsymbol{e}_3\times\bar{X}^\theta + \boldsymbol{N}_\theta(\bar{X}^\theta)\widehat{\rho}) \\ &= \boldsymbol{N}_\theta(\bar{X}^\theta)\cdot(\boldsymbol{e}_3\times\bar{X}^\theta) = h_\theta(\bar{X}^\theta) = h_0(\overline{\mathcal{Z}(\lambda)X}). \end{aligned}$$

Thus, $(\boldsymbol{u},q,r)$ can be regarded as a solution to the problem

$$\begin{gathered}\boldsymbol{u}_t+2\omega(\boldsymbol{e}_3\times\boldsymbol{u})-\nu\nabla^2\boldsymbol{u}+\nabla q=\boldsymbol{l}_1(\boldsymbol{u},q),\\ \nabla\cdot\boldsymbol{u}=l_2(\boldsymbol{u}),\qquad \xi\in\Omega_0,\ \ t>0,\\ \Pi_0S(\boldsymbol{u})\boldsymbol{n}_0=\boldsymbol{l}_3(\boldsymbol{u}),\\ -q+\nu\boldsymbol{n}_0\cdot S(\boldsymbol{u})\boldsymbol{n}_0+B_0'(\xi)r=l_4(\boldsymbol{u})+l_5(\boldsymbol{u},r),\\ r_t(\xi,t)=\boldsymbol{N}_0(\bar\xi)\cdot\boldsymbol{u}-\frac{h_0(\bar\xi)}{\|h_0(\bar\xi)\|^2_{L_2(\Gamma_0)}}\int_{\Gamma_0}\boldsymbol{u}(\eta,t)\cdot\boldsymbol{N}_0(\bar\eta)h_0(\bar\eta)dS+l_6(\boldsymbol{u}),\quad \xi\in\Gamma_0,\\ \boldsymbol{u}(\xi,0)=\boldsymbol{w}_0(\xi),\quad \xi\in\Omega_0,\quad r(\xi,0)=\rho_0(\bar\xi),\quad \xi\in\Gamma_0.\end{gathered}\tag{3.7}$$

The expressions $\boldsymbol{l}_1$, $l_2$, $\boldsymbol{l}_3$, $l_4$, $l_5$, $l_6$ are nonlinear (at least quadratic) with respect to $\boldsymbol{u}$, $q$, $r$; they are given by the formulas

$$\begin{gathered}\boldsymbol{l}_1(\boldsymbol{u},q)=\nu(\nabla_u^2\boldsymbol{u}-\nabla^2\boldsymbol{u})+\nabla q-\nabla_u q,\\ l_2(\boldsymbol{u})=(\nabla-\nabla_u)\cdot\boldsymbol{u},\\ \boldsymbol{l}_3(\boldsymbol{u})=\Pi_0(\Pi_0S(\boldsymbol{u})\boldsymbol{n}_0-\Pi S_u(\boldsymbol{u})\boldsymbol{n}),\\ l_4(\boldsymbol{u})=\nu(\boldsymbol{n}_0\cdot S(\boldsymbol{u})\boldsymbol{n}_0-\boldsymbol{n}\cdot S_u(\boldsymbol{u})\boldsymbol{n}),\end{gathered}\tag{3.8}$$

$$l_5(\boldsymbol{u},R)=\frac{\omega^2}{2}|\boldsymbol{N}_0'(\overline{\mathcal{Z}X})|^2r^2(\xi,t)+\int_0^1(1-s)\frac{d^2U_s}{ds^2}ds+B_1(r,\boldsymbol{u}),\tag{3.9}$$

and, in view of (2.18), (2.19),

$$l_6(\boldsymbol{u})=(\mathcal{Z}^{-1}(\lambda(t))\boldsymbol{N}_0(\overline{\mathcal{Z}X})-\boldsymbol{N}_0(\bar\xi))\cdot\boldsymbol{u}(\xi,t)+(h_0(\overline{\mathcal{Z}X})-h_0(\bar\xi))\lambda'(t)+h_0(\bar\xi)m(t).\tag{3.10}$$

Owing to the Piola identity $\nabla\cdot A^T=\Big(\sum_{j=1}^3\frac{\partial}{\partial x_j}A_{ij}\Big)_{i=1,2,3}=0$, where $A^T$ means the transposed matrix $A$, we have

$$l_2(\boldsymbol{u})=\nabla\cdot\boldsymbol{L}(\boldsymbol{u}),\quad \boldsymbol{L}(\boldsymbol{u})=(I-A^T)\boldsymbol{u}.\tag{3.11}$$

Now we outline the proof of Theorem 1.1. As in [1], we use maximum regularity estimates for the solutions of the linear problem

$$\begin{gathered}\boldsymbol{v}_t+2\omega(\boldsymbol{e}_3\times\boldsymbol{v})-\nu\nabla^2\boldsymbol{v}+\nabla p=\boldsymbol{f}(\xi,t),\\ \nabla\cdot\boldsymbol{v}=f(\xi,t),\qquad \xi\in\Omega_0,\ \ t\in(0,T),\\ \Pi_0S(\boldsymbol{v})\boldsymbol{n}_0=\Pi_0\boldsymbol{d}(\xi,t),\\ -q+\nu\boldsymbol{n}_0\cdot S(\boldsymbol{v})\boldsymbol{n}_0+B_0'(\xi)r=d(\xi,t),\\ r_t(\xi,t)=\boldsymbol{N}_0(\bar\xi)\cdot\boldsymbol{v}-\frac{h_0(\bar\xi)}{\|h_0(\bar\xi)\|^2_{L_2(\Gamma_0)}}\int_{\Gamma_0}\boldsymbol{v}(\eta,t)\cdot\boldsymbol{N}_0(\bar\eta)h_0(\bar\eta)dS+g(\xi,t),\quad \xi\in\Gamma_0,\\ \boldsymbol{v}(\xi,0)=\boldsymbol{v}_0(\xi),\quad \xi\in\Omega_0,\quad r(\xi,0)=r_0(\xi),\quad \xi\in\Gamma_0.\end{gathered}\tag{3.12}$$

and of a similar problem in $\mathcal{F}_0$:

$$\begin{gathered}\boldsymbol{v}_t'+2\omega(\boldsymbol{e}_3\times\boldsymbol{v}')-\nu\nabla^2\boldsymbol{v}'+\nabla p'=\boldsymbol{f}'(x,t),\quad \nabla\cdot\boldsymbol{v}'=f'(x,t)\quad x\in\mathcal{F}_0,\\ \Pi_{\mathcal{G}}S(\boldsymbol{v})\boldsymbol{N}_0(x)=\Pi_{\mathcal{G}}\boldsymbol{d}'(x,t),\\ -p'+\nu\boldsymbol{N}_0\cdot S(\boldsymbol{v}')\boldsymbol{N}_0+B_0(x)r'(x,t)=d'(x,t),\end{gathered}\tag{3.13}$$

$$r'_t = \boldsymbol{N}_0(x) \cdot \boldsymbol{v}'(x,t) - \frac{h_0(x)}{\|h_0\|^2_{L_2(\mathcal{G}_0)}} \int_{\mathcal{G}_0} \boldsymbol{v}'(y,t) \cdot \boldsymbol{N}_0(y) h_0(y) dS + g'(x,t), \quad x \in \mathcal{G}_0,$$

$$\boldsymbol{v}'(x,0) = \boldsymbol{v}'_0(x), \qquad x \in \mathcal{F}_0, \quad r'(x,0) = r'_0(x), \quad x \in \mathcal{G}_0,$$

where $\Pi_{\mathcal{G}} \boldsymbol{f} = \boldsymbol{f} - \boldsymbol{N}_0(\boldsymbol{N}_0 \cdot \boldsymbol{f})$. In comparison with the case of axially symmetric $\mathcal{F}$, these problems contain an extra integral term in the boundary conditions.

We consider at first the problem (3.13).

**Theorem 3.1.** *Let $l \in (1, 3/2)$, $\mathfrak{Q}_T = \mathcal{F}_0 \times (0,T)$, $\mathfrak{G}_T = \mathcal{G}_0 \times (0,T)$ and let the data of the problem* (3.13) *possess the following regularity properties:*

$$\boldsymbol{f}' \in W_2^{l,l/2}(\mathfrak{Q}_T), \quad f' \in W_2^{1+l,0}(\mathfrak{Q}_T), \quad f' = \nabla \cdot \boldsymbol{F}', \ \boldsymbol{F}' \in W_2^{0,1+l/2}(\mathfrak{Q}_T),$$

$$\boldsymbol{v}'_0 \in W_2^{1+l}(\mathcal{F}), \quad r'_0 \in W_2^{l+1}(\mathcal{G}_0), \quad \boldsymbol{d}' \in W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T),$$

$$d' \in W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T), \quad g' \in W_2^{l+3/2,l/2+3/4}(\mathfrak{G}_T).$$

*Assume also that the compatibility conditions*

$$\nabla \cdot \boldsymbol{v}'_0 = f'(x,0), \quad x \in \mathcal{F}_0, \quad \Pi_{\mathcal{G}} S(\boldsymbol{v}'_0) \boldsymbol{N}_0 = \Pi_{\mathcal{G}} \boldsymbol{d}'(x,0), \quad x \in \mathcal{G}_0$$

*are satisfied. Then the problem* (3.13) *has a unique solution $\boldsymbol{v}' \in W_2^{2+l,1+l/2}(\mathfrak{Q}_T)$, $\nabla p' \in W_2^{l,l/2}(\mathfrak{Q}_T)$, $r' \in W_2^{l+1/2,0}(\mathfrak{G}_T)$, such that $p'|_{\mathfrak{G}_T} \in W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)$, $r'(\cdot,t) \in W_2^{l+1}(\mathcal{G}_0)$ for arbitrary $t \in (0,T)$, and*

$$\begin{aligned}
\mathcal{Y}(T) \equiv{}& \|\boldsymbol{v}'\|_{W_2^{2+l,1+l/2}(\mathfrak{Q}_T)} + \|\nabla p'\|_{W_2^{l,l/2}(\mathfrak{Q}_T)} + \|p'\|_{W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)} \\
&+ \|r'\|_{W_2^{l+1/2,0}(\mathfrak{G}_T)} + \sup_{t<T} \|r'(\cdot,t)\|_{W_2^{l+1}(\mathcal{G}_0)} \\
\le{}& c\Big(\mathcal{N}(T) + \Big(\int_0^T (\|\boldsymbol{v}'\|^2_{L_2(\mathcal{F}_0)} + \|r'\|^2_{W_2^{-1/2}(\mathcal{G}_0)}) dt\Big)^{1/2}\Big),
\end{aligned} \tag{3.14}$$

*where*

$$\begin{aligned}
\mathcal{N}(T) ={}& \|\boldsymbol{f}'\|_{W_2^{l,l/2}(\mathfrak{Q}_T)} + \|f'\|_{W_2^{l+1,0}(\mathfrak{Q}_T)} + \|\boldsymbol{F}'\|_{W_2^{0,1+l/2}(\mathfrak{Q}_T)} \\
&+ \|r'_0\|_{W_2^{l+1}(\mathcal{G}_0)} + \|\boldsymbol{v}'_0\|_{W_2^{1+l}(\mathcal{F}_0)} + \|\boldsymbol{d}'\|_{W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)} \\
&+ \|d'\|_{W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)} + \|g'\|_{W_2^{l+3/2,l/2+3/4}(\mathfrak{G}_T)}.
\end{aligned}$$

*Moreover, if*

$$\boldsymbol{f}' \in \widetilde{W}_2^{l,l/2}(\mathfrak{Q}_T), \quad \boldsymbol{d}' \in \widetilde{W}_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T), \quad d' \in \widetilde{W}_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T),$$

$$g' \in \widetilde{W}_2^{l+3/2,l/2+3/4}(\mathfrak{G}_T), \quad f' \in \widetilde{W}_2^{1+l,0}(\mathfrak{Q}_T), \quad \boldsymbol{F}' \in \widetilde{W}_2^{0,1+l/2}(\mathfrak{Q}_T)$$

(*this means that*

$$f' \in W_2^{1+l,0}(\mathfrak{Q}_T), \quad tf' \in W_2^{l,0}(\mathfrak{Q}_T),$$

$$\boldsymbol{F}' \in W_2^{0,1+l/2}(\mathfrak{Q}_T), \quad t\boldsymbol{F}' \in W_2^{0,(l+1)/2}(\mathfrak{Q}_T)),$$

*then*

$$
\begin{aligned}
\widetilde{\mathcal{Y}}(T) \equiv &\|\boldsymbol{v}'\|_{\widetilde{W}_2^{2+l,1+l/2}(\mathfrak{Q}_T)} + \|\nabla p'\|_{\widetilde{W}_2^{l,l/2}(\mathfrak{Q}_T)} + \|p'\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)} \\
&+ \|r'\|_{\widetilde{W}_2^{l+1/2,0}(\mathfrak{G}_T)} + \sup_{t<T}\|r'(\cdot,t)\|_{W_2^{l+1}(\mathcal{G}_0)} + \sup_{t<T} t\|r'(\cdot,t)\|_{W_2^{l}(\mathcal{G}_0)} \\
&\leq c\Big(\widetilde{\mathcal{N}}(T) + \Big(\int_0^T (1+t^2)(\|\boldsymbol{v}'\|^2_{L_2(\mathcal{F}_0)} + \|r'\|^2_{W_2^{-1/2}(\mathcal{G}_0)})dt\Big)^{1/2}\Big), \qquad (3.15)
\end{aligned}
$$

*where*

$$
\begin{aligned}
\widetilde{\mathcal{N}}(T) = &\|\boldsymbol{f}'\|_{\widetilde{W}_2^{l,l/2}(\mathfrak{Q}_T)} + \|f'\|_{\widetilde{W}_2^{l+1,0}(\mathfrak{Q}_T)} + \|\boldsymbol{F}'\|_{\widetilde{W}_2^{0,1+l/2}(\mathfrak{Q}_T)} \\
&+ \|r_0'\|_{W_2^{l+1}(\mathcal{G}_0)} + \|\boldsymbol{v}_0'\|_{W_2^{1+l}(\mathcal{F})} + \|\boldsymbol{d}'\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)} \\
&+ \|d'\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)} + \|g'\|_{\widetilde{W}_2^{l+3/2,l/2+3/4}(\mathfrak{G}_T)}.
\end{aligned}
$$

*The constants in* (3.14), (3.15) *are independent of* $T$.

In fact, the inequality (3.14) is valid for $l \in (0, 5/2)$, and (3.15) is obtained by combination of (3.14) with $l$ and $l-1$. The proof is given in [9, 10]. The problem (3.12) reduces to (3.13) by the transformation

$$\xi = x + \boldsymbol{N}_0^*(x)\rho_0^*(x) \equiv e_{\rho_0}(x), \;\; x \in \mathcal{F}_0, \qquad (3.16)$$

where $\boldsymbol{N}_0^*$ and $\rho_0^*$ are extensions of $\boldsymbol{N}_0$ and $\rho_0$ from $\mathcal{G}_0$ into $\mathcal{F}_0$ such that $\boldsymbol{N}_0^*$ is sufficiently regular and

$$\|\rho_0^*\|_{W_2^{l+2}(\mathcal{F}_0)} \leq c\|\rho_0\|_{W_2^{l+3/2}(\mathcal{G}_0)}. \qquad (3.17)$$

This transformation converts (3.12) to

$$
\begin{gathered}
\boldsymbol{v}_t' + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{v}') - \nu\nabla^2\boldsymbol{v}' + \nabla p' = \boldsymbol{f}'(x,t) + \boldsymbol{m}_1(\boldsymbol{v}', p'), \\
\nabla \cdot \boldsymbol{v}' = L_0 f'(x,t) + m_2(\boldsymbol{v}'), \quad x \in \mathcal{F}, \\
\Pi_{\mathcal{G}} S(\boldsymbol{v}')\boldsymbol{N}_0 = \Pi_{\mathcal{G}}\Pi_0\boldsymbol{d}' + \boldsymbol{m}_3(\boldsymbol{v}'), \\
-p' + \nu\boldsymbol{N}_0 \cdot S(\boldsymbol{v}')\boldsymbol{N}_0 + B_0(x)r'(x,t) = d'(x,t) + m_4(\boldsymbol{v}'), \qquad (3.18) \\
r_t' = \boldsymbol{N}_0(x) \cdot \boldsymbol{v}'(x,t) \\
- \frac{h_0(x)}{\|h_0\|^2_{L_2(\mathcal{G}_0)}} \int_{\mathcal{G}_0} \boldsymbol{v}'(y,t) \cdot \boldsymbol{N}_0(y)h_0(y)dS + g'(x,t) + m_5(\boldsymbol{v}', r'), \quad x \in \mathcal{G}, \\
\boldsymbol{v}'(x,0) = \boldsymbol{v}_0'(x), \qquad x \in \mathcal{F}, \quad r'(x,0) = r_0'(x), \quad x \in \mathcal{G},
\end{gathered}
$$

where "′" denotes the change of variables (3.16): $f'(x,t) = f(e_{\rho_0}^{-1}(\xi), t)$. The expressions $m_i$ are given by

$$
\begin{gathered}
\boldsymbol{m}_1(\boldsymbol{v}', p') = \nu(\widehat{\nabla}^2 - \nabla^2)\boldsymbol{v}'(y,t) + (\nabla - \widehat{\nabla})p'(y,t), \\
m_2(\boldsymbol{v}') = (\nabla - L_0\widehat{\nabla}) \cdot \boldsymbol{v}', \\
\boldsymbol{m}_3(\boldsymbol{v}') = \Pi_{\mathcal{G}}(\Pi_{\mathcal{G}} S(\boldsymbol{v}')\boldsymbol{N}_0 - \Pi_0\widehat{S}(\boldsymbol{v}')\boldsymbol{n}_0), \qquad (3.19)
\end{gathered}
$$

$$m_4(\boldsymbol{v}', r') = \nu(\boldsymbol{N}_0 \cdot S(\boldsymbol{v}')\boldsymbol{N}_0 - \boldsymbol{n}_0 \cdot \widehat{S}(\boldsymbol{v}')\boldsymbol{n}_0) + \kappa \int_{\mathcal{G}_0} \frac{r'(z,t)}{|y-z|}(|\widehat{\mathcal{L}}_0^T(z,\rho_0)\boldsymbol{N}_0(z)| - 1)dS,$$

$$m_5(\boldsymbol{v}', r') = h_0(x)\Big(\Big(\int_{\mathcal{G}_0} h_0^2(y)dS\Big)^{-1} - \Big(\int_{\mathcal{G}_0} h_0^2(y)|\widehat{\mathcal{L}}_0^T(y,\rho_0)\boldsymbol{N}_0(y)|dS\Big)^{-1}\Big)$$
$$\times \int_{\mathcal{G}_0} \boldsymbol{v}' \cdot \boldsymbol{N}_0(y)h_0(y)dS$$
$$+ h_0(x)\Big(\int_{\mathcal{G}_0} h_0^2(y)|\widehat{\mathcal{L}}_0^T(y,\rho_0)\boldsymbol{N}_0(y)|dS\Big)^{-1}$$
$$\times \int_{\mathcal{G}_0} \boldsymbol{v}' \cdot \boldsymbol{N}_0(y)h_0(y)(1 - |\widehat{\mathcal{L}}_0^T(y,\rho_0)\boldsymbol{N}_0(y)|)dS.$$

By $L_0 = \det \mathcal{L}_0$ we mean the Jacobian of the transformation $e_{\rho_0}$, $\mathcal{L}_0$ is its Jacobian matrix, $\widehat{\mathcal{L}}_0 = L_0\mathcal{L}_0^{-1}$, $\widehat{\nabla} = \mathcal{L}_0^{-T}\nabla$ is a transformed gradient with respect to $\xi$, $\nabla = \Big(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3}\Big)$, $\widehat{S}(\boldsymbol{v}) = \widehat{\nabla}\boldsymbol{v} + (\widehat{\nabla}\boldsymbol{v})^T$ is a transformed rate-of-strain tensor. The normals $\boldsymbol{N}_0$ and $\boldsymbol{n}_0$ are connected with each other by

$$\boldsymbol{n}_0(e_{\rho_0}(y)) = \frac{\widehat{\mathcal{L}}_0^T\boldsymbol{N}_0(y)}{|\widehat{\mathcal{L}}_0^T\boldsymbol{N}_0(y)|}.$$

We notice that $m_2(\boldsymbol{v}')$ is representable in the divergence form:

$$(\nabla - L_0\widehat{\nabla}) \cdot \boldsymbol{v}' = (\nabla - \widehat{\mathcal{L}}^T\nabla) \cdot \boldsymbol{v}' = \nabla \cdot (I - \widehat{\mathcal{L}}^T)\boldsymbol{v}' \equiv \nabla \cdot \boldsymbol{M},$$

where $\boldsymbol{M} = (I - \widehat{\mathcal{L}})\boldsymbol{v}'$.

The expressions (3.19) are linear functions of their arguments with small coefficients proportional to the derivatives of $\rho_0$. Under the assumption (1.19) they satisfy the inequality

$$\|\boldsymbol{m}_1\|_{\widetilde{W}_2^{l,l/2}(\mathfrak{Q}_T)} + \|m_2\|_{\widetilde{W}_2^{l+1,0}(\mathfrak{Q}_T)} + \|\boldsymbol{M}\|_{\widetilde{W}_2^{0,1+l/2}(\mathfrak{Q}_T)}$$
$$+ \|\boldsymbol{m}_3(\boldsymbol{v})\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)} + \|m_4(\boldsymbol{v},r)\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)}$$
$$+ \|m_5(\boldsymbol{v},r)\|_{\widetilde{W}_2^{l+3/2,l/2+3/4}(\mathfrak{G}_T)} \le c\epsilon\widetilde{\mathcal{Y}}(T), \tag{3.20}$$

that can be obtained with the help of Proposition 4.1 in [1]. The estimate of $m_5$ follows from

$$\Big|1 - \Big|\widehat{\mathcal{L}}_0^T(z,\rho_0)\boldsymbol{N}_0(z)\Big|\Big| \le c\epsilon. \tag{3.18}$$

Using (3.14) and (3.20), it is possible to prove the solvability of the problem (3.12) and estimate the solution in a standard way, provided $\epsilon$ is sufficiently small (the details are omitted). We obtain the following result.

**Theorem 3.2.** *Let $l \in (1,3/2)$, $Q_T = \Omega_0\times(0,T)$, $G_T = \Gamma_0\times(0,T)$ and let the data of the problem (3.12) possess the following regularity properties: $\boldsymbol{f} \in W_2^{l,l/2}(Q_T)$, $f \in W_2^{1+l,0}(Q_T)$, $f = \nabla \cdot \boldsymbol{F}$, $\boldsymbol{F} \in W_2^{0,1+l/2}(Q_T)$, $\boldsymbol{v}_0 \in W_2^{1+l}(\Omega_0)$, $r_0 \in W_2^{l+1}(\Gamma_0)$, $\boldsymbol{d} \in W_2^{l+1/2,l/2+1/4}(G_T)$, $d \in W_2^{l+1/2,l/2+1/4}(G_T)$, $g \in W_2^{l+3/2,l/2+3/4}(G_T)$. Assume also that the compatibility conditions*

$$\nabla \cdot \boldsymbol{v}_0 = f(\xi,0), \quad \xi \in \Omega_0, \quad \Pi_0 S(\boldsymbol{v}_0)\boldsymbol{n}_0 = \Pi_0\boldsymbol{d}(\xi,0), \quad \xi \in \Gamma_0$$

*are satisfied. Then the problem* (3.12) *has a unique solution* $\boldsymbol{v} \in W_2^{2+l,1+l/2}(Q_T)$, $\nabla p \in W_2^{l,l/2}(Q_T)$, $r \in W_2^{l+1/2,0}(G_T)$, *such that* $p|_{G_T} \in W_2^{l+1/2,l/2+1/4}(G_T)$, $r(\cdot,t) \in W_2^{l+1}(\Gamma_0)$ *for arbitrary* $t \in (0,T)$, *and*

$$Y(T) \equiv \|\boldsymbol{v}\|_{W_2^{2+l,1+l/2}(Q_T)} + \|\nabla p\|_{W_2^{l,l/2}(Q_T)} + \|p\|_{W_2^{l+1/2,l/2+1/4}(G_T)} + \|r\|_{W_2^{l+1/2,0}(G_T)}$$
$$+ \sup_{t<T} \|r(\cdot,t)\|_{W_2^{l+1}(\Gamma_0)} \le c\Big(N(T) + \Big(\int_0^T (\|\boldsymbol{v}\|^2_{L_2(\Omega_0)} + \|r\|^2_{W_2^{-1/2}(\Gamma_0)})dt\Big)^{1/2}\Big), \tag{3.21}$$

*where*

$$N(T) = \|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T)} + \|f\|_{W_2^{l+1,0}(Q_T)} + \|\boldsymbol{F}\|_{W_2^{0,1+l/2}(Q_T)} + \|r_0\|_{W_2^{l+1}(\Gamma_0)}$$
$$+ \|\boldsymbol{v}_0\|_{W_2^{1+l}(\Omega_0)} + \|\boldsymbol{d}\|_{W_2^{l+1/2,l/2+1/4}(G_T)} + \|d\|_{W_2^{l+1/2,l/2+1/4}(G_T)} + \|g\|_{W_2^{l+3/2,l/2+3/4}(G_T)}.$$

*Moreover, if*

$$\boldsymbol{f} \in \widetilde{W}_2^{l,l/2}(Q_T), \quad \boldsymbol{d} \in \widetilde{W}_2^{l+1/2,l/2+1/4}(G_T), \quad d \in \widetilde{W}_2^{l+1/2,l/2+1/4}(G_T),$$
$$g \in \widetilde{W}_2^{l+3/2,l/2+3/4}(G_T), \quad f \in \widetilde{W}_2^{1+l,0}(Q_T), \quad \boldsymbol{F} \in \widetilde{W}_2^{0,1+l/2}(Q_T)$$

(*this means that*

$$f \in W_2^{1+l,0}(Q_T), \quad tf \in W_2^{1,0}(Q_T),$$
$$\boldsymbol{F} \in W_2^{0,1+l/2}(Q_T), \quad t\boldsymbol{F} \in W_2^{0,(l+1)/2}(Q_T)),$$

*then*

$$\widetilde{Y}(T) \equiv \|\boldsymbol{v}\|_{\widetilde{W}_2^{2+l,1+l/2}(Q_T)} + \|\nabla p\|_{\widetilde{W}_2^{l,l/2}(Q_T)} + \|p\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)}$$
$$+ \|r\|_{\widetilde{W}_2^{l+1/2,0}(G_T)} + \sup_{t<T}\|r(\cdot,t)\|_{W_2^{l+1}(\Gamma_0)} + \sup_{t<T} t\|r(\cdot,t)\|_{W_2^{l}(\Gamma_0)}$$
$$\le c\Big(\widetilde{N}(T) + \Big(\int_0^T (1+t^2)(\|\boldsymbol{v}\|^2_{L_2(\Omega_0)} + \|r\|^2_{W_2^{-1/2}(\Gamma_0)})dt\Big)^{1/2}\Big), \tag{3.22}$$

*where*

$$\widetilde{N}(T) = \|\boldsymbol{f}\|_{\widetilde{W}_2^{l,l/2}(Q_T)} + \|f\|_{\widetilde{W}_2^{l+1,0}(Q_T)} + \|\boldsymbol{F}\|_{\widetilde{W}_2^{0,1+l/2}(Q_T)}$$
$$+ \|r_0\|_{W_2^{l+1}(\Gamma_0)} + \|\boldsymbol{v}_0\|_{W_2^{1+l}(\mathcal{G})} + \|\boldsymbol{d}\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)}$$
$$+ \|d\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)} + \|g\|_{\widetilde{W}_2^{l+3/2,l/2+3/4}(G_T)}.$$

*The constants in* (3.21), (3.22) *are independent of* $T$.

The norm $\|r\|_{W_2^{-1/2}(\Gamma_0)}$ is defined in a standard way:

$$\|r\|_{W_2^{-1/2}(\Gamma_0)} = \sup_{\varphi \in W_2^{1/2}(\Gamma_0)} \frac{\left|\int_{\Gamma_0} r(x)\varphi(x)dx\right|}{\|\varphi\|_{W_2^{1/2}(\Gamma_0)}}.$$

In order to be able to apply the inequality (3.22) to the problem (3.7), we need to estimate the nonlinear terms in (3.7) and the lower order norms in (3.21), (3.22).

**Theorem 3.3.** *If* $(\boldsymbol{u}, q, r)$ *satisfy the inequality*

$$\widetilde{Y}(T) \le \delta \ll 1, \tag{3.23}$$

*where* $\widetilde{Y}(T)$ *is the norm of* $(\boldsymbol{u}, q, r)$ *defined in* (3.22), *then*

$$\begin{aligned} &\|\boldsymbol{l}_1\|_{\widetilde{W}_2^{l,l/2}(Q_T)} + \|l_2\|_{\widetilde{W}_2^{1+l,0}(Q_T)} + \|\boldsymbol{L}\|_{\widetilde{W}_2^{0,(l+1)/2}(Q_T)} \\ &\qquad + \|\boldsymbol{l}_3\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)} + \|l_4\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)} \\ &\qquad + \|l_5\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)} + \|l_6\|_{\widetilde{W}_2^{l+3/2,l/2+3/4}(G_T)} \le c\widetilde{Y}^2(T). \end{aligned} \tag{3.24}$$

*with the constant c independent of* $T \ge 1$.

**Theorem 3.4**. *If the solution of the problem* (3.7) *is defined for* $t \in (0, T)$ *and* (3.23) *holds, then* $\boldsymbol{w}$ *and* $\widehat{\rho}$ *satisfy the inequality*

$$\begin{aligned} \|\boldsymbol{w}(\cdot,t)\|^2_{L_2(\Omega_t)} + \|\widehat{\rho}(\cdot,t)\|^2_{L_2(\mathcal{G}_{\theta(t)})} + \int_0^t \Big(\|\boldsymbol{w}(\cdot,t)\|^2_{L_2(\Omega_\tau)} + \|\widehat{\rho}(\cdot,t)\|^2_{W_2^{-1/2}(\mathcal{G}_{\theta(t)})}\Big) d\tau \\ \le c\Big(\|\boldsymbol{w}_0\|^2_{L_2(\Omega_0)} + \|\rho_0\|^2_{L_2(\mathcal{G}_0)}\Big) \end{aligned} \tag{3.25}$$

*with the constant independent of* $T$.

The proof of Theorem 3.4 is given in Section 5. By Proposition 4.6 in [1], (3.25) implies

$$\begin{aligned} \|\boldsymbol{u}(\cdot,t)\|^2_{L_2(\Omega_0)} + \|r(\cdot,t)\|^2_{L_2(\Gamma_0)} + \int_0^t \Big(\|\boldsymbol{u}(\cdot,t)\|^2_{L_2(\Omega_0)} + \|r(\cdot,t)\|^2_{W_2^{-1/2}(\Gamma_0)}\Big) d\tau \\ \le c\Big(\|\boldsymbol{w}_0\|^2_{L_2(\Omega_0)} + \|\rho_0\|^2_{L_2(\mathcal{G}_0)}\Big). \end{aligned} \tag{3.26}$$

As in the case of axially symmetric $\mathcal{F}$, inequalities (3.22), (3.24), (3.26) allow us to obtain the following uniform estimate of the solution of (3.7) playing a crucial role in the analysis of the problem (1.9) (cf. [1], Theorem 2.3).

**Theorem 3.5**. *Assume that the assumptions of Theorem* 1.1 *are satisfied. If the solution of* (3.7) *is defined for* $t \in (0, T)$ *and* (3.23) *holds, then*

$$\widetilde{Y}(T) \le c\Big(\|\boldsymbol{w}_0\|_{W_2^{l+1}(\Omega_0)} + \|\rho_0\|_{W_2^{l+1}(\mathcal{G}_0)}\Big). \tag{3.27}$$

Inequality (3.23) is verified in the process of the proof of the solvability of the problem (1.14). As in [1], the proof is carried out in two steps. First, using the maximum regularity estimates for the Neumann problem

$$\boldsymbol{v}_t - \nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{f}(x,t), \quad \nabla\cdot\boldsymbol{v} = f(x,t) \quad x \in \Omega_0,$$

$$T(\boldsymbol{v},p)\boldsymbol{n}_0 = \boldsymbol{d}(x,t), \quad x \in \Gamma_0,$$

$$\boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \qquad x \in \Omega_0,$$

and the estimate (3.24) of the nonlinear terms, we prove the solvability of the problem (1.14) in the interval $t \in (0,1)$, and obtain the estimate

$$\begin{aligned}&\|\boldsymbol{u}\|_{W_2^{2+l,1+l/2}(Q_1)} + \|\nabla q\|_{W_2^{l,l/2}(Q_1)} + \|q\|_{W_2^{l+1,(l+1)/2}(G_1)}\\ &\quad\leq c\Big(\|\boldsymbol{w}_0\|_{W_2^{l+1}(\Omega_0)} + \|\rho_0\|_{W_2^{l+1}(\mathcal{G}_0)}\Big)\end{aligned}$$

for the solution (cf. [1], Theorem 3.1). Then we construct $\theta(t) = -\lambda(t)$, as made in Propositions 2.1, 2.2, and estimate the function

$$r(\xi,t) = \rho_0(\bar{\xi}) + \int_0^t \Big(\boldsymbol{N}_0(\overline{\mathcal{Z}(\lambda(\tau)X(\xi,\tau))}\cdot \mathcal{Z}\boldsymbol{u} + h_0(\overline{\mathcal{Z}X})\lambda'(\tau)\Big)d\tau.$$

If $\epsilon$ in (1.19) is small, then we arrive at (3.23) and, by Theorem 3.5, at (3.27) for $t \in (0,1)$. Now we can make one more step and define the solution for $t \in (T, 2T)$. Assume that the solution of (3.7), as well as the function $\theta(t)$, is defined for $t \in (0,T)$ and inequalities (3.23) and (2.17) are satisfied. Then it is possible to extend the solution in the time interval $t \in (0, T+1)$. As in [1] (see Theorem 3.2), this reduces to the problem (3.5) in [1], slightly more complicated than (1.14). It is essential that in the proof of Theorems 3.1 and 3.2 in [1] the symmetry properties of $\mathcal{F}$ are not used. If $\boldsymbol{u}$ and $q$ are constructed for $t \in (0, T+1)$, then it is possible to define $\theta(t),\ t \in (0, T+1)$, satisfying (2.17), and estimate

$$r(\xi,t) = r(\xi,T) + \int_T^t \Big(\boldsymbol{N}_0(\overline{\mathcal{Z}X})\cdot \mathcal{Z}\boldsymbol{u} + h_0(\overline{\mathcal{Z}X})\lambda'(\tau)\Big)d\tau,$$

$t \in (T, T+1)$. By Theorem 3.5, the extended functions satisfy (3.27), (2.17) with constants independent of $T$, as in the symmetric case. In this way we construct the solution in the infinite time interval and conclude the proof of Theorem 1.1.

## 4. Proof of Proposition 2.2 and of the estimate (3.24)

This section is devoted to some estimates presented in Sections 2 and 3.

*Proof of Proposition* 2.2. We consider the function $f(t,\lambda)$ defined in (2.8). When we extend $h_0$ from $\mathcal{G}_0$ in the $\delta_1$-neighborhood of $\mathcal{G}_0$ so that this function remains smooth (which reduces to the extension of $\boldsymbol{N}_0$, as it has been done above) and take account of the relation $h_0(\bar{y}) = h_0(\mathfrak{R}(y))$, then we can write $f(t,\lambda)$ as

$$f(t,\lambda) = \int_{\Gamma_0} F(\mathcal{Z}(\lambda)X(\xi,t))dS_\xi, \tag{4.1}$$

where $F$ is a smooth function in a certain neighborhood of $\Gamma_0$. The partial derivatives of $f$ with respect to $\lambda$ are given by

$$\begin{aligned}
f_\lambda(t,\lambda) &= \int_{\Gamma_0} \nabla F(\mathcal{Z}(\lambda)X(\xi,t))\cdot \mathcal{Z}' X dS_\xi = \int_{\Gamma_0} \nabla F(\mathcal{Z}(\lambda)X)\cdot \mathcal{Z}'\mathcal{Z}^{-1}\mathcal{Z}X dS_\xi \\
&= \int_{\Gamma_0} \nabla F(\mathcal{Z}(\lambda)X(\xi,t))\cdot (\boldsymbol{e}_3\times \mathcal{Z}X) dS_\xi \equiv \int_{\Gamma_0} F_1(\mathcal{Z}X)dS, \\
f_{\lambda\lambda}(t,\lambda) &= \int_{\Gamma_0} \nabla F_1(\mathcal{Z}(\lambda)X)\cdot (\boldsymbol{e}_3\times \mathcal{Z}X))dS \equiv \int_{\Gamma_0} F_2(\mathcal{Z}X)dS_\xi,
\end{aligned} \tag{4.2}$$

where $F_1$ and $F_2$ are also smooth functions. Moreover,

$$\begin{aligned}
f_t(t,\lambda) &= \int_{\Gamma_0} \nabla F(\mathcal{Z}(\lambda)X)\cdot \mathcal{Z}\boldsymbol{u}(\xi,t)dS_\xi, \\
f_{t\lambda}(t,\lambda) &= \int_{\Gamma_0} \nabla F_1(\mathcal{Z}(\lambda)X)\cdot \mathcal{Z}\boldsymbol{u}(\xi,t)dS_\xi, \qquad (4.3)\\
f_{t\lambda\lambda}(t,\lambda) &= \int_{\Gamma_0} \nabla F_2(\mathcal{Z}(\lambda)X)\cdot \mathcal{Z}\boldsymbol{u}(\xi,t)dS_\xi, \\
f_{tt}(t,\lambda) &= \int_{\Gamma_0} \nabla F(\mathcal{Z}(\lambda)X)\cdot \mathcal{Z}\boldsymbol{u}_t dS_\xi + \int_{\Gamma_0} \mathcal{Z}\boldsymbol{u}\cdot\nabla\nabla F(\mathcal{Z}X)\cdot \mathcal{Z}\boldsymbol{u}dS_\xi \\
&\equiv \phi_1(t)+\phi_2(t). \qquad (4.4)
\end{aligned}$$

Differentiating (2.13) with respect to $t$, we obtain

$$\lambda''(t) = -\Big(\frac{\partial}{\partial t}\frac{f_t}{f_\lambda}\Big)_{\lambda=\lambda(t)} - \Big(\frac{\partial}{\partial \lambda}\frac{f_t}{f_\lambda}\Big)_{\lambda=\lambda(t)}\lambda'(t) = \lambda_1(t)+\lambda_2(t). \tag{4.5}$$

Since $X(\xi,t)$ and $\boldsymbol{u}(\xi,t)$ are bounded uniformly with respect to $t$ and $f_\lambda$ satisfies (2.10), we have

$$\begin{aligned}
&\Big|\Big(\frac{\partial}{\partial \lambda}\frac{f_t}{f_\lambda}\Big)_{\lambda=\lambda(t)}\Big| \le c, \\
&\Big|\Big(\frac{\partial}{\partial t}\frac{f_t}{f_\lambda}\Big)_{\lambda=\lambda(t)}\Big| \le c\int_{\Gamma_0} (|\boldsymbol{u}_t(\xi,t)|+|\boldsymbol{u}(\xi,t)|)dS,
\end{aligned}$$

and in view of (2.15)

$$\begin{aligned}
|\lambda''(t)| &\le c\int_{\Gamma_0} (|\boldsymbol{u}_t(\xi,t)|+|\boldsymbol{u}(\xi,t)|)dS, \\
\|\lambda''\|_{L_2(0,T)} &\le c\|\boldsymbol{u}\|_{W_2^{0,1}(G_T)}.
\end{aligned} \tag{4.6}$$

Now we estimate

$$\Big(\int_0^{\min(T,1)} \frac{dh}{h^{1+2\mu}} \int_h^T |\Delta_t(-h)\lambda''(t)|^2 dt\Big)^{1/2},$$

where $\mu = l/2-1/4$, $\Delta_t(-h)\lambda''(t) = \lambda''(t-h)-\lambda''(t)$. Using (4.2) it is not difficult to verify that

$$\Big(\int_0^{\min(T,1)} \frac{dh}{h^{1+2\mu}} \int_h^T |\Delta_t(-h)\lambda_2(t)|^2 dt\Big)^{1/2} \le \|\lambda_2\|_{W_2^1(0,T)} \le c\|\boldsymbol{u}\|_{W_2^{0,1}(G_T)}. \tag{4.7}$$

The function $\lambda_1(t)$ is given by

$$\lambda_1(t) = -\Big(\frac{f_{tt}}{f_\lambda} - \frac{f_t f_{\lambda t}}{f_\lambda^2}\Big)_{\lambda=\lambda(t)} \equiv \lambda_3(t)+\lambda_4(t)$$

with $\lambda_4$ also satisfying (4.7). Now we consider the difference

$$\begin{aligned}\Delta_t(-h)\lambda_3(t) = &-\frac{1}{f_\lambda(t-h,\lambda(t-h))}\Delta_t(-h)f_{tt}(t,\lambda(t))\\ &- f_{tt}(t,\lambda(t))\Delta_t(-h)\frac{1}{f_\lambda(t,\lambda(t))}.\end{aligned}$$

Since $|f_\lambda(t,\lambda(t))| \ge c > 0$ and $|\frac{\partial}{\partial t}\frac{1}{f_\lambda(t,\lambda(t))}| \le c$, we have

$$\begin{aligned}&\Big(\int_0^{\min(1,T)} \frac{dh}{h^{1+2\mu}} \int_h^T |f_{tt}|^2\Big|\Delta_t(-h)\frac{1}{f_\lambda}\Big|^2 dt\Big)^{1/2} \le c\|f_{tt}\|_{L_2(0,T)},\\ &\Big(\int_0^{\min(1,T)} \frac{dh}{h^{1+2\mu}} \int_h^T |\Delta_t(-h)f_{tt}|^2\frac{1}{|f_\lambda|^2} dt\Big)^{1/2}\\ &\quad\le c\Big(\int_0^{\min(1,T)} \frac{dh}{h^{1+2\mu}} \int_h^T |\Delta_t(-h)f_{tt}|^2 dt\Big)^{1/2},\end{aligned}$$

which implies

$$\begin{aligned}&\|\lambda_3\|_{W_2^\mu(0,T)} \le c\|f_{tt}\|_{W_2^\mu(0,T)},\\ &\|\lambda''\|_{W_2^\mu(0,T)} \le c\Big(\|f_{tt}\|_{W_2^\mu(0,T)} + \|\boldsymbol{u}\|_{W_2^{0,1}(Q_T)}\Big).\end{aligned}$$

The function $f_{tt}(t)$ is representable in the form (4.4) with $\phi_2$ satisfying

$$\|\phi_2\|_{W_2^\mu(0,T)} \le c\|\phi_2\|_{W_2^1(0,T)} \le c\|\boldsymbol{u}\|_{W_2^{0,1}(G_T)} \tag{4.8}$$

and

$$\phi_1(t) = \int_{\Gamma_0} \boldsymbol{b}(\xi,t)\cdot\boldsymbol{u}_t(\xi,t)dS,$$

where $\boldsymbol{b} = \mathcal{Z}^{-1}(\lambda(t))\nabla F(\mathcal{Z}(\lambda(t))X(\xi,t))$ is the function such that

$$\sup_{G_T}|\boldsymbol{b}(\xi,t)| + \sup_{G_T}|\boldsymbol{b}_t(\xi,t)| \le c.$$

Hence

$$\begin{aligned}&\|\phi_{1t}\|_{W_2^\mu(0,T)} \le c\|\boldsymbol{u}_t\|_{W_2^{0,\mu}(G_T)},\\ &\|f_{tt}\|_{W_2^\mu(0,T)} \le c\|\boldsymbol{u}\|_{W_2^{0,1+\mu}(G_T)}.\end{aligned}$$

Together with (4.8), this inequality implies

$$\|\lambda'\|_{W_2^{l/2+3/4}(0,T)} \le c\|\boldsymbol{u}\|_{W_2^{0,l/2+3/4}(G_T)}. \tag{4.9}$$

In order to conclude the proof of (2.17), we need to estimate the norm

$$\|(1+t)\lambda'\|_{W_2^{\mu_1}(0,T)}$$

with $\mu_1 = l/2 + 1/4$. This can be done by repeating the above arguments. In view of (2.15), we have

$$\begin{aligned}
&\|(1+t)\lambda'\|_{L_2(0,T)} \le c\|(1+t)\boldsymbol{u}\|_{L_2(G_T)},\\
&\Big(\int_0^{\min(1,T)} \frac{dh}{h^{1+2\mu_1}} \int_h^T (1+t)^2 \Big|\Delta_t(-h)\lambda'(t)\Big|^2 dt\Big)^{1/2}\\
&\qquad \le c\Big(\int_0^{\min(1,T)} \frac{dh}{h^{1+2\mu_1}} \int_h^T (1+t)^2 \Big|\Delta_t(-h) f_t(t)\Big|^2 dt\Big)^{1/2} + c\|(1+t)f_t\|_{L_2(0,T)}\\
&\qquad \le c\|(1+t)\boldsymbol{u}\|_{W_2^{0,\mu_1}(G_T)},
\end{aligned}$$

which concludes the proof of (2.17) and of Proposition 2.2. □

*On the estimate* (3.24). The expressions $\boldsymbol{l}_1$, $l_2$, $\boldsymbol{l}_3$, $l_4$ are the same as in the symmetric case, and they have been estimated in [9], Propositions 5.5 and 5.6, but $l_5$ and $l_6$ are somewhat different. As in the symmetric case, the main technical difficulties arise in the estimate of $l_5$, in particular, of the second derivative $\frac{\partial^2 U_s}{\partial s^2}$ of the potential (3.3). It has the same form as the analogous function in [1], only the role of $\rho$ is played by $\widehat{\rho}$ or $\widetilde{\rho}$. We have:

$$\begin{aligned}
\frac{\partial^2 U_s(z,t)}{\partial s^2} &= V_1(z,t) + V_2(z,t) - \boldsymbol{W}_1(z,t)\cdot \boldsymbol{N}_{\theta(t)}(z)\widehat{\rho}(z,t)\\
&\quad - \boldsymbol{W}_2(z,t)\cdot \boldsymbol{N}_\theta(z)\widehat{\rho}(z,t), \qquad\qquad (4.10)\\
V_1(z,t) &= \int_{\mathcal{G}_{\theta(t)}} \widehat{\rho}(\zeta,t)\frac{\partial \Lambda(\zeta, s\widehat{\rho})}{\partial s}\frac{dS_\zeta}{|e_{s\widehat{\rho}}(z) - e_{s\widehat{\rho}}(\zeta)|},\\
V_2(z,t) &= \int_{\mathcal{G}_\theta} \widehat{\rho}(\zeta,t)\Lambda(\zeta, s\widehat{\rho})\frac{\partial}{\partial s}\frac{1}{|e_{s\widehat{\rho}}(z) - e_{s\widehat{\rho}}(\zeta)|}dS,\\
\boldsymbol{W}_1(z,t) &= \int_{\mathcal{F}_\theta} \frac{\partial L(\zeta, s\widehat{\rho}^*)}{\partial s}\frac{e_{s\widehat{\rho}}(z) - e_{s\widehat{\rho}}(\zeta)}{|e_{s\widehat{\rho}}(z) - e_{s\widehat{\rho}}(\zeta)|^3}d\zeta,\\
\boldsymbol{W}_2(z,t) &= \int_{\mathcal{F}_\theta} L(\zeta, s\widehat{\rho}^*)\frac{\partial}{\partial s}\frac{e_{s\widehat{\rho}}(z) - e_{s\widehat{\rho}}(\zeta)}{|e_{s\widehat{\rho}}(z) - e_{s\widehat{\rho}}(\zeta)|^3}d\zeta.
\end{aligned}$$

Since $\widetilde{\rho}(y,t) = \widehat{\rho}(\mathcal{Z}(\theta(t))y,t)$, the formula (4.10) is equivalent to

$$\begin{aligned}
&\frac{\partial^2 U_s(\mathcal{Z}(\theta(t))y,t)}{\partial s^2}\\
&\quad = V_1^{(0)}(y) + V_2^{(0)}(y,t) - \boldsymbol{W}_1^{(0)}(y,t)\cdot \boldsymbol{N}_0(y)\widetilde{\rho}(y,t) - \boldsymbol{W}_2^{(0)}(y,t)\cdot \boldsymbol{N}_0(y)\widetilde{\rho}(y,t),
\end{aligned}$$

where $y \in \mathcal{G}_0$,

$$V_1^{(0)}(y,t) = \int_{\mathcal{G}_0} \widetilde{\rho}(\eta,t) \frac{\partial \Lambda(\eta, s\widetilde{\rho})}{\partial s} \frac{dS_\eta}{|e_{s\widetilde{\rho}}(y) - e_{s\widetilde{\rho}}(\eta)|},$$
$$V_2^{(0)}(y,t) = \int_{\mathcal{G}_0} \widetilde{\rho}(\eta,t) \Lambda(\eta, s\widetilde{\rho}) \frac{\partial}{\partial s} \frac{1}{|e_{s\widetilde{\rho}}(y) - e_{s\widetilde{\rho}}(\eta)|} dS,$$
$$\boldsymbol{W}_1^{(0)}(y,t) = \int_{\mathcal{F}_0} \frac{\partial L(\eta, s\widetilde{\rho}^*)}{\partial s} \frac{e_{s\widetilde{\rho}}(y) - e_{s\widetilde{\rho}}(\eta)}{|e_{s\widetilde{\rho}}(y) - e_{s\widetilde{\rho}}(\eta)|^3} d\eta,$$
$$\boldsymbol{W}_2^{(0)}(y,t) = \int_{\mathcal{F}_0} L(\eta, s\widetilde{\rho}^*) \frac{\partial}{\partial s} \frac{e_{s\widetilde{\rho}}(y) - e_{s\widetilde{\rho}}(\eta)}{|e_{s\widetilde{\rho}}(y) - e_{s\widetilde{\rho}}(\eta)|^3} d\eta.$$

Estimates of these potentials are made exactly as in [11] and they lead to the inequality analogous to (3.20) in [11], namely,

$$\begin{aligned} &\Big\| \frac{\partial^2 U_s}{\partial s^2}\Big|_{y=\overline{\mathcal{Z}X}} \Big\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)} \\ &\quad \leq c \sup_{t<T} \|r(\cdot,t)\|_{W_2^{l+1}(\Gamma_0)} \Big( \|r\|_{W_2^{l+1/2,0}(G_T)} + \|(1+t)\boldsymbol{u}\|_{W_2^{1/2,0}(G_T)} \Big). \end{aligned} \tag{4.11}$$

The proof is based on the estimates of the Newtonian and single layer potentials obtained in [12]. We also make use of the estimate of the time derivative $\widetilde{\rho}_t$. Let $V_n'$ be the velocity of the evolution of the surface $\mathcal{Z}(\lambda(t))X(\xi,t)$ in the direction of the exterior normal $\boldsymbol{n}'$. We have

$$\begin{aligned} V_n' &= \frac{\partial}{\partial t} \mathcal{Z}(\lambda(t))X(\xi,t) \cdot \boldsymbol{n}' = \mathcal{Z}\boldsymbol{u} \cdot \boldsymbol{n}' + \mathcal{Z}' X \boldsymbol{n}' \lambda'(t) \\ &= \boldsymbol{u} \cdot \boldsymbol{n} + (\boldsymbol{e}_3 \times X) \cdot \boldsymbol{n} \lambda'(t). \end{aligned}$$

Hence

$$\begin{aligned} \widetilde{\rho}_t(y,t) &= \frac{V_n'}{\boldsymbol{n}' \cdot \boldsymbol{N}_0} = \frac{\boldsymbol{u} \cdot \boldsymbol{n}}{\boldsymbol{n}' \cdot \boldsymbol{N}_0} + \frac{(\boldsymbol{e}_3 \times X) \cdot \boldsymbol{n}}{\boldsymbol{n}' \cdot \boldsymbol{N}_0} \lambda'(t) \\ &= \frac{\boldsymbol{u}(\xi,t) \cdot \widehat{\mathcal{L}}^T(z) \boldsymbol{N}_\theta(z)}{\Lambda(z,\widehat{\rho})} + \frac{(\boldsymbol{e}_3 \times X) \cdot \widehat{\mathcal{L}}^T(z) \boldsymbol{N}_\theta(z)}{\Lambda(z,\widehat{\rho})} \lambda'(t). \end{aligned} \tag{4.12}$$

Here, as usual, the points $y$ and $z$ are connected with $\mathcal{Z}(\theta(t))y = z$ and $\Lambda(z,\widehat{\rho}) = 1 - \widehat{\rho}\mathcal{H}_\theta(z) + \widehat{\rho}^2 \mathcal{K}_\theta(z)$, where $\mathcal{H}_\theta$ is the doubled mean curvature and $\mathcal{K}_\theta$ is the Gaussian curvature of $\mathcal{G}_\theta$. From (4.12) and (2.15) it follows that

$$\|\widetilde{\rho}_t(\cdot,t)\|_{W_2^{1/2}(\mathcal{G}_0)} \leq c \|\boldsymbol{u}(\cdot,t)\|_{W_2^{1/2}(\Gamma_0)},$$

which is analogous to the estimate (3.19) in [11] for $\rho_t$. This allows us to obtain (4.11).

Now we turn our attention to $B_1(r,\boldsymbol{u})$. According to (3.5),

$$B_1(r,\boldsymbol{u}) = B_2(r,\boldsymbol{u}) - \kappa B_3(r,\boldsymbol{u}) - \kappa B_4(r,\boldsymbol{u}) - \kappa B_5(r,\boldsymbol{u}),$$

where

$$\begin{aligned}
B_2(r,\boldsymbol{u}) &= (b(\overline{\mathcal{Z}(\lambda(t)X)}) - b(\bar{\xi}))r(\xi,t),\\
B_3(r,\boldsymbol{u}) &= \int_{\Gamma_0} \frac{r(\eta,t)(\Psi(\eta,t)-1)dS_\eta}{|\overline{\mathcal{Z}X(\xi,t)} - \overline{\mathcal{Z}X(\eta,t)}|},\\
B_4(r,\boldsymbol{u}) &= \int_0^1 ds \int_{\Gamma_0} r(\eta,t)\frac{\partial}{\partial s}\frac{1}{|\bar{X}_s(\xi,t) - \bar{X}_s(\eta,t)|}dS,\\
B_5(r,\boldsymbol{u}) &= \int_0^1 ds \int_{\Gamma_0} r(\eta,t)\frac{\partial}{\partial s}\frac{1}{|\overline{\mathcal{Z}(s\lambda(t))X(\xi,t)} - \overline{\mathcal{Z}(s\lambda)X(\eta,t)}|}dS.
\end{aligned}$$

In the case of axially symmetric $\mathcal{F}$ we have $\mathcal{Z} = I$ and the term $B_5$ drops out.

We start with the estimate of $B_2$ and show that

$$\|B_2\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)} \le c\Big(\|\boldsymbol{u}\|_{\widetilde{W}_2^{l+2,0}(Q_T)} + \|r\|_{\widetilde{W}_2^{l+1/2,0}(G_T)}\Big)\|\boldsymbol{u}\|_{\widetilde{W}_2^{l+2,0}(Q_T)}. \tag{4.13}$$

Following the arguments in the proof of Proposition 5.7 in [9], we estimate the difference

$$b_0(\overline{\mathcal{Z}X}) - b_0(\bar{\xi}). \tag{4.14}$$

In Proposition 5.7 it is proved that

$$\|b_0(\bar{X}(\xi,t)) - b_0(\bar{\xi})\|_{W_2^{l+1/2}(\Gamma_0)} \le c\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,0}(Q_t)}, \quad \forall t \in (0,T). \tag{4.15}$$

The difference (4.14) satisfies the same inequality; indeed,

$$\begin{aligned}
b_0(\overline{\mathcal{Z}X}) - b_0(\bar{\xi}) &= (b_0(\overline{\mathcal{Z}X}) - b_0(\bar{X})) + (b_0(\bar{X}) - b_0(\bar{\xi})),\\
b_0(\overline{\mathcal{Z}X}) - b_0(\bar{X}) &= \int_0^1 \frac{\partial}{\partial s}\mathfrak{b}_0(\mathcal{Z}(s\lambda)X)ds = \int_0^1 \nabla\mathfrak{b}_0(\mathcal{Z}(s\lambda)X)ds\lambda(t),
\end{aligned}$$

where $\mathfrak{b}_0(\cdot) = b_0(\Re(\cdot))$. Hence, by (2.15),

$$\begin{aligned}
\|b_0(\overline{\mathcal{Z}X}) - b_0(\bar{X})\|_{W_2^{l+1/2}(\Gamma_0)} &\le c\Big(1 + \|\boldsymbol{u}\|_{\widetilde{W}_2^{l+2,0}(Q_t)}\Big)|\lambda(t)| \\
&\le c\int_0^t \int_{\Gamma_0} |\boldsymbol{u}(\xi,\tau)|dSd\tau \le c\|\boldsymbol{u}\|_{\widetilde{W}_2^{l+2,0}(Q_t)}.
\end{aligned} \tag{4.16}$$

We also need to estimate the time derivative

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big(b_0(\overline{\mathcal{Z}X}) - b_0(\bar{\xi})\Big)r\\
&\quad = \Big(b_0(\overline{\mathcal{Z}X}) - b_0(\bar{\xi})\Big)r_t + r\nabla\mathfrak{b}_0(\mathcal{Z}X)\Big(\mathcal{Z}(\lambda)\boldsymbol{u}(\xi,t) + \mathcal{Z}'(\lambda)X\lambda'(t)\Big).
\end{aligned}$$

Using the inequalities (4.16), (2.15), we obtain

$$\begin{aligned}&\Big\|\frac{\partial}{\partial t}\Big(b_0(\overline{\mathcal{Z}X})-b_0(\bar\xi)\Big)r\Big\|_{L_2(\Gamma_0)}\\&\quad\le\sup_{\Gamma_0}|(b_0(\overline{\mathcal{Z}X})-b_0(\bar\xi)|\|r_t(\cdot,t)\|_{L_2(\Gamma_0)}+\Big\|\frac{\partial b_0(\overline{\mathcal{Z}X})}{\partial t}\Big\|_{L_2(\Gamma_0)}\sup_{\Gamma_0}|r(\xi,t)|\\&\quad\le c\|\boldsymbol{u}(\cdot,t)\|_{L_2(\Gamma_0)}\Big(\|\boldsymbol{u}\|_{\widetilde{W}_2^{l+2,0}(Q_t)}+\sup_{\Gamma_0}|r(\xi,t)|\Big).\end{aligned}$$

Together with (4.15), (4.16), this estimate implies (4.13).

For the estimate of $B_3,B_4,B_5$ we can use Proposition 2.10 in [11]. It concerns the surface integrals of the form

$$\begin{aligned}v(y,t)&=\int_{\Gamma_0}|\mathcal{T}(y,t)-\mathcal{T}(\eta,t)|^{-1}g(\eta,t)dS,\\v_1(y,t)&=\int_{\Gamma_0}\frac{\mathcal{T}(y,t)-\mathcal{T}(\eta,t)}{|\mathcal{T}(y,t)-\mathcal{T}(\eta,t)|^3}\cdot(\boldsymbol{a}(y,t)-\boldsymbol{a}(\eta,t))g(\eta,t)dS,\end{aligned}$$

where $\mathcal{T}(y,t)$ is an invertible mapping of class $W_2^{l+3/2-\epsilon}(\Omega_0)$, $\epsilon\in(0,l-1)$, with $\mathcal{T}_t\in W_2^1(\Omega_0)$ and $\boldsymbol{a}$ is as regular as $\widetilde{\rho}$. It is easily seen that the transformation $\mathcal{T}=\mathcal{Z}(\lambda(t))X(\xi,t)$ possesses these properties and $\overline{\mathcal{Z}X}=e_{\widetilde{\rho}}^{-1}X$. Therefore the application of Proposition 2.10 leads to the same estimates for $B_1$ (and for $l_5$) as in [11], namely,

$$\begin{aligned}&\|l_5\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)}\\&\quad\le c\Big(\sup_{t<T}\|r(\cdot,t)\|_{W_2^{l+1}(\Gamma_0)}+\|r(\cdot,t)\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)}+\|\boldsymbol{u}\|_{\widetilde{W}_2^{l+2,l/2+1}(Q_T)}\Big)^2.\end{aligned}\tag{4.17}$$

Now we pass to the estimate of

$$\begin{aligned}l_6(\boldsymbol{u})&=(\mathcal{Z}^{-1}(\lambda(t))\boldsymbol{N}_0(\overline{\mathcal{Z}X})-\boldsymbol{N}_0(\bar\xi))\cdot\boldsymbol{u}(\xi,t)\\&\quad+(h_0(\overline{\mathcal{Z}X})-h_0(\bar\xi))\lambda'(t)+h_0(\bar\xi)m(t)\\&\equiv l_6^{(1)}(\boldsymbol{u})+l_6^{(2)}(\boldsymbol{u})+l_6^{(3)}(\boldsymbol{u})\end{aligned}$$

where $m(t)$ is defined in (2.19). We have already seen above that

$$\begin{aligned}\|h_0(\overline{\mathcal{Z}X})-h_0(\bar\xi)\|_{W_2^{l+1/2}(\Gamma_0)}&\le c\|\boldsymbol{u}\|_{\widehat{W}_2^{l+2.0}(Q_t)},\qquad(4.18)\\\|\mathcal{Z}^{-1}(\lambda(t))\boldsymbol{N}_0(\overline{\mathcal{Z}X})-\boldsymbol{N}_0(\bar\xi))\|_{W_2^{l+1/2}(\Gamma_0)}&\le c|\lambda(t)|\|\boldsymbol{N}_0(\overline{\mathcal{Z}X})\|_{W_2^{l+1/2}(\Gamma_0)}\\&\quad+\|\boldsymbol{N}_0(\overline{\mathcal{Z}X})-\boldsymbol{N}_0(\bar\xi)\|_{W_2^{l+1/2}(\Gamma_0)}\\&\le c\|\boldsymbol{u}\|_{\widehat{W}_2^{l+2.0}(Q_t)}.\end{aligned}$$

Exactly in the same way we obtain

$$\begin{aligned}&\|h_0(\overline{\mathcal{Z}X})-h_0(\bar\xi)\|_{W_2^{l+3/2}(\Gamma_0)}+\|\mathcal{Z}^{-1}(\lambda(t))\boldsymbol{N}_0(\overline{\mathcal{Z}X})-\boldsymbol{N}_0(\bar\xi))\|_{W_2^{l+1/2}(\Gamma_0)}\\&\quad\le c(1+\sqrt{t})\|\boldsymbol{u}\|_{\widetilde{W}_2^{l+2,0}(Q_t)}.\end{aligned}$$

In addition, we have

$$\Big\|\frac{\partial}{\partial t}h_0(\overline{\mathcal{Z}X})\Big\|_{L_2(\Gamma_0)}+\Big\|\frac{\partial}{\partial t}\mathcal{Z}^{-1}\boldsymbol{N}_0(\overline{\mathcal{Z}X})\Big\|_{L_2(\Gamma_0)}\le c\|\boldsymbol{u}\|_{L_2(\Gamma_0)},$$
$$\Big\|\frac{\partial^2}{\partial t^2}h_0(\overline{\mathcal{Z}X})\Big\|_{L_2(\Gamma_0)}+\Big\|\frac{\partial^2}{\partial t^2}\mathcal{Z}^{-1}\boldsymbol{N}_0(\overline{\mathcal{Z}X})\Big\|_{L_2(\Gamma_0)}\le c\Big(\|\boldsymbol{u}_t\|_{L_2(\Gamma_0}+\|\boldsymbol{u}\|_{L_2(\Gamma_0)}\Big).$$

These inequalities allow us to estimate $l_6^{(1)}$ and $l_6^{(2)}$ exactly in the same way as $l_6$ has been estimated in [9], Proposition 5.8:

$$\|l_6^{(1)}(\boldsymbol{u})\|_{\widetilde{W}_2^{3/2+l,3/4+l/2}(G_T)}+\|l_6^{(2)}(\boldsymbol{u})\|_{\widetilde{W}_2^{3/2+l,3/4+l/2}(G_T)}$$
$$\le c\|\boldsymbol{u}\|_{\widetilde{W}_2^{l+2,l/2+1}(Q_T)}\Big(\|\boldsymbol{u}\|_{\widetilde{W}_2^{l+2,0}(Q_T)}+\sup_{Q_T}|\boldsymbol{u}(\xi,t)|\Big).$$

The proof reduces to repeating the arguments in this Proposition. Finally, it is easily seen that

$$\|l_6^{(3)}(\boldsymbol{u})\|_{\widetilde{W}_2^{3/2+l,3/4+l/2}(G_T)}\le c\|m\|_{\widetilde{W}_2^{l/2+3/4}(0,T)}.$$

According to (2.19), $m(t)=m_1(t)+m_2(t)$, where

$$m_1(t)=-\frac{1}{f_\lambda(t,\lambda(t))}\int_{\Gamma_0}\Big(\mathcal{Z}^{-1}(\lambda(t))\boldsymbol{N}_0(\overline{\mathcal{Z}X})h_0(\overline{\mathcal{Z}X})-\boldsymbol{N}_0(\bar\xi)h_0(\bar\xi)\Big)\cdot\boldsymbol{u}(\xi,t)dS_\xi,$$
$$m_2(t)=\frac{\int_{\Gamma_0}\boldsymbol{N}_0(\bar\xi)\cdot\boldsymbol{u}(\xi,t)h_0(\bar\xi)dS}{f_\lambda(t,\lambda(t))\int_{\Gamma_0}h_0^2(\bar\xi)dS}\int_{\Gamma_0}r(\xi,t))\nabla h_0(\overline{\mathcal{Z}X})\cdot\Big(\nabla\mathfrak{R}(\mathcal{Z}X)(\boldsymbol{e}_3\times\mathcal{Z}X)\Big)dS$$
$$-\frac{1}{f_\lambda(t,\lambda(t))}\int_{\Gamma_0}r(\xi,t)\nabla h_0(\overline{\mathcal{Z}X})\cdot\nabla\mathfrak{R}(\mathcal{Z}X)\mathcal{Z}(\lambda(t))\boldsymbol{u}(\xi,t)dS. \qquad (4.19)$$

In view of (4.18), (2.15),

$$\Big|\mathcal{Z}^{-1}(\lambda(t))\boldsymbol{N}_0(\overline{\mathcal{Z}X})h_0(\overline{\mathcal{Z}X})-\boldsymbol{N}_0(\bar\xi)h_0(\bar\xi)\Big|\le c\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,0}(Q_t)},$$
$$\Big\|\frac{\partial}{\partial t}\Big(\mathcal{Z}^{-1}(\lambda(t))\boldsymbol{N}_0(\overline{\mathcal{Z}X})h_0(\overline{\mathcal{Z}X})-\boldsymbol{N}_0(\bar\xi)h_0(\bar\xi)\Big)\Big\|_{L_2(\Gamma_0)}\le c\|\boldsymbol{u}(\cdot,t)\|_{L_2(\Gamma_0)},$$
$$\Big\|\frac{\partial^2}{\partial t^2}\Big(\mathcal{Z}^{-1}(\lambda(t))\boldsymbol{N}_0(\overline{\mathcal{Z}X})h_0(\overline{\mathcal{Z}X})-\boldsymbol{N}_0(\bar\xi)h_0(\bar\xi)\Big)\Big\|_{L_2(\Gamma_0)}$$
$$\le c\big(\|\boldsymbol{u}_t(\cdot,t)\|_{L_2(\Gamma_0)}+\|\boldsymbol{u}(\cdot,t)\|_{L_2(\Gamma_0)}\big),$$

hence

$$m_1'(t)$$
$$=\frac{f_{t\lambda}(t,\lambda(t))}{f_\lambda^2(t,\lambda(t)}\Big|_{\lambda=\lambda(t)}\int_{\Gamma_0}\Big(\mathcal{Z}^{-1}(\lambda(t))\boldsymbol{N}_0(\overline{\mathcal{Z}X})h_0(\overline{\mathcal{Z}X})-\boldsymbol{N}_0(\bar\xi)h_0(\bar\xi)\Big)\cdot\boldsymbol{u}(\xi,t)dS_\xi$$
$$-\frac{1}{f_\lambda(t,\lambda(t))}\Big(\int_{\Gamma_0}\frac{\partial}{\partial t}\Big(\mathcal{Z}^{-1}(\lambda(t))\boldsymbol{N}_0(\overline{\mathcal{Z}X})h_0(\overline{\mathcal{Z}X})-\boldsymbol{N}_0(\bar\xi)h_0(\bar\xi)\Big)\cdot\boldsymbol{u}(\xi,t)dS_\xi$$
$$-\frac{1}{f_\lambda(t,\lambda(t))}\int_{\Gamma_0}\Big(\mathcal{Z}^{-1}(\lambda(t))\boldsymbol{N}_0(\overline{\mathcal{Z}X})h_0(\overline{\mathcal{Z}X})-\boldsymbol{N}_0(\bar\xi)h_0(\bar\xi)\Big)\cdot\boldsymbol{u}_t(\xi,t)dS_\xi\Big)$$

satisfies the inequalities

$$\|m_1'\|_{L_2(0,T)} \le c\|\boldsymbol{u}\|_{W_2^{0,1}(G_T)}\Big(\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,0}(Q_T)} + \sup_{t<T}\|\boldsymbol{u}\|_{L_2(\Gamma_0)}\Big),$$

$$\Big(\int_0^1 \frac{dh}{h^{1+2\mu}}\int_h^T |\Delta_t(-h)m_1'(t)|^2 dt\Big)^{1/2}$$
$$\le c\|\boldsymbol{u}\|_{W_2^{0,1+\mu}(G_T)}\Big(\|\boldsymbol{u}\|_{\widetilde{W}_2^{l+2,0}(Q_T)} + \sup_{t<T}\|\boldsymbol{u}\|_{L_2(\Gamma_0)}\Big),$$

and

$$\Big(\int_0^1 \frac{dh}{h^{1+2\mu_1}}\int_h^T (1+t)^2|\Delta_t(-h)m_1(t)|^2 dt\Big)^{1/2} + \|(1+t)m_1\|_{L_2(0,T)}$$
$$\le c\|(1+t)\boldsymbol{u}\|_{W_2^{0,\mu_1}(G_T)}\Big(\|\boldsymbol{u}\|_{\widetilde{W}_2^{l+2,0}(Q_T)} + \sup_{t<T}\|\boldsymbol{u}\|_{L_2(\Gamma_0)}\Big),$$

established in the same way as (2.17). Hence

$$\|m_1\|_{\widetilde{W}_2^{l/2+3/4}(0,T)} \le c\|\boldsymbol{u}\|_{\widetilde{W}_2^{0,l/2+3/4}(G_T)}\Big(\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,0}(Q_T)} + \sup_{t<T}\|\boldsymbol{u}\|_{L_2(\Gamma_0)}\Big). \quad (4.20)$$

The function $m_2(t)$ is estimated by similar arguments. Taking (3.6) into account we obtain

$$\|m_2\|_{\widetilde{W}_2^{l/2+3/4}(0,T)} \le c\|\boldsymbol{u}\|_{\widetilde{W}_2^{0,l/2+3/4}(G_T)}\Big(\sup_{t<T}\|\boldsymbol{u}\|_{L_2(\Gamma_0)} + \sup_{t<T}\|r\|_{L_2(\Gamma_0)}\Big).$$

This implies

$$\|l_6(\boldsymbol{u})\|_{\widetilde{W}_2^{l+3/2,l/2+1/2}(G_T)}$$
$$\le c\|\boldsymbol{u}\|_{\widetilde{W}_2^{l+2,l/2+1}(Q_T)}\Big(\|\boldsymbol{u}\|_{\widetilde{W}_2^{l+2,0}(Q_T)} + \sup_{Q_T}|\boldsymbol{u}(\xi,t)| + \sup_{t<T}\|r\|_{L_2(\Gamma_0)}\Big).$$

Thus (3.24) is proved.

## 5. Proof of Theorem 3.4

We start by obtaining some auxiliary relations and estimates. Let $\boldsymbol{w}$ be a solution of the problem (1.9) and let $\boldsymbol{w}^\perp$ be a projection of $\boldsymbol{w}$ on the subspace of vector fields orthogonal to all rigid displacements. In view of (1.11),

$$\boldsymbol{w}(x,t) = \boldsymbol{w}^\perp(x,t) + \sum_{k=1}^3 g_k(t)\boldsymbol{\eta}_k(x), \quad (5.1)$$

where $\boldsymbol{\eta}_k(x) = \boldsymbol{e}_k \times x$, $\boldsymbol{e}_k = (\delta_{ik})_{i=1,2,3}$. and $g_k(t)$ are functions defined as a solution of a linear algebraic system

$$\sum_{k=1}^3 S_{ik}(t)g_k(t) = \int_{\Omega_t}\boldsymbol{w}(x,t)\cdot\boldsymbol{\eta}_i(x)dx = I_i(t) \quad (5.2)$$

with

$$S_{ik}(t) = \int_{\Omega_t} \boldsymbol{\eta}_i(x) \cdot \boldsymbol{\eta}_k(x) dx,$$

$$I_i(t) = -\omega\Big( \int_{\Omega_t} \boldsymbol{\eta}_3(x) \cdot \boldsymbol{\eta}_i(x) dx - \int_{\mathcal{F}} \boldsymbol{\eta}_3(x) \cdot \boldsymbol{\eta}_i(x) dx \Big); \tag{5.3}$$

by $\mathcal{F}$ we mean arbitrary $\mathcal{F}_\theta$. Since

$$-\int_{\mathcal{F}} \boldsymbol{\eta}_3 \cdot \boldsymbol{\eta}_j dx = \int_{\mathcal{F}} x_j x_3 dx = 0, \quad j = 1, 2 \tag{5.4}$$

(see [2]), we have

$$I_i(t) = \beta \delta_{i3} - S_{i3}(t)\omega, \tag{5.5}$$

where $\beta = \omega \int_{\mathcal{F}} |x'|^2 dx$ is the magnitude of the angular momentum of the rotating liquid. The matrix $\mathcal{S} = (S_{ik}(t))_{i,k=1,2,3}$ is symmetric and positive definite, because, for arbitrary real $\xi_k$,

$$\sum_{i,k=1}^{3} S_{ik}(t)\xi_i\xi_k = \int_{\Omega_t} \Big| \sum_{i=1}^{3} \xi_i \boldsymbol{\eta}_i(x) \Big|^2 dx = \int_{\Omega_t} |\xi \times x|^2 dx \geq c|\xi|^2.$$

Hence there exists the inverse matrix $\mathcal{S}^{-1} = (S^{ik}(t))_{i,k=1,2,3}$, and

$$g_k(t) = \sum_{m=1}^{3} S^{km}(t)(\beta\delta_{m3} - S_{m3}(t)\omega) = \beta S^{k3}(t) - \delta_{k3}\omega. \tag{5.6}$$

We recall that $\Gamma_t$ is given by the equation (1.21) with $\widehat{\rho}$ satisfying (1.25). We compute the projection $\widehat{\rho}^{\perp}$ of $\widehat{\rho}$ on the subspace of $L_2(\mathcal{G}_\theta)$ orthogonal to the functions $(1, x_1, x_2, x_3, h_{\theta(t)}(x))$. It is clear that $(1, x_1, x_2, x_3)$ are linearly independent functions of $x \in \mathcal{G}$ and $h_\theta(x) = \boldsymbol{N}_\theta(x) \cdot \boldsymbol{\eta}_3(x)$ is orthogonal to them, because

$$\int_{\mathcal{G}_\theta} \boldsymbol{N}_\theta(z) \cdot \boldsymbol{\eta}_3(z) dS = \int_{\mathcal{F}_\theta} \nabla \cdot \boldsymbol{\eta}_3(x) dx = 0, \tag{5.7}$$

$$\int_{\mathcal{G}_\theta} z_i \boldsymbol{N}_\theta(z) \cdot \boldsymbol{\eta}_3(z) dS = \int_{\mathcal{F}_\theta} \nabla \cdot x_i \boldsymbol{\eta}_3(x) dx = 0. \tag{5.8}$$

We have

$$\widehat{\rho} = \widehat{\rho}^{\perp} + \sum_{k=0}^{4} c_k(t)\varphi_k,$$

where $\varphi_0(x) = 1$, $\varphi_i(x) = x_i$, $i = 1, 2, 3$, $\varphi_4(x) = h_\theta(x)$. By (5.7), (5.8),

$$\int_{\mathcal{G}_\theta} \widehat{\rho}\varphi_a dS = \sum_{b=0}^{3} c_b(t) \int_{\mathcal{G}_\theta} \varphi_a(x)\varphi_b(x) dS, \quad a = 0, 1, 2, 3,$$

and

$$c_a(t) = \sum_{b=0}^{3} \phi^{ab}(t) \int_{\mathcal{G}_\theta} \widehat{\rho}\varphi_b dS,$$

where $\phi^{ab}(t)$ are elements of the matrix inverse to $\Phi = \Big(\int_{\mathcal{G}_\theta}\varphi_a\varphi_b dS\Big)_{a,b=0,1,2,3}$. It follows that

$$\widehat{\rho} = \widehat{\rho}^\perp + \sum_{a,b=0}^{3}\phi^{ab}\int_{\mathcal{G}_\theta}\widehat{\rho}\varphi_b dS\varphi_a(x) + h_\theta(x)\|h_\theta\|^{-2}_{L_2(\mathcal{G}_\theta)}\int_{\mathcal{G}_\theta}\widehat{\rho}h_\theta(y)dS.$$

Conditions (1.28) for $\widehat{\rho}$ imply

$$\int_{\mathcal{G}_\theta}\widehat{\rho}(x,t)dS = \int_{\mathcal{G}_\theta}\widehat{\rho}(x,t)(1-\varphi(x,\widehat{\rho}))dS,$$
$$\int_{\mathcal{G}_\theta}\widehat{\rho}(x,t)x_i dS = \int_{\mathcal{G}_\theta}\widehat{\rho}(x,t)(x_i-\psi_i(x,\widehat{\rho}))dS,$$
$$\int_{\mathcal{G}_\theta}\widehat{\rho}(x,t)h_\theta(x)dS = \int_{\mathcal{G}_\theta}\widehat{\rho}(x,t)h_\theta(x)(1-\Psi)dS,$$

and, as a consequence,

$$\Big|\int_{\mathcal{G}_\theta}\widehat{\rho}(x,t)dS\Big| \le \|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)}\|1-\varphi\|_{W_2^{1/2}(\mathcal{G}_\theta)} \le c\delta\|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)},$$
$$\Big|\int_{\mathcal{G}_\theta}\widehat{\rho}(x,t)x_i dS\Big| \le c\delta\|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)};$$

moreover, since

$$|1-\Psi| \le c\Big(|1-|A\boldsymbol{n}_0|| + |1-|\widehat{\mathcal{L}}^T(z,\widehat{\rho})\boldsymbol{N}_\theta||\Big)$$
$$\le c\Big(\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,0}(G_t)} + \|\widehat{\rho}\|_{W_2^{l+1-\epsilon}(\mathcal{G}_\theta)}\Big) \le c\delta,$$

we have

$$\Big|\int_{\mathcal{G}_\theta}\widehat{\rho}(x,t)h_\theta(x)dS\Big| \le c\delta\|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)}.$$

Hence

$$\|\widehat{\rho}-\widehat{\rho}^\perp\|_{W_2^{-1/2}(\mathcal{G}_\theta)} \le c\delta\|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)},$$

which means that for small $\delta$ the norms $\|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)}$ and $\|\widehat{\rho}^\perp\|_{W_2^{-1/2}(\mathcal{G}_\theta)}$ are equivalent to each other:

$$c_1\|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)} \le \|\widehat{\rho}^\perp\|_{W_2^{-1/2}(\mathcal{G}_\theta)} \le c_2\|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)}. \tag{5.9}$$

Now we proceed to the proof of Theorem 3.4, assuming that the solution of (1.9), (1.14) is constructed for $t\in(0,T)$ and the condition (3.23) is satisfied. The following propositions play an important role in the proof.

**Proposition 5.1.** *Given the function $f_0\in W_2^{1/2}(\mathcal{G}_0)$ such that $\int_{\mathcal{G}_0}f_0 dS = 0$, there exists a divergence free vector field $\boldsymbol{W}\in W_2^1(\Omega_t)$ satisfying the conditions*

$$\boldsymbol{W}(x,t)\cdot\boldsymbol{n} = \frac{f_0(y,t)}{|\widehat{\mathcal{L}}^T(y,\widetilde{\rho})\boldsymbol{N}_0(y)|}, \quad x = e_{\widetilde{\rho}}(\mathcal{Z}(\theta(t))y), \quad y\in\mathcal{G}_0,$$

$$\int_{\Omega_t} \boldsymbol{W}(x,t)\cdot\boldsymbol{\eta}_j(x)dx = 0, \quad j = 1,2,3, \tag{5.10}$$

*and the inequalities*

$$\begin{aligned} \|\boldsymbol{W}\|_{W_2^1(\Omega_t)} &\le c\|f_0\|_{W_2^{1/2}(\mathcal{G}_0)}, \\ \|\boldsymbol{W}\|_{L_2(\Omega_t)} &\le c\|f_0\|_{L_2(\mathcal{G}_0)}, \\ \|\boldsymbol{W}_t\|_{L_2(\Omega_t)} &\le c\Big(\|f_0\|_{W_2^{1/2}(\mathcal{G}_0)} + \|f_{0t}\|_{L_2(\mathcal{G}_0)}\Big). \end{aligned} \tag{5.11}$$

*Sketch of the proof.* At first we construct a divergence free $\widetilde{\boldsymbol{W}}$ in the domain $\widetilde{\Omega}_t = \mathcal{Z}(\lambda(t))\Omega_t$ with the normal component on $\partial\widetilde{\Omega}_t$ equal to $f_0(e_{\widetilde{\rho}}(y))|\widehat{\mathcal{L}}^T(y,\widetilde{\rho})\boldsymbol{N}_0|^{-1}$ that satisfies inequalities (5.11) and the condition (5.10) in $\widetilde{\Omega}_t$. The construction (for a particular $f_0$) is given in [13], Lemma 4.1, and it is valid for arbitrary $f_0 \in W_2^{1/2}(\mathcal{G}_0)$. The vector field $\boldsymbol{W}$ is defined by

$$\boldsymbol{W}(x,t) = \mathcal{Z}^{-1}(\lambda(t))\widetilde{\boldsymbol{W}}(\mathcal{Z}(\lambda(t))x,t), \quad x \in \Omega_t.$$

Direct computation shows that $\boldsymbol{W}$ satisfies (5.10). Inequalities (5.11) follow from similar inequalities for $\widetilde{\boldsymbol{W}}$.

**Proposition 5.2.** *Let $U_s$ be a potential defined in* (3.3)*. For arbitrary $f_1 \in W_2^{1/2}(\mathcal{G}_0)$ the following inequality holds:*

$$\Big|\int_{\mathcal{G}_\theta} \frac{\partial^2 U_s}{\partial s^2} f_1(z)dS\Big| \le c\delta\|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)}\|f_1\|_{W_2^{1/2}(\mathcal{G}_\theta)}. \tag{5.12}$$

*Proof.* According to (4.10),

$$\begin{aligned} \int_{\mathcal{G}_\theta} \frac{\partial^2 U_s}{\partial s^2} f_1(z)dS &= \int_{\mathcal{G}_\theta} \Big(V_1(z,t) + V_2(z,t) - \boldsymbol{W}_1(z,t)\cdot\boldsymbol{N}_{\theta(t)}(z)\widehat{\rho}(z,t) \\ &\qquad - \boldsymbol{W}_2(z,t)\cdot\boldsymbol{N}_\theta(z)\widehat{\rho}(z,t)\Big) f_1 dS \\ &= \int_{\mathcal{G}_\theta} \widehat{\rho}(z,t)\Big(\frac{\partial\Lambda(z,s\widehat{\rho})}{\partial s}V_3[f_1] + \Lambda(z,s\widehat{\rho})V_4[f_1]\Big)dS \\ &\qquad - \int_{\mathcal{G}_\theta} \widehat{\rho}(z,t)f_1(z)\boldsymbol{N}_\theta(z)\cdot(\boldsymbol{W}_1(z,t) + \boldsymbol{W}_2(z,t))dS, \end{aligned} \tag{5.13}$$

where

$$V_3[f_1] = \int_{\mathcal{G}_\theta} \frac{f_1(\zeta)dS}{|e_{s\widehat{\rho}}(z) - e_{s\widehat{\rho}}(\zeta)|}, \quad V_4[f_1] = \int_{\mathcal{G}_\theta} f_1(\zeta)\frac{\partial}{\partial s}\frac{1}{|e_{s\widehat{\rho}}(z) - e_{s\widehat{\rho}}(\zeta)|}dS.$$

The right-hand side of (5.13) does not exceed

$$\begin{aligned} &\|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)}\Big(\Big\|\frac{\partial\Lambda}{\partial s}V_3[f_1]\Big\|_{W_2^{1/2}(\mathcal{G}_\theta)} + \|\Lambda V_4[f_1]\|_{W_2^{1/2}(\mathcal{G}_\theta)} \\ &\quad + \|f_1\boldsymbol{N}_\theta\cdot(\boldsymbol{W}_1 + \boldsymbol{W}_2)\|_{W_2^{1/2}(\mathcal{G}_\theta)}\Big). \end{aligned}$$

Applying Proposition 4.1 in [1] that concerns the estimate of the product of two functions, we obtain

$$\Big|\int_{\mathcal{G}_\theta}\frac{\partial^2 U_s}{\partial s^2}f_1(z)dS\Big|\leq c\|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)}\Big(\Big\|\frac{\partial\Lambda}{\partial s}\Big\|_{W_2^{l+1/2}(\mathcal{G}_\theta)}\|V_3[f_1]\|_{W_2^{1/2}(\mathcal{G}_\theta)}$$
$$+\|\Lambda\|_{W_2^{l+1/2}(\mathcal{G}_\theta)}\|V_4[f_1]\|_{W_2^{1/2}(\mathcal{G}_\theta)}$$
$$+\|f_1\|_{W_2^{1/2}(\mathcal{G}_\theta)}(\|\boldsymbol{W}_1\|\|_{W_2^{l+1/2}(\mathcal{G}_\theta)}+\|\boldsymbol{W}_1\|\|_{W_2^{l+1/2}(\mathcal{G}_\theta)})\Big).$$

In view of the estimates of the volume and surface potentials obtained in [11], Section 3, this inequality implies (5.12). The proposition is proved. □

Inequality (3.25) follows from the estimate of a "generalized energy". We multiply the first equation in (1.9) by $\boldsymbol{w}$ and integrate over $\Omega_t$. Making use of the transport theorem and of the boundary conditions, we arrive at the energy relation

$$\frac{1}{2}\frac{d}{dt}\Big(\|\boldsymbol{w}\|^2_{L_2(\Omega_t)}-\omega^2\int_{\Omega_t}|x'|^2dx-\kappa\int_{\Omega_t}U(x,t)dx\Big)+\frac{\nu}{2}\int_{\Omega_t}|S(\boldsymbol{w})|^2dx=0. \quad (5.14)$$

By (5.1) and (5.5),

$$\begin{aligned}\|\boldsymbol{w}\|^2_{L_2(\Omega_t)}&=\|\boldsymbol{w}^\perp\|^2_{L_2(\Omega_t)}+\sum_{k,j=1}^3 S_{kj}(t)g_k(t)g_j(t)\\&=\|\boldsymbol{w}^\perp\|^2_{L_2(\Omega_t)}+\sum_{k,j=1}^3 S_{kj}(\beta S^{k3}(t)-\delta_{k3}\omega)(\beta S^{j3}(t)-\delta_{j3}\omega)\\&=\|\boldsymbol{w}^\perp\|^2_{L_2(\Omega_t)}+S^{33}(t)\beta^2+S_{33}(t)\omega^2-2\beta\omega\\&=\|\boldsymbol{w}^\perp\|^2_{L_2(\Omega_t)}+\frac{\beta^2}{\int_{\Omega_t}|x'|^2dx}+\beta^2(S^{33}-S^{-1}_{33})+\omega^2\int_{\Omega_t}|x'|^2dx-2\beta\omega.\end{aligned}$$

The expression

$$\begin{aligned}\beta^2(S^{33}-S^{-1}_{33})&=-\beta^2S^{-1}_{33}\sum_{j=1}^2 S^{3j}S_{j3}\\&=\frac{\beta^2}{S_{33}\det\mathcal{S}}(S_{11}S^2_{23}+S_{22}S^2_{13}-2S_{12}S_{13}S_{23})\equiv Q(t)\end{aligned}$$

is a positive definite quadratic form with respect to $S_{13},S_{23}$, since

$$2S_{12}\leq\sqrt{S_{11}}\sqrt{S_{22}}.$$

Hence (5.14) may be written in the form

$$\frac{d}{dt}\Big(\frac{1}{2}\|\boldsymbol{w}^\perp\|^2_{L_2(\Omega_t)}+Q(t)+\mathcal{R}(t)-\mathcal{R}_0\Big)+\frac{\nu}{2}\int_{\Omega_t}|S(\boldsymbol{w}^\perp)|^2dx=0, \quad (5.15)$$

where $\mathcal{R}(t)$ is defined in (1.27) with $\Omega=\Omega_t$ and $\mathcal{R}_0=\mathcal{R}|_{\Omega=\mathcal{F}}$.

Now we use the relations

$$2(\boldsymbol{e}_3 \times \boldsymbol{\eta}_i) = -\nabla(\boldsymbol{\eta}_i \cdot \boldsymbol{\eta}_3) + \boldsymbol{\eta}^i, \quad i = 1, 2, 3,$$

where $\boldsymbol{\eta}^1 = \boldsymbol{\eta}_2$, $\boldsymbol{\eta}^2 = -\boldsymbol{\eta}_1$, $\boldsymbol{\eta}^3 = 0$, and write the first equation in (1.9) in the form

$$\begin{aligned} &\boldsymbol{w}_t^\perp + (\boldsymbol{w} \cdot \nabla)\boldsymbol{w}^\perp + (\boldsymbol{w} \cdot \nabla)\boldsymbol{w}' + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{w}^\perp) - \nu\nabla^2\boldsymbol{w}^\perp \\ &\quad + \nabla\Big(p - \omega\sum_{j=1}^3 g_j(t)\boldsymbol{\eta}_3 \cdot \boldsymbol{\eta}_j\Big) = -\boldsymbol{w}_t' - \omega\sum_{\alpha=1}^2 g_\alpha\boldsymbol{\eta}^\alpha(x), \end{aligned} \tag{5.16}$$

where $\boldsymbol{w}' = \sum_{j=1}^3 g_j(t)\boldsymbol{\eta}_j(x)$. Since $(\boldsymbol{w}' \cdot \nabla)\boldsymbol{w}' = -\frac{1}{2}\nabla|\boldsymbol{w}'|^2$, (5.16) is equivalent to

$$\begin{aligned} &\boldsymbol{w}_t^\perp + (\boldsymbol{w} \cdot \nabla)\boldsymbol{w}^\perp + (\boldsymbol{w}^\perp \cdot \nabla)\boldsymbol{w}' + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{w}^\perp) - \nu\nabla^2\boldsymbol{w}^\perp \\ &\quad + \nabla\Big(p - \omega\sum_{j=1}^3 g_j(t)\boldsymbol{\eta}_3 \cdot \boldsymbol{\eta}_j - \frac{1}{2}|\boldsymbol{w}'|^2\Big) = -\boldsymbol{w}_t' - \omega\sum_{\alpha=1}^2 g_\alpha\boldsymbol{\eta}^\alpha(x). \end{aligned} \tag{5.17}$$

We multiply (5.17) by the auxiliary vector field $\boldsymbol{W}$ constructed in Proposition 5.2 leaving for the moment the function $f_0$ indefinite. Then we integrate the product over $\Omega_t$. Elementary calculations lead to

$$\begin{aligned} &\frac{d}{dt}\int_{\Omega_t} \boldsymbol{w}^\perp \cdot \boldsymbol{W}dx - \int_{\Omega_t} \boldsymbol{w}^\perp \cdot (\boldsymbol{W}_t + (\boldsymbol{w} \cdot \nabla)\boldsymbol{W})dx + 2\omega\int_{\Omega_t} (\boldsymbol{e}_3 \times \boldsymbol{w}^\perp) \cdot \boldsymbol{W}dx \\ &\quad + \frac{\nu}{2}\int_{\Omega_t} S(\boldsymbol{w}^\perp) \cdot S(\boldsymbol{W})dx + \int_{\Omega_t} (\boldsymbol{w}^\perp \cdot \nabla)\boldsymbol{w}' \cdot \boldsymbol{W}dx \\ &\quad - \int_{\Gamma_t} \Big(M + \omega\sum_{j=1}^3 g_j(t)\boldsymbol{\eta}_3(x) \cdot \boldsymbol{\eta}_j(x) + \frac{1}{2}|\boldsymbol{w}'|^2\Big)\boldsymbol{W} \cdot \boldsymbol{n}dS = 0. \end{aligned} \tag{5.18}$$

We multiply (5.18) by a small positive $\gamma$ and add to (5.15). As a result we obtain

$$\frac{dE(t)}{dt} + E_1(t) = 0 \tag{5.19}$$

with

$$E(t) = \frac{1}{2}\|\boldsymbol{w}\|^2_{L_2(\Omega_t)} + Q + (\mathcal{R} - \mathcal{R}_0) + \gamma\int_{\Omega_t} \boldsymbol{w}^\perp \cdot \boldsymbol{W}dx, \tag{5.20}$$

$$\begin{aligned} E_1(t) &= \frac{\nu}{2}\|S(\boldsymbol{w}^\perp)\|^2_{L_2(\Omega_t)} \\ &\quad - \gamma\int_{\Omega_t} \boldsymbol{w}^\perp \cdot (\boldsymbol{W}_t + (\boldsymbol{w} \cdot \nabla)\boldsymbol{W})dx + 2\omega\gamma\int_{\Omega_t} (\boldsymbol{e}_3 \times \boldsymbol{w}^\perp) \cdot \boldsymbol{W}dx \\ &\quad + \frac{\nu\gamma}{2}\int_{\Omega_t} S(\boldsymbol{w}^\perp) \cdot S(\boldsymbol{W})dx + \gamma\int_{\Omega_t} (\boldsymbol{w}^\perp \cdot \nabla)\boldsymbol{w}' \cdot \boldsymbol{W}dx - \gamma\mathcal{J}, \end{aligned} \tag{5.21}$$

where $\mathcal{J}$ is the surface integral in (5.18).

We pass to the estimates of $E$ and $E_1$. At first we consider the integral $\mathcal{J}$. It can be written in the form

$$\mathcal{J} = -\int_{\mathcal{G}_{\theta(t)}} (M + \omega \sum_{j=1}^{3} g_j(t)\boldsymbol{\eta}_3(x) \cdot \boldsymbol{\eta}_j(x) + \frac{1}{2}|\boldsymbol{w}'|^2)\Big|_{x=e_{\widehat{\rho}-1}(z)} f_1 dS_z$$

where $f_1 = \boldsymbol{W} \cdot \boldsymbol{n}|_{x=e_{\widehat{\rho}}(z)} |\widehat{\mathcal{L}}^T(z,\widehat{\rho})\boldsymbol{N}_\theta(z)|$.

We introduce the matrix $\mathcal{S}_0 = (S^0_{jk})_{j,k=1,2,3}$ with the elements

$$S^0_{jk} = \int_{\mathcal{F}_{\theta(t)}} \boldsymbol{\eta}_j(x) \cdot \boldsymbol{\eta}_k(x) dx.$$

In view of (5.4), $S^0_{\alpha 3}$ and $S^0_{3\alpha}$ vanish, $S^0_{33} = \int_{\mathcal{F}_\theta} |x'|^2 dx$ and the matrix $(S^0_{\alpha\beta})_{\alpha,\beta=1,2}$ is positive definite. We make use of the relation

$$\begin{aligned} M + \omega \sum_{j=1}^{3} g_j(t)\boldsymbol{\eta}_3(x) \cdot \boldsymbol{\eta}_j(x) + \frac{1}{2}|\boldsymbol{w}'|^2\Big|_{x=e_{\widehat{\rho}}(z)} \\ = -B_0(z)\widehat{\rho}(z,t) + \omega \sum_{k,j=1}^{3} S_0^{jk} d_k(t)\boldsymbol{\eta}_3(z) \cdot \boldsymbol{\eta}_j(z) + M', \end{aligned} \tag{5.22}$$

where

$$\begin{aligned} B_0(z) &= b_0(z)\widehat{\rho} - \kappa \int_{\mathcal{G}_\theta} \frac{\widehat{\rho}(\zeta,t)dS}{|z-\zeta)|}, \\ d_k(t) &= -\omega \int_{\mathcal{G}_\theta} \widehat{\rho}(z,t)\boldsymbol{\eta}_3(z) \cdot \boldsymbol{\eta}_k(z) dS, \\ M' &= \frac{\omega^2}{2}|\boldsymbol{N}'_\theta(z)|^2\widehat{\rho}^2(z,t) + \kappa \int_0^1 (1-s)\frac{\partial^2 U_s}{\partial s^2} ds + \frac{1}{2}|\boldsymbol{w}'|^2 \\ &\quad + \omega \sum_{j,k=1}^{3} (S^{jk}(t)I_k(t) - S_0^{jk}(t)d_k(t))\boldsymbol{\eta}_3(z) \cdot \boldsymbol{\eta}_j(z) \qquad (5.23) \\ &\quad + \sum_{j,k=1}^{3} S^{jk}(t)I_k(t))(\boldsymbol{\eta}_3(x) \cdot \boldsymbol{\eta}_j(x) - \boldsymbol{\eta}_3(z) \cdot \boldsymbol{\eta}_j(z)), \quad x = e_{\widehat{\rho}}(z) \end{aligned}$$

is the sum of nonlinear terms with respect to $\widehat{\rho}$ in (5.22). Let

$$\begin{aligned} B(z)\widehat{\rho}(z,t) &= B_0(z)\widehat{\rho}(z,t) - \omega S_0^{33} d_3(t)|\boldsymbol{\eta}_3(z)|^2 \\ &= B_0(z)\widehat{\rho}(z,t) + \frac{\omega^2|z'|^2}{\int_{\mathcal{G}_\theta}|\zeta'|^2 dS} \int_{\mathcal{G}_\theta} \widehat{\rho}(\zeta,t)|\zeta'|^2 dS, \end{aligned}$$

$$B_1(z)\widehat{\rho}(z,t) = B_0(z)\widehat{\rho}(z,t) - \omega \sum_{k,j=1}^{3} S_0^{jk} d_k(t)\boldsymbol{\eta}_3(z)\cdot\boldsymbol{\eta}_j(z)$$
$$= B(z)\widehat{\rho}(z,t) + \omega^2 \sum_{\alpha,\beta=1}^{2} S_0^{\alpha\beta} z_\alpha z_3 \int_{\mathcal{G}_\theta} \widehat{\rho}(\zeta,t)\zeta_\beta\zeta_3 dS,$$
$$\mathcal{B}_1\widehat{\rho} = PB_1P\widehat{\rho} + \sum_{k=1}^{4}\varphi_k(z)\int_{\mathcal{G}_\theta}\widehat{\rho}(\zeta,t)\varphi_k(\zeta)dS,$$

where $P$ is the projection on the subspace of $L_2(\mathcal{G}_\theta)$ orthogonal to the functions $\varphi_k$, i.e., to $(1, z_1, z_2, z_3, h_{\theta(t)}(z))$ defined on $\mathcal{G}_\theta$. The quadratic form $\int_{\mathcal{G}_\theta}\rho B\rho dS$ of the operator $B$ coincides with the form (1.4), hence, for $\rho$ satisfying (1.7), (1.8) we have $\int_{\mathcal{G}_\theta} B_1(\rho)\rho dS \geq c\|\rho\|^2_{L_2(\mathcal{G}_\theta)}$. It follows that $\int_{\mathcal{G}_\theta}\rho\mathcal{B}_1(\rho)dS \geq c\|\rho\|^2_{L_2(\mathcal{G}_\theta)}$ for arbitrary $\rho \in L_2(\mathcal{G}_\theta)$. The integral equation

$$\mathcal{B}_1 f = g$$

of the Fredholm type is uniquely solvable for arbitrary $g \in L_2(\mathcal{G}_\theta)$; moreover, if $g = Pg$, then $f = Pf$ and the equation $PB_1 f = g$ holds. Finally, if $g \in W_2^{1/2}(\mathcal{G}_\theta)$, then $f \in W_2^{1/2}(\mathcal{G}_\theta)$, and

$$\|f\|_{W_2^{1/2}(\mathcal{G}_\theta)} \leq c\|g\|_{W_2^{1/2}(\mathcal{G}_\theta)}. \tag{5.24}$$

Now we define $f_1$ as the solution of the equation

$$\mathcal{B}_1 f_1 = P(-\Delta_\theta)^{-1/2}P\widehat{\rho} = P(-\Delta_\theta)^{-1/2}\widehat{\rho}^{\perp},$$

where $\Delta_\theta$ is the Laplace-Beltrami operator on $\mathcal{G}_\theta$ (in fact, the equation $PB_1f_1 = P(-\Delta_\theta)^{-1/2}\widehat{\rho}^{\perp}$ is satisfied). By virtue of (5.24) and (5.9),

$$\|f_1\|_{W_2^{1/2}(\mathcal{G}_\theta)} \leq c\|(-\Delta_\theta)^{-1/2}\widehat{\rho}^{\perp}\|_{W_2^{1/2}(\mathcal{G}_\theta)}$$
$$\leq c\|\widehat{\rho}^{\perp}\|_{W_2^{-1/2}(\mathcal{G}_\theta)} \leq c\|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)}.$$

The function $f_0(y) = f_1(\mathcal{Z}(\theta(t))y, t)$, $y \in \mathcal{G}_0$, is a solution of the equation

$$P_0B_1(y)f_0 = P_0(-\Delta_0)^{-1/2}\widetilde{\rho}^{\perp}, \quad y \in \mathcal{G}_0,$$

where $\Delta_0$ is the Laplace-Beltrami operator on $\mathcal{G}_0$ and $P_0$ is a projection on the subspace of functions orthogonal to $(1, y_1, y_2, y_3, h_0(y))$. Hence

$$\|\widetilde{\rho}\|_{W_2^{-1/2}(\mathcal{G}_0)} = \|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)}.$$

We set $\boldsymbol{W}\cdot\boldsymbol{n}|_{x=\widehat{\rho}(z)} = f_1(z,t)|\widehat{\mathcal{L}}^T(z,\widehat{\rho})\boldsymbol{N}_\theta|^{-1}$.

$$\begin{aligned}
-\mathcal{J}' &\equiv \int_{\mathcal{G}_\theta(t)} \Big(B_0(z)\widehat{\rho}(z,t) - \omega\sum_{j=1}^3 S_0^{jk} d_k(t)\boldsymbol{\eta}_3(z)\cdot\boldsymbol{\eta}_j(z)\Big) f_1(z,t)dS \\
&= \int_{\mathcal{G}_\theta} B_1(z)\widehat{\rho} f_1(z,t)dS = \int_{\mathcal{G}_\theta} \widehat{\rho} B_1 f_1 dS \\
&= \int_{\mathcal{G}_\theta} \widehat{\rho}^\perp(-\Delta_\theta)^{-1}\widehat{\rho}^\perp dS \geq c\|\widehat{\rho}\|^2_{W_2^{-1/2}(\mathcal{G}_\theta)}.
\end{aligned}$$

Now we consider the contribution of the nonlinear terms (5.23) into $-\mathcal{J}$, i.e., the integral

$$-\mathcal{J}'' = \int_{\mathcal{G}_\theta} M' f_1(z,t)dS.$$

We have

$$\begin{aligned}
\Big|\int_{\mathcal{G}_\theta} |\boldsymbol{N}'(z)|^2\widehat{\rho}^2(z,t)f_1(z,t)dS\Big| &\leq \|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)}\||\boldsymbol{N}'(z)|^2\widehat{\rho}f_1\|_{W_2^{1/2}(\mathcal{G}_\theta)} \\
&\leq c\delta\|\rho\|_{W_2^{-1/2}(\mathcal{G}_\theta)}\|f_1\|_{W_2^{1/2}(\mathcal{G}_\theta)} \leq c\delta\|\widehat{\rho}\|^2_{W_2^{-1/2}(\mathcal{G}_\theta)}.
\end{aligned}$$

From the formula (2.9) in [14] it follows that (5.3) can be written in the form

$$I_k(t) = -\omega\int_0^1 ds\int_{\mathcal{G}_\theta} \widehat{\rho}(z,t)\boldsymbol{\eta}_3(e_{s\widehat{\rho}}(z))\cdot\boldsymbol{\eta}_k(e_{s\widehat{\rho}}(z))\Lambda(z,s\widehat{\rho})dS,$$

which implies

$$I_k(t) - d_k(t) = -\omega\int_0^1 ds\int_{\mathcal{G}_\theta} \widehat{\rho}(z,t)(\boldsymbol{\eta}_3(e_{s\widehat{\rho}}(z))\cdot\boldsymbol{\eta}_k(e_{s\widehat{\rho}}(z))\Lambda(z,s\widehat{\rho}) - \boldsymbol{\eta}_3(z)\cdot\boldsymbol{\eta}_k(z))dS,$$

$$\begin{aligned}
|d_k(t)| + |I_k(t)| &\leq c\|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)}, \\
|I_k(t) - d_k(t)| &\leq c\delta\|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)}.
\end{aligned}$$

For the estimate of $S^{jk}(t) - S_0^{jk}(t)$ we use the relations $\mathcal{S}^{-1} - \mathcal{S}_0^{-1} = \mathcal{S}_0^{-1}(\mathcal{S}_0 - \mathcal{S})\mathcal{S}^{-1}$ and

$$S_{jk}(t) - S^0_{jk} = \int_0^1 ds\int_{\mathcal{G}_\theta} \widehat{\rho}(z,t)\boldsymbol{\eta}_j(e_{s\widehat{\rho}}(z))\cdot\boldsymbol{\eta}_k(e_{s\widehat{\rho}}(z))\Lambda(z,s\widehat{\rho})dS.$$

It follows that

$$|S_{jk}(t) - S^0_{jk}| \leq c\|\rho\|_{W_2^{-1/2}(\mathcal{G}_\theta)}. \tag{5.25}$$

From the above inequalities it is easy to conclude that

$$\Big|\int_{\mathcal{G}_\theta}\Big(\frac{1}{2}|\boldsymbol{w}'^2| + \omega\sum_{j,k=1}^{3}(S^{jk}I_k - S_0^{jk}d_k(t))\boldsymbol{\eta}_3(z)\cdot\boldsymbol{\eta}_j)(z)$$
$$+\sum_{j,k=1}^{3}S^{jk}I_k(t)(\boldsymbol{\eta}_3(e_{s\widehat{\rho}}(z))\cdot\boldsymbol{\eta}_k(e_{s\widehat{\rho}}(z)) - \boldsymbol{\eta}_3(z)\cdot\boldsymbol{\eta}_k(z)\Big)f_1(z,t)dS\Big|$$
$$\leq c\delta\|\widehat{\rho}\|^2_{W_2^{-1/2}(\mathcal{G}_\theta)}.$$

Finally, by Proposition 5.2,

$$\Big|\int_0^1(1-s)ds\int_{\mathcal{G}_\theta}\frac{\partial^2 U_s}{\partial s^2}f_1 dS\Big| \leq c\delta\|\widehat{\rho}\|^2_{W_2^{-1/2}(\mathcal{G}_\theta)}.$$

Putting all the estimates together we see that for small $\delta$,

$$-\gamma\mathcal{J} \geq c\gamma\|\widehat{\rho}\|^2_{W_2^{-1/2}(\mathcal{G}_\theta)} = c\gamma\|\widetilde{\rho}\|^2_{W_2^{-1/2}(\mathcal{G}_0)}.$$

We pass to the estimates of the volume integrals in (5.21). By Proposition 5.1,

$$\Big|\int_{\Omega_t}\boldsymbol{w}^\perp\cdot\boldsymbol{W}_t dx\Big| \leq c\|\boldsymbol{w}^\perp\|_{L_2(\Omega_t)}\Big(\|f_0\|_{W_2^{1/2}(\mathcal{G}_0)} + \|f_{0t}\|_{L_2(\mathcal{G})}\Big)$$
$$\leq c\|\boldsymbol{w}^\perp\|_{L_2(\Omega_t)}\Big(\|\widetilde{\rho}\|_{W_2^{-1/2}(\mathcal{G}_0)} + \|\widetilde{\rho}_t\|_{L_2(\mathcal{G})}\Big),$$

and since

$$\|\widetilde{\rho}_t\|_{L_2(\mathcal{G}_0)} \leq c\|\boldsymbol{w}\|_{L_2(\Gamma_t)} \leq c\|\boldsymbol{w}^\perp\|_{L_2(\Gamma_t)} + c\sum_{k=1}^{3}|I_k(t)|$$
$$\leq c\Big(\|\boldsymbol{w}^\perp\|_{W_2^(\Omega_t)} + \|\widetilde{\rho}\|_{W_2^{-1/2}(\mathcal{G}_0)}\Big),$$

we have

$$\gamma\Big|\int_{\Omega_t}\boldsymbol{w}^\perp\cdot\boldsymbol{W}_t dx\Big| \leq c\gamma\|\boldsymbol{w}^\perp\|_{W_2^1(\Omega_t)}\Big(\|\boldsymbol{w}^\perp\|_{W_2^1(\Omega_t)} + \|\widetilde{\rho}\|_{W_2^{-1/2}(\mathcal{G}_0)}\Big).$$

In view of Proposition 5.1, other integrals in (5.21) do not exceed

$$c\gamma\|\boldsymbol{w}^\perp\|_{W_2^1(\Omega_t)}\|f_0\|_{W_2^{1/2}(\mathcal{G}_0)} \leq c\gamma\|\boldsymbol{w}^\perp\|_{W_2^1(\Omega_t)}\|\widetilde{\rho}\|_{W_2^{-1/2}(\mathcal{G}_0)}$$
$$= c\gamma\|\boldsymbol{w}^\perp\|_{W_2^1(\Omega_t)}\|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)},$$

which allows us to conclude, taking the Korn inequality into account, that for small $\gamma$,

$$E_1(t) \geq c\Big(\nu\|\boldsymbol{w}^\perp\|^2_{W_2^1(\Omega_t)} + \gamma\|\widehat{\rho}\|^2_{W_2^{-1/2}(\mathcal{G}_\theta)}\Big).$$

As for $E(t)$, this function satisfies (also for small $\gamma$) the inequality

$$c_1\Big(\|\boldsymbol{w}^\perp\|^2_{L_2(\Omega_t)} + \|\widehat{\rho}\|^2_{L_2(\mathcal{G}_\theta)}\Big) \leq E(t) \leq c_2\Big(\|\boldsymbol{w}^\perp\|^2_{L_2(\Omega_t)} + \|\widehat{\rho}\|^2_{L_2(\mathcal{G}_\theta)}\Big)$$

that is a consequence of the estimate

$$c_3\|\widetilde{\rho}\|^2_{L_2(\mathcal{G}_0)} = c_1\|\widehat{\rho}\|^2_{L_2(\mathcal{G}_\theta)} \le \mathcal{R} - \mathcal{R}_0 \le c_4\|\widehat{\rho}\|^2_{L_2(\mathcal{G})_\theta} = c_2\|\widetilde{\rho}\|^2_{L_2(\mathcal{G}_0)}$$

(see the remark at the end of Section 1). When we integrate (5.19), we arrive at (3.25). Thus Theorem 3.4 is proved.

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# Dynamics of a Non-fixed Elastic Body

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*Dedicated to the memory of Alexander Vasilievich Kazhikhov*

**Abstract.** In this paper, we present a model of the motion of a non-fixed elastic body. The word "non-fixed" means that the body can move as a whole under the action of external bulk and surface forces. Besides, these forces cause the elastic deformation of the body. We decompose the motion of the body on the rigid and deformation parts and write down governing equations for them.

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## 1. Introduction

The goal of this paper is to write down equations of the motion of an elastic body under the action of prescribed external surface and bulk forces. The body is assumed to change its form as well as its position in a designated space; therefore, the work done by the forces can be split into two parts. The first one is spent on the displacement of the body as a whole and the second part produces elastic deformations. So, the main task consists in decomposing the motion of the body, considered as a continuum, into a mean rigid motion and deformations. Investigation of various mechanical systems leads to solving of such a problem. For instance, the most useful information for captains of submarines or for pilots is related to the position and orientation of their ships in their designated space, i.e., to the rigid part of the motion, while the designer would be also interested to know the elastic deformations and the stresses.

In what follows, we will not focus on the smoothness of the functions encountered and will assume that they are smooth enough to perform all necessary operations. Let $S_*$ be a domain in $\mathbb{R}^3$ occupied by the body at the time

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moment $t_*$. Suppose that the motion of the body is prescribed by the mapping $\boldsymbol{\phi} : S_* \times [t_*, \infty) \to \mathbb{R}^3$ and at the time moment $t \in (t_*, \infty)$ the body occupies the domain $S(t) = \boldsymbol{\phi}(S_*, t)$. If $t = t_*$, then $\boldsymbol{\phi}(\cdot, t_*)$ is the identical mapping and $\boldsymbol{\phi}(S_*, t_*) = S_*$. We will denote by $\boldsymbol{\xi}$ and $\boldsymbol{x}$ the points in $S_*$ and in the actual configuration $S(t)$, respectively. Thus, $\boldsymbol{\phi}(\boldsymbol{\xi}, t_*) = \boldsymbol{\xi}$ and for every $\boldsymbol{x} \in S(t)$ there exists $\boldsymbol{\xi} \in S_*$ such that $\boldsymbol{\phi}(\boldsymbol{\xi}, t) = \boldsymbol{x}$. Our task is to represent $\boldsymbol{\phi}$ in the form

$$\boldsymbol{\phi}(\boldsymbol{\xi}, t) = \boldsymbol{\psi}(\boldsymbol{\xi}, t) + \boldsymbol{\delta}(\boldsymbol{\xi}, t), \qquad \boldsymbol{\xi} \in S_*, \quad t \in (t_*, \infty), \tag{1.1}$$

where the mappings $\boldsymbol{\psi} : S_* \times [t_*, \infty) \to \mathbb{R}^3$ and $\boldsymbol{\delta} : S_* \times [t_*, \infty) \to \mathbb{R}^3$ represent a rigid motion and an elastic deformation, respectively. Besides, we have to derive the governing equations for these mappings. It is clear that such a representation can be done by numerous methods, however, a simple and at the same time not too artificial one would be most preferable.

At first, we define the notion of rigid motion. The point is that in many papers this term is used for infinitesimal rigid displacements that are characterized by the zero linear strain tensor. We say that a mapping $\boldsymbol{\psi} : S_* \times [t_*, \infty) \to \mathbb{R}^3$ determines a rigid motion if it can be represented as a composition of a translation and a rotation for every $t \in [t_*, \infty)$. The space of rigid displacements will be denoted by $\mathcal{R}_d$. Thus, $\boldsymbol{\psi}(\cdot, t) \in \mathcal{R}_d$ for $t \in [t_*, \infty)$ if and only if there exist a vector $\boldsymbol{\tau}(t) \in \mathbb{R}^3$ and a rotation matrix $\boldsymbol{R}(t) \in SO(3)$ such that $\boldsymbol{\psi}(\boldsymbol{\xi}, t) = \boldsymbol{\tau}(t) + \boldsymbol{R}(t)\boldsymbol{\xi}$ for all $\boldsymbol{\xi} \in S_*$. Notice that a matrix $\boldsymbol{M}$ from $SO(3)$ is characterized by the following properties: $\boldsymbol{M}\boldsymbol{M}^T = \boldsymbol{M}^T\boldsymbol{M} = \boldsymbol{I}$ and $\det \boldsymbol{M} = 1$, where $\boldsymbol{M}^T$ is the transpose of $\boldsymbol{M}$ and $\boldsymbol{I}$ is the identity matrix. We suppose also that $\boldsymbol{\tau}(t_*) = 0$ and $\boldsymbol{R}(t_*) = \boldsymbol{I}$, i.e., $\boldsymbol{\psi}(\boldsymbol{\xi}, t_*) = \boldsymbol{\xi}$.

In a very interesting paper [1], the authors propose a decomposition of the form (1.1) by defining $\boldsymbol{\psi}(\cdot, t)$ as the minimizer of the functional $\|\boldsymbol{\phi}(\cdot, t) - \overline{\boldsymbol{\psi}}(\cdot, t)\|^2$ among all $\overline{\boldsymbol{\psi}} \in \mathcal{R}_d$ for each $t \in (t_*, \infty)$, where $\|f\|^2 = \int_{S_*} |f(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi}$. That is, $\boldsymbol{\psi}$ is the orthogonal projection of $\boldsymbol{\phi}$ in $L^2(S_*)$ onto $\mathcal{R}_d$. Such a method has both advantages and disadvantages. One can read about its advantages in the paper [2]. Its disadvantage is the complexity of the obtained model as well as the following observation. The point is that the space $\mathcal{R}_d$ is not linear, therefore the suggested minimization problem admits in general a non-unique solution. The solution will be unique if $\boldsymbol{\phi}$ is sufficiently close to $\mathcal{R}_d$. For this reason, some additional restrictions on the mapping $\boldsymbol{\phi}$ are imposed in [1]. From a mechanical point of view, these restrictions mean that the elastic deformations of the body should be small enough. Such a suggestion is quite acceptable of course.

In this work, we employ another method of constructing the decomposition (1.1). The basic idea consists in the fact that unlike $\mathcal{R}_d$ the space represented the Eulerian coordinates velocities of rigid displacements is linear. Therefore, the orthogonal projection in $L^2(S(t))$ of the Eulerian velocity field onto this space will be uniquely defined and we do not need the restrictions on the mapping $\boldsymbol{\phi}$ imposed in [1]. Notice also that the system of equations derived in the present work looks simpler than that in [1].

## 2. Decomposition of the motion

Let us denote by $\varrho_* = \varrho_*(\boldsymbol{\xi})$ and $\varrho = \varrho(\boldsymbol{x}, t)$ the functions of the density distribution in the body at the time moments $t_*$ and $t > t_*$, respectively. These functions satisfy the following equation:

$$\varrho_*(\boldsymbol{\xi}) = \varrho(\boldsymbol{\phi}(\boldsymbol{\xi}), t)\, J(\boldsymbol{\xi}, t), \qquad \boldsymbol{\xi} \in S_*, \quad t \geqslant t_*, \tag{2.1}$$

where $J(\boldsymbol{\xi}, t) = \det(\nabla_\xi \boldsymbol{\phi}(\boldsymbol{\xi}, t))$. Further on, we will always suppose that the mapping $\boldsymbol{\phi}(\cdot, t)$ is invertible for each $t > t_*$, i.e., the mapping $\boldsymbol{\phi}^{-1}(\cdot, t) : S(t) \to S_*$ is well defined. Recall that we assumed these mappings to be smooth enough. This means in particular that the Jacobian $J$ does not vanish and is always positive because $J(\boldsymbol{\xi}, t_*) = 1$ for all $\boldsymbol{\xi} \in S_*$. We can rewrite (2.1) in Eulerian coordinates:

$$\varrho(\boldsymbol{x}, t) = \varrho_*(\boldsymbol{\phi}^{-1}(\boldsymbol{x}, t))\, J^{-1}(\boldsymbol{\phi}^{-1}(\boldsymbol{x}, t), t), \qquad \boldsymbol{x} \in S(t), \quad t \geqslant t_*.$$

Denote by $\boldsymbol{x}_c(t)$ the mass center of the body at the time moment $t$:

$$\boldsymbol{x}_c(t) = m^{-1} \int_{S(t)} \varrho(\boldsymbol{x}, t)\, \boldsymbol{x}\, d\boldsymbol{x},$$

where $m = \int_{S(t)} \varrho(\boldsymbol{x}, t)\, d\boldsymbol{x}$ is the mass of the body. Notice that $m = \int_{S_*} \varrho_*(\boldsymbol{\xi})\, d\boldsymbol{\xi}$, which can be obtained from the definition of $m$ by changing the variables $\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{\xi}, t)$ and taking into account (2.1). This in particular implies that $m$ is independent of $t$. Without loss of generality, we suppose that the mass center coincides with the origin of the coordinate system at the time moment $t_*$, i.e., $\boldsymbol{x}_c(t_*) = \mathbf{0}$.

Let us define the space $\mathcal{R}_v$ of the Eulerian velocities of rigid displacements. We say that a vector field $\boldsymbol{u}(\cdot, t) : S(t) \to \mathbb{R}^3$ is an element of the space $\mathcal{R}_v$ at the instant $t$ if there exist a vector $\boldsymbol{a}(t)$ and a skew-symmetric matrix $\boldsymbol{Q}(t)$ (i.e., $\boldsymbol{Q}^T = -\boldsymbol{Q}$) such that

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{a}(t) + \boldsymbol{Q}(t)\big(\boldsymbol{x} - \boldsymbol{x}_c(t)\big) \quad \text{for all } \boldsymbol{x} \in S(t). \tag{2.2}$$

Every skew-symmetric matrix $\boldsymbol{Q}$ can be associated with a vector $\boldsymbol{\omega}$ such that $\boldsymbol{Q}\boldsymbol{b} = \boldsymbol{\omega} \times \boldsymbol{b}$ for all $\boldsymbol{b} \in \mathbb{R}^3$. This correspondence is bijective and will be denoted later on as $\boldsymbol{Q} = [\boldsymbol{\omega}]$. Therefore, (2.2) can be rewritten as

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{a}(t) + [\boldsymbol{\omega}(t)]\big(\boldsymbol{x} - \boldsymbol{x}_c(t)\big) = \boldsymbol{a}(t) + \boldsymbol{\omega}(t) \times \big(\boldsymbol{x} - \boldsymbol{x}_c(t)\big) \quad \text{for all } \boldsymbol{x} \in S(t).$$

Since the space $\mathcal{R}_v$ is a closed linear subspace in $L^2(S(t))$, for every vector field $\boldsymbol{v} \in L^2(S(t))$ there exists a unique vector field $\boldsymbol{V} \in \mathcal{R}_v$ which is the orthogonal projection of $\boldsymbol{v}$ in $L^2(S(t))$ onto $\mathcal{R}_v$.

The vector field $\boldsymbol{v}_*(\boldsymbol{\xi}, t) = \partial_t \boldsymbol{\phi}(\boldsymbol{\xi}, t)$, $\boldsymbol{\xi} \in S_*$, is the velocity field in the Lagrangian description of the motion of the body. The Eulerian velocity is defined as follows: $\boldsymbol{v}(\boldsymbol{x}, t) = \boldsymbol{v}_*(\boldsymbol{\phi}^{-1}\big(\boldsymbol{x}, t), t\big)$, $\boldsymbol{x} \in S(t)$. In order to find the orthogonal projection $\boldsymbol{V}$ of the velocity field $\boldsymbol{v}$ in $L^2(S(t))$ onto $\mathcal{R}_v$ we have to solve the following minimization problem: it is necessary to find $\boldsymbol{V} \in \mathcal{R}_v$ such that

$$\int_{S(t)} \varrho(\boldsymbol{x}, t)\, |\boldsymbol{v}(\boldsymbol{x}, t) - \boldsymbol{V}(\boldsymbol{x}, t)|^2 d\boldsymbol{x} \leqslant \int_{S(t)} \varrho(\boldsymbol{x}, t)\, |\boldsymbol{v}(\boldsymbol{x}, t) - \boldsymbol{u}(\boldsymbol{x}, t)|^2 d\boldsymbol{x} \tag{2.3}$$

for all $\boldsymbol{u} \in \mathcal{R}_v$. Notice that we have defined the norm in $L^2(S(t))$ as

$$\|f\|^2_{L^2(S(t))} = \int_{S(t)} \varrho(\boldsymbol{x},t)\,|f(\boldsymbol{x})|^2\,d\boldsymbol{x} = \int_{S_*} \varrho_*(\boldsymbol{\xi})\,|f(\boldsymbol{\phi}(\boldsymbol{\xi},t))|^2\,d\boldsymbol{\xi}.$$

Problem (2.3) has a unique solution $\boldsymbol{V}$ and there exist vectors $\boldsymbol{a}(t)$ and $\boldsymbol{\omega}(t)$ such that $\boldsymbol{V}(\boldsymbol{x},t) = \boldsymbol{a}(t)+[\boldsymbol{\omega}(t)]\,\big(\boldsymbol{x}-\boldsymbol{x}_c(t)\big)$. It is not difficult to see that for an arbitrary vector $\boldsymbol{b} \in \mathbb{R}^3$ these vectors satisfy the following relations:

$$\int_{S(t)} \varrho(\boldsymbol{x},t)\,\big(\boldsymbol{v}(\boldsymbol{x},t) - \boldsymbol{a}(t)\big)\cdot\boldsymbol{b}\,d\boldsymbol{x} = 0, \tag{2.4}$$

$$\int_{S(t)} \varrho(\boldsymbol{x},t)\,\boldsymbol{v}(\boldsymbol{x},t)\cdot[\boldsymbol{\omega}(t)]\,\big(\boldsymbol{x}-\boldsymbol{x}_c(t)\big)\,d\boldsymbol{x} + \int_{S(t)} \varrho(\boldsymbol{x},t)\,\big(\boldsymbol{x}-\boldsymbol{x}_c(t)\big)\cdot\Big([\boldsymbol{\omega}(t)]^T[\boldsymbol{b}] + [\boldsymbol{b}]^T[\boldsymbol{\omega}(t)]\Big)\big(\boldsymbol{x}-\boldsymbol{x}_c(t)\big)\,d\boldsymbol{x} = 0. \tag{2.5}$$

Since $\boldsymbol{r}\cdot\Big([\boldsymbol{\omega}(t)]^T[\boldsymbol{b}] + [\boldsymbol{b}]^T[\boldsymbol{\omega}(t)]\Big)\boldsymbol{r} = 2\,[\boldsymbol{\omega}(t)]\boldsymbol{r}\cdot[\boldsymbol{b}]\boldsymbol{r}$ for every $\boldsymbol{r} \in \mathbb{R}^3$, equality (2.5) can be rewritten:

$$\int_{S(t)} \varrho(\boldsymbol{x},t)\,\Big([\boldsymbol{\omega}(t)]\big(\boldsymbol{x}-\boldsymbol{x}_c(t)\big) - \boldsymbol{v}(\boldsymbol{x},t)\Big)\cdot[\boldsymbol{b}]\big(\boldsymbol{x}-\boldsymbol{x}_c(t)\big)\,d\boldsymbol{x} = 0. \tag{2.6}$$

Due to the arbitrariness of $\boldsymbol{b}$, (2.4) and (2.6) imply that

$$\boldsymbol{a}(t) = m^{-1}\int_{S(t)} \varrho(\boldsymbol{x},t)\,\boldsymbol{v}(\boldsymbol{x},t)\,d\boldsymbol{x}, \tag{2.7}$$

$$\boldsymbol{\omega}(t) = \boldsymbol{J}_c^{-1}(t)\int_{S(t)} \varrho(\boldsymbol{x},t)\,\big(\boldsymbol{x}-\boldsymbol{x}_c(t)\big)\times\boldsymbol{v}(\boldsymbol{x},t)\,d\boldsymbol{x}, \tag{2.8}$$

where

$$\boldsymbol{J}_c(t) = \int_{S(t)} \varrho(\boldsymbol{x},t)\,\Big(\boldsymbol{I}\,\big|\boldsymbol{x}-\boldsymbol{x}_c(t)\big|^2 - \big(\boldsymbol{x}-\boldsymbol{x}_c(t)\big)\otimes\big(\boldsymbol{x}-\boldsymbol{x}_c(t)\big)\Big)\,d\boldsymbol{x}$$

is the matrix of the inertia moments of the body $S(t)$ with respect to its mass center $\boldsymbol{x}_c(t)$.

The vector field $\boldsymbol{V}$ is a velocity field of some rigid motion that will be denoted by $\boldsymbol{\psi} = \boldsymbol{\psi}(\boldsymbol{\xi},t)$. The field $\boldsymbol{\psi}$ is the unique solution of the following Cauchy problem:

$$\partial_t\boldsymbol{\psi}(\boldsymbol{\xi},t) = \boldsymbol{V}\big(\boldsymbol{\psi}(\boldsymbol{\xi},t),t\big), \quad \boldsymbol{\psi}(\boldsymbol{\xi},t_*) = \boldsymbol{\xi}, \quad \boldsymbol{\xi}\in S_*. \tag{2.9}$$

Since $\boldsymbol{\psi} \in \mathcal{R}_d$, there exist a vector $\boldsymbol{\tau}(t)$ and a rotation matrix $\boldsymbol{R}(t)$ such that

$$\boldsymbol{\psi}(\boldsymbol{\xi},t) = \boldsymbol{\tau}(t) + \boldsymbol{R}(t)\boldsymbol{\xi}, \quad \text{for all } \boldsymbol{\xi}\in S_*.$$

Let us derive relations satisfied by $\boldsymbol{\tau}$ and $\boldsymbol{R}$.

At first we note that

$$\boldsymbol{x}_c(t) = m^{-1}\int_{S_*} \varrho_*(\boldsymbol{\xi})\,\boldsymbol{\phi}(\boldsymbol{\xi},t)\,d\boldsymbol{\xi}. \tag{2.10}$$

The change of the variable $\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{\xi}, t)$ in (2.7) gives:

$$\boldsymbol{a}(t) = m^{-1} \int_{S_*} \varrho_*(\boldsymbol{\xi})\, \boldsymbol{v}_*(\boldsymbol{\xi}, t)\, d\boldsymbol{\xi} = m^{-1} \int_{S_*} \varrho_*(\boldsymbol{\xi})\, \partial_t \boldsymbol{\phi}(\boldsymbol{\xi}, t)\, d\boldsymbol{\xi}$$
$$= \frac{d}{dt}\Big(m^{-1} \int_{S_*} \varrho_*(\boldsymbol{\xi})\, \boldsymbol{\phi}(\boldsymbol{\xi}, t)\, d\boldsymbol{\xi}\Big) = \dot{\boldsymbol{x}}_c(t), \quad (2.11)$$

where $\dot{\boldsymbol{x}}_c = d\boldsymbol{x}_c/dt$. Since the point $\boldsymbol{\xi} = \boldsymbol{0}$ is the mass center of the body $S_*$, multiplying (2.9) by $\varrho_*$ and integrating over $S_*$ give:

$$\dot{\boldsymbol{\tau}}(t) = m^{-1} \int_{S_*} \varrho_*(\boldsymbol{\xi})\, \big(\dot{\boldsymbol{x}}_c(t) + [\boldsymbol{\omega}(t)]\big(\boldsymbol{\psi}(\boldsymbol{\xi}, t) - \boldsymbol{x}_c(t)\big)\, d\boldsymbol{\xi}$$
$$= \dot{\boldsymbol{x}}_c(t) + m^{-1}[\boldsymbol{\omega}(t)] \int_{S_*} \varrho_*(\boldsymbol{\xi})\, \big(\boldsymbol{\tau}(t) + \boldsymbol{R}(t)\boldsymbol{\xi} - \boldsymbol{x}_c(t)\big)\, d\boldsymbol{\xi}$$
$$= \dot{\boldsymbol{x}}_c(t) + [\boldsymbol{\omega}(t)]\big(\boldsymbol{\tau}(t) - \boldsymbol{x}_c(t)\big).$$

Thus, we have obtained an ordinary differential equation

$$\frac{d}{dt}\big(\boldsymbol{\tau}(t) - \boldsymbol{x}_c(t)\big) = [\boldsymbol{\omega}(t)]\big(\boldsymbol{\tau}(t) - \boldsymbol{x}_c(t)\big)$$

with the initial condition $\boldsymbol{\tau}(t_*) - \boldsymbol{x}_c(t_*) = \boldsymbol{0}$ that follows from the fact that $\boldsymbol{x}_c(t_*) = \boldsymbol{\tau}(t_*) = \boldsymbol{0}$. As $[\boldsymbol{\omega}(t)]$ is a skew-symmetric matrix, the unique solution of this problem is zero. That is

$$\boldsymbol{\tau}(t) = \boldsymbol{x}_c(t) \quad \text{for all } t \geqslant t_*.$$

Therefore,

$$\boldsymbol{\psi}(\boldsymbol{\xi}, t) = \boldsymbol{x}_c(t) + \boldsymbol{R}(t)\boldsymbol{\xi}, \qquad \boldsymbol{\xi} \in S_*, \quad t \in [t_*, \infty). \quad (2.12)$$

This relation implies in particular that

$$\int_{S_*} \varrho_*(\boldsymbol{\xi})\, \big(\boldsymbol{\phi}(\boldsymbol{\xi}, t) - \boldsymbol{\psi}(\boldsymbol{\xi}, t)\big)\, d\boldsymbol{\xi} = \boldsymbol{0}, \qquad t \in [t_*, \infty).$$

As for the matrix $\boldsymbol{R}$, it follows from (2.9), (2.11), and (2.12) that

$$\dot{\boldsymbol{R}}(t) = [\boldsymbol{\omega}(t)]\boldsymbol{R}(t), \quad (2.13)$$

or, in other words,

$$\dot{\boldsymbol{R}}\boldsymbol{R}^T = [\boldsymbol{\omega}]. \quad (2.14)$$

Let us write

$$\boldsymbol{\eta}(\boldsymbol{\xi}, t) = \boldsymbol{R}^T(t)\big(\boldsymbol{\phi}(\boldsymbol{\xi}, t) - \boldsymbol{\psi}(\boldsymbol{\xi}, t)\big).$$

The vector field $\boldsymbol{\eta}$ is the elastic displacement of the body's particles in the frame whose motion is determined by the mapping $\boldsymbol{\psi}$. Thus, solving problem (2.3) gives us the following decomposition of the motion $\boldsymbol{\phi}$ of the body $S_*$ into the rigid motion ($\boldsymbol{\psi}$) and the deformation ($\boldsymbol{\eta}$):

$$\boldsymbol{\phi}(\boldsymbol{\xi}, t) = \boldsymbol{\psi}(\boldsymbol{\xi}, t) + \boldsymbol{R}(t)\boldsymbol{\eta}(\boldsymbol{\xi}, t) = \boldsymbol{x}_c(t) + \boldsymbol{R}(t)\big(\boldsymbol{\xi} + \boldsymbol{\eta}(\boldsymbol{\xi}, t)\big), \quad (2.15)$$

where $\boldsymbol{x}_c(t)$ and $\boldsymbol{R}(t)$ are defined by (2.10), (2.14), and (2.8). Here, $\boldsymbol{R}\,\boldsymbol{\eta}$ plays the role of $\boldsymbol{\delta}$ in (1.1).

Let us investigate properties of the vector field $\boldsymbol{\eta}$. As it follows directly from the definition of $\boldsymbol{\eta}$,

$$\boldsymbol{\eta}(\boldsymbol{\xi}, t_*) = \mathbf{0} \quad \text{for all } \boldsymbol{\xi} \in S_*$$

and

$$\int_{S_*} \varrho_*(\boldsymbol{\xi})\, \boldsymbol{\eta}(\boldsymbol{\xi}, t)\, d\boldsymbol{\xi} = \mathbf{0} \quad \text{for all } t \geqslant t_*. \tag{2.16}$$

**Proposition 2.1.** *For each $t \geqslant t_*$ the vector field $\boldsymbol{\eta}$ defined above satisfies the following relations:*

$$\int_{S_*} \varrho_*(\boldsymbol{\xi})\, \partial_t \boldsymbol{\eta}(\boldsymbol{\xi}, t)\, d\boldsymbol{\xi} = \mathbf{0}, \tag{2.17}$$

$$\int_{S_*} \varrho_*(\boldsymbol{\xi})\, \big(\boldsymbol{\xi} + \boldsymbol{\eta}(\boldsymbol{\xi}, t)\big) \times \partial_t \boldsymbol{\eta}(\boldsymbol{\xi}, t)\, d\boldsymbol{\xi} = \mathbf{0}. \tag{2.18}$$

*Equation* (2.18) *can be rewritten as*

$$\int_{S_*} \varrho_*(\boldsymbol{\xi})\, \boldsymbol{R}(t)\, \partial_t \boldsymbol{\eta}(\boldsymbol{\xi}, t) \cdot [\boldsymbol{b}] \boldsymbol{R}(t) \big(\boldsymbol{\xi} + \boldsymbol{\eta}(\boldsymbol{\xi}, t)\big)\, d\boldsymbol{\xi} = \mathbf{0} \quad \textit{for all } \boldsymbol{b} \in \mathbb{R}^3. \tag{2.19}$$

*Proof.* Relation (2.17) for $t > t_*$ is a direct consequence of (2.16).

Let us show that it holds true also for $t = t_*$. Due to (2.7) and (2.11), $\dot{\boldsymbol{x}}_c(t_*) = m^{-1} \int_{S_*} \varrho_*(\boldsymbol{\xi}) \partial_t \boldsymbol{\phi}(\boldsymbol{\xi}, t_*)\, d\boldsymbol{\xi}$. Besides that, $\partial_t \boldsymbol{\phi}(\boldsymbol{\xi}, t_*) = \dot{\boldsymbol{x}}_c(t_*) + [\boldsymbol{\omega}(t_*)]\, \boldsymbol{\xi} + \partial_t \boldsymbol{\eta}(\boldsymbol{\xi}, t_*)$ and $\int_{S_*} \varrho_*(\boldsymbol{\xi})\, \boldsymbol{\xi}\, d\boldsymbol{\xi} = \mathbf{0}$. Therefore, we obtain (2.17) for $t = t_*$.

Let us prove the equivalence of (2.18) and (2.19). Since the matrix $\boldsymbol{R}^T [\boldsymbol{b}] \boldsymbol{R}$ is skew-symmetric, there exists a unique vector $\boldsymbol{q}$ such that $\boldsymbol{R}^T [\boldsymbol{b}] \boldsymbol{R} = [\boldsymbol{q}]$. Therefore, (2.19) implies that

$$\boldsymbol{q} \cdot \int_{S_*} \varrho_*\, (\boldsymbol{\xi} + \boldsymbol{\eta}) \times \partial_t \boldsymbol{\eta}\, d\boldsymbol{\xi} = \mathbf{0}.$$

Due to the arbitrariness of $\boldsymbol{q}$ we obtain (2.18). Here, we used the equality $\boldsymbol{u} \cdot [\boldsymbol{q}] \boldsymbol{v} = \boldsymbol{u} \cdot (\boldsymbol{q} \times \boldsymbol{v}) = \boldsymbol{q} \cdot (\boldsymbol{v} \times \boldsymbol{u})$ that holds true for all vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$. In order to derive (2.19) from (2.18), multiply (2.18) by an arbitrary vector $\boldsymbol{q}$. If we denote by $\boldsymbol{b}$ the vector such that $[\boldsymbol{b}] = \boldsymbol{R}[\boldsymbol{q}]\boldsymbol{R}^T$, then we obtain (2.19).

Thus, we have only to establish (2.19). The passage in (2.6) to the Lagrangian coordinates gives:

$$\int_{S_*} \varrho_*(\boldsymbol{\xi}) \Big( [\boldsymbol{\omega}(t)] \big(\boldsymbol{\phi}(\boldsymbol{\xi}, t) - \boldsymbol{x}_c(t)\big) - \partial_t (\boldsymbol{\phi}(\boldsymbol{\xi}, t) - \boldsymbol{x}_c(t)) \Big) \cdot [\boldsymbol{b}] \big(\boldsymbol{\phi}(\boldsymbol{\xi}, t) - \boldsymbol{x}_c(t)\big)\, d\boldsymbol{\xi} = \mathbf{0}$$

for every $\boldsymbol{b} \in \mathbb{R}^3$. Taking into account (2.15) and (2.13), we find that $[\boldsymbol{\omega}](\boldsymbol{\phi} - \boldsymbol{x}_c) - \partial_t(\boldsymbol{\phi} - \boldsymbol{x}_c) = -\boldsymbol{R}\, \partial_t \boldsymbol{\eta}$. The substitution of this expression into the last integral identity yields (2.19). The proposition is proved. □

Conditions (2.17)–(2.19) are necessary for the vector field $\dot{\boldsymbol{x}}_c + [\boldsymbol{\omega}](\boldsymbol{x} - \boldsymbol{x}_c)$ to be the solution of problem (2.3). The next proposition shows that, in a certain sense, they are also sufficient.

**Proposition 2.2.** *Suppose that we have functions* $\boldsymbol{x}_c : [t_*, \infty) \to \mathbb{R}^3$, $\boldsymbol{\omega} : [t_*, \infty) \to \mathbb{R}^3$, $\boldsymbol{\eta} : S_* \times [t_*, \infty) \to \mathbb{R}^3$, *and* $\boldsymbol{R} : [t_*, \infty) \to SO(3)$ *such that:*

1. $\boldsymbol{x}_c(t_*) = \boldsymbol{0}$, $\boldsymbol{R}(t_*) = \boldsymbol{I}$, *and* $\boldsymbol{\eta}(\boldsymbol{\xi}, t_*) = \boldsymbol{0}$ *for all* $\boldsymbol{\xi} \in S_*$*;*
2. $\dot{\boldsymbol{R}}(t) = [\boldsymbol{\omega}(t)]\boldsymbol{R}(t)$ *for* $t \in [t_*, \infty)$*;*
3. *the mapping* $\boldsymbol{\phi}(\cdot, t) : S_* \to \mathbb{R}^3$ *defined as* $\boldsymbol{\phi}(\boldsymbol{\xi}, t) = \boldsymbol{x}_c(t) + \boldsymbol{R}(t)(\boldsymbol{\xi} + \boldsymbol{\eta}(\boldsymbol{\xi}, t))$ *is invertible;*
4. *either pair of conditions* (2.17) *and* (2.18) *or* (2.17)*and* (2.19) *is satisfied.*

*Then* $\boldsymbol{x}_c(t)$ *is the mass center of the body* $S(t) = \boldsymbol{\phi}(S_*, t)$ *and* $\boldsymbol{V}(\boldsymbol{x}, t) = \dot{\boldsymbol{x}}_c(t) + [\boldsymbol{\omega}(t)](\boldsymbol{x} - \boldsymbol{x}_c(t))$ *is the solution of the minimization problem* (2.3) *with* $\boldsymbol{v}(\boldsymbol{x}, t) = \partial_t \boldsymbol{\phi}(\boldsymbol{\xi}, t)\big|_{\boldsymbol{\xi} = \boldsymbol{\phi}^{-1}(\boldsymbol{x}, t)}$ *and* $\varrho(\boldsymbol{x}, t) = \varrho_*\big(\boldsymbol{\phi}^{-1}(\boldsymbol{x}, t)\big) \det(\nabla_\xi \boldsymbol{\phi}(\boldsymbol{\xi}, t))$.

*Proof.* As problem (2.3) has a unique solution, it is enough to verify that

$$\frac{d}{d\lambda} \int_{S(t)} \varrho \left| \dot{\boldsymbol{x}}_c + \lambda\, \boldsymbol{a} + [\boldsymbol{\omega} + \lambda\, \boldsymbol{b}](\boldsymbol{x} - \boldsymbol{x}_c) - \boldsymbol{v} \right|^2 d\boldsymbol{x}\, \Big|_{\lambda=0} = \boldsymbol{0}$$

for arbitrary vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. This condition is equivalent to the following two equations:

$$\int_{S_*} \varrho_* \left(\dot{\boldsymbol{x}}_c - \partial_t \boldsymbol{\phi}\right) d\boldsymbol{\xi} = \boldsymbol{0}, \tag{2.20}$$

$$\int_{S_*} \varrho_* \Big([\boldsymbol{\omega}](\boldsymbol{\phi} - \boldsymbol{x}_c) - \partial_t \boldsymbol{\phi}\Big) \cdot [\boldsymbol{b}](\boldsymbol{\phi} - \boldsymbol{x}_c)\, d\boldsymbol{\xi} = \boldsymbol{0}. \tag{2.21}$$

Relation (2.20) follows directly from the facts that the point $\boldsymbol{\xi} = \boldsymbol{0}$ is the mass center of the body $S_*$, $\int_{S_*} \varrho_*(\boldsymbol{\xi})\, \boldsymbol{\eta}(\boldsymbol{\xi}, t)\, d\boldsymbol{\xi} = \boldsymbol{0}$ for all $t \geqslant t_*$ (due to the first assumption of the proposition and (2.17)), and $\partial_t(\boldsymbol{\phi} - \boldsymbol{x}_c) = \dot{\boldsymbol{R}}(\boldsymbol{\xi} + \boldsymbol{\eta}) + \boldsymbol{R}\, \partial_t \boldsymbol{\eta}$.

In order to prove (2.21), we make use of equality (2.19) that can be rewritten as

$$\int_{S_*} \varrho_* \Big(\partial_t(\boldsymbol{\phi} - \boldsymbol{x}_c) - [\boldsymbol{\omega}]\, (\boldsymbol{\phi} - \boldsymbol{x}_c)\Big) \cdot [\boldsymbol{b}]\, (\boldsymbol{\phi} - \boldsymbol{x}_c)\, d\boldsymbol{\xi} = \boldsymbol{0}.$$

Since $\int_{S_*} \varrho_*\, \dot{\boldsymbol{x}}_c \cdot [\boldsymbol{b}]\, (\boldsymbol{\phi} - \boldsymbol{x}_c)\, d\boldsymbol{\xi} = \boldsymbol{0}$, we immediately obtain (2.21). The proposition is proved. ☐

Notice that the function $\boldsymbol{\eta}$ in the constructed decomposition (2.15) must satisfy the nonlinear condition (2.18). This fact was a cause for negative evaluation of this method in [2]. The author gave preference to another decomposition that was thoroughly investigated subsequently in [1].

## 3. Equations of motion

Proposition 2.2 implies that if we know the functions $\boldsymbol{x}_c$, $\boldsymbol{\omega}$, and $\boldsymbol{\eta}$, then we are able to reconstruct the motion $\boldsymbol{\phi}$ of the body. Of course, the mapping $\boldsymbol{\phi}(\cdot, t)$ must be invertible for all $t$. Let us derive the governing equations for these functions.

The general equations of body dynamics in Eulerian variables read as follows:

$$\varrho\big(\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\big) = \operatorname{div}\boldsymbol{T} + \varrho\boldsymbol{f}, \tag{3.1}$$

$$\partial_t\varrho + \operatorname{div}(\varrho\,\boldsymbol{v}) = 0, \tag{3.2}$$

where $\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{x},t)$ is the Cauchy stress tensor and $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x},t)$ is the external mass force. Notice that the tensor $\boldsymbol{T}$ is symmetric: $\boldsymbol{T} = \boldsymbol{T}^T$. These equations should be supplemented with boundary and initial conditions:

$$\boldsymbol{T}(\boldsymbol{x},t)\boldsymbol{n}(\boldsymbol{x},t) = \boldsymbol{g}(\boldsymbol{x},t), \qquad \boldsymbol{x}\in\partial S(t), \quad t \geqslant t_*,$$

$$S(t_*) = S_*, \qquad \boldsymbol{v}(\boldsymbol{\xi},t_*) = \boldsymbol{v}^*(\boldsymbol{\xi}), \qquad \varrho(\boldsymbol{\xi},t_*) = \varrho_*(\boldsymbol{\xi}), \qquad \boldsymbol{\xi}\in S_*,$$

where $\boldsymbol{g} = \boldsymbol{g}(\boldsymbol{x},t)$ is the vector of a force acting on the boundary of the body. More precisely, it is the density of the force, i.e., the force acting on the unit area of the boundary.

The multiplication of (3.1) by an arbitrary function $\boldsymbol{h}$ that is equal to zero at $t = T > t_*$ and $t = t_*$ and the integration over $\{(\boldsymbol{x},t) \mid t\in[t_*,T],\ \boldsymbol{x}\in S(t)\}$ give us the following integral identity:

$$\int_{t_*}^T\int_{S(t)}\Big(\varrho\,\boldsymbol{v}\cdot\big(\partial_t\boldsymbol{h} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{h}\big) - \boldsymbol{T}:\nabla\boldsymbol{h} + \varrho\boldsymbol{f}\cdot\boldsymbol{h}\Big)\,d\boldsymbol{x}dt = -\int_{t_*}^T\int_{\partial S(t)}\boldsymbol{g}\cdot\boldsymbol{h}\,ds_x dt, \tag{3.3}$$

where $ds_x$ is the area measure on the surface $\partial S(t)$.

At first, we take an (independent of $\boldsymbol{x}$) test function $\boldsymbol{h}$: $\boldsymbol{h}(\boldsymbol{x},t) = \boldsymbol{a}(t)$ for all $\boldsymbol{x}$. Then

$$\int_{t_*}^T\dot{\boldsymbol{a}}\cdot\int_{S(t)}\varrho\,\boldsymbol{v}\,d\boldsymbol{x}\,dt = -\int_{t_*}^T\boldsymbol{a}\cdot\Big(\int_{\partial S(t)}\boldsymbol{g}\,ds_x + \int_{S(t)}\varrho\boldsymbol{f}\,d\boldsymbol{x}\Big)\,dt.$$

But, as it follows from (2.7) and (2.11), $\int_{S(t)}\varrho\,\boldsymbol{v}\,d\boldsymbol{x} = m\,\dot{\boldsymbol{x}}_c$. Therefore, due to the arbitrariness of $\boldsymbol{a}$, we obtain:

$$m\,\ddot{\boldsymbol{x}}_c(t) = \int_{\partial S(t)}\boldsymbol{g}(\boldsymbol{x},t)\,ds_x + \int_{S(t)}\varrho(\boldsymbol{x},t)\,\boldsymbol{f}(\boldsymbol{x},t)\,d\boldsymbol{x}. \tag{3.4}$$

If we take $\boldsymbol{h}(\boldsymbol{x},t) = [\boldsymbol{b}(t)]\,\big(\boldsymbol{x}-\boldsymbol{x}_c(t)\big)$ in (3.3) with an arbitrary vector $\boldsymbol{b}(t)\in\mathbb{R}^3$, then we find that

$$\int_{t_*}^T\int_{S(t)}\Big(\varrho\,\boldsymbol{v}\cdot\big([\dot{\boldsymbol{b}}](\boldsymbol{x}-\boldsymbol{x}_c) - [\boldsymbol{b}]\,\dot{\boldsymbol{x}}_c\big) + \big(\varrho\,(\boldsymbol{v}\otimes\boldsymbol{v}) - \boldsymbol{T}\big):[\boldsymbol{b}]\Big)\,d\boldsymbol{x}dt$$
$$= -\int_{t_*}^T\int_{\partial S(t)}\boldsymbol{g}\cdot[\boldsymbol{b}](\boldsymbol{x}-\boldsymbol{x}_c)\,ds_x dt - \int_{t_*}^T\int_{S(t)}\varrho\boldsymbol{f}\cdot[\boldsymbol{b}](\boldsymbol{x}-\boldsymbol{x}_c)\,d\boldsymbol{x}dt.$$

Since the matrix $\big(\varrho\,(\boldsymbol{v}\otimes\boldsymbol{v}) - \boldsymbol{T}\big)$ is symmetric, $\big(\varrho\,(\boldsymbol{v}\otimes\boldsymbol{v}) - \boldsymbol{T}\big):[\boldsymbol{b}] = 0$. Moreover, using (2.6) and the fact that the matrices $[\boldsymbol{b}]$ and $[\dot{\boldsymbol{b}}]$ are skew-symmetric, we obtain that

$$\int_{S(t)}\varrho\,\boldsymbol{v}\cdot[\boldsymbol{b}]\dot{\boldsymbol{x}}_c\,d\boldsymbol{x} = \int_{S(t)}\varrho\,\boldsymbol{v}\,d\boldsymbol{x}\cdot[\boldsymbol{b}]\dot{\boldsymbol{x}}_c = m\,\dot{\boldsymbol{x}}_c\cdot[\boldsymbol{b}]\dot{\boldsymbol{x}}_c = 0,$$

$$\int_{S(t)} \varrho \boldsymbol{v}\cdot[\dot{\boldsymbol{b}}](\boldsymbol{x}-\boldsymbol{x}_c)\,d\boldsymbol{x} = \int_{S(t)} \varrho\,[\boldsymbol{\omega}](\boldsymbol{x}-\boldsymbol{x}_c)\cdot[\dot{\boldsymbol{b}}](\boldsymbol{x}-\boldsymbol{x}_c)\,d\boldsymbol{x}$$
$$= \int_{S(t)} \varrho\,\big(\boldsymbol{\omega}\times(\boldsymbol{x}-\boldsymbol{x}_c)\big)\cdot\big(\dot{\boldsymbol{b}}\times(\boldsymbol{x}-\boldsymbol{x}_c)\big)\,d\boldsymbol{x} = \dot{\boldsymbol{b}}\cdot\boldsymbol{J}_c\,\boldsymbol{\omega},$$

where $\boldsymbol{J}_c$ is the matrix of the inertia moments of the body $S(t)$ with respect to its mass center $\boldsymbol{x}_c(t)$ (see (2.8)). Thus,

$$\int_{t_*}^{T} \dot{\boldsymbol{b}}\cdot\boldsymbol{J}_c\,\boldsymbol{\omega}\,dt = -\int_{t_*}^{T} \boldsymbol{b}\cdot\Big(\int_{\partial S(t)} (\boldsymbol{x}-\boldsymbol{x}_c)\times\boldsymbol{g}\,ds_x + \int_{S(t)} \varrho\,(\boldsymbol{x}-\boldsymbol{x}_c)\times\boldsymbol{f}\,d\boldsymbol{x}\Big)\,dt$$

and as a consequence

$$\frac{d}{dt}\big(\boldsymbol{J}_c(t)\,\boldsymbol{\omega}(t)\big) = \int_{\partial S(t)} \big(\boldsymbol{x}-\boldsymbol{x}_c(t)\big)\times\boldsymbol{g}(\boldsymbol{x},t)\,ds_x + \int_{S(t)} \varrho(\boldsymbol{x},t)\,\big(\boldsymbol{x}-\boldsymbol{x}_c(t)\big)\times\boldsymbol{f}(\boldsymbol{x},t)\,d\boldsymbol{x}. \tag{3.5}$$

**Remark.** If we consider the body as a closed system of material points, then the vectors $m\dot{\boldsymbol{x}}_c$ and $\boldsymbol{J}_c\boldsymbol{\omega}$ are its vectors of linear and angular momenta with respect to the mass center $\boldsymbol{x}_c$, respectively. Therefore, equations (3.4) and (3.5) are entirely in accordance with the fundamental theorems of classical mechanics. •

Let us obtain an equation for $\boldsymbol{\eta}$. The change of the variable $\boldsymbol{x}=\boldsymbol{\phi}(\boldsymbol{\xi},t)$ in (3.3) gives the following identity:

$$\int_{t_*}^{T}\int_{S_*} \Big(\varrho_*\,\partial_t\boldsymbol{\phi}\cdot\partial_t\boldsymbol{h}_* - (\nabla_\xi\boldsymbol{\phi}\,\boldsymbol{T}^{(2)}) : \nabla_\xi\boldsymbol{h}_* + \varrho_*\boldsymbol{f}_*\cdot\boldsymbol{h}_*\Big)\,d\boldsymbol{\xi}dt = -\int_{t_*}^{T}\int_{\partial S_*} \boldsymbol{g}_*\cdot\boldsymbol{h}_*\,ds_\xi dt, \tag{3.6}$$

where $\boldsymbol{h}_*(\boldsymbol{\xi},t)=\boldsymbol{h}\big(\boldsymbol{\phi}(\boldsymbol{\xi},t),t\big)$, $\boldsymbol{f}_*(\boldsymbol{\xi},t)=\boldsymbol{f}\big(\boldsymbol{\phi}(\boldsymbol{\xi},t),t\big)$,

$$\boldsymbol{T}^{(2)}(\boldsymbol{\xi},t) = \nabla_\xi\boldsymbol{\phi}^{-1}(\boldsymbol{\xi},t)\,\boldsymbol{T}^{(1)}(\boldsymbol{\xi},t)$$

is the second Piola–Kirchhoff stress tensor (see [3]),

$$\boldsymbol{T}^{(1)}(\boldsymbol{\xi},t) = \boldsymbol{T}\big(\boldsymbol{\phi}(\boldsymbol{\xi},t),t\big)\,\nabla_\xi\boldsymbol{\phi}^{-T}(\boldsymbol{\xi},t)\det\nabla_\xi\boldsymbol{\phi}(\boldsymbol{\xi},t)$$

is the first Piola–Kirchhoff stress tensor,

$$\boldsymbol{g}_*(\boldsymbol{\xi},t) = \det\nabla_\xi\boldsymbol{\phi}(\boldsymbol{\xi},t)\,\big|\nabla_\xi\boldsymbol{\phi}^{-T}(\boldsymbol{\xi},t)\,\boldsymbol{n}_*\big|\,\boldsymbol{g}\big(\boldsymbol{\phi}(\boldsymbol{\xi},t),t\big)$$

is the density of the surface force in the Lagrangian variables, and $\boldsymbol{n}_*=\boldsymbol{n}_*(\boldsymbol{\xi},t)$ is the external normal to $\partial S_*$. Notice that the second Piola–Kirchhoff stress tensor is symmetric: $\boldsymbol{T}^{(2)}={\boldsymbol{T}^{(2)}}^T$.

When deriving equations (3.4) and (3.5), we took an arbitrary function from the space $\mathcal{R}_v$ as the test function $\boldsymbol{h}$ in (3.3). Generally speaking, in order to obtain an equation for $\boldsymbol{\eta}$, it suffices to take $\boldsymbol{h}$ from ${\mathcal{R}_v}^\perp$, i.e., from the space that is orthogonal to $\mathcal{R}_v$ in $L^2\big(S(t)\big)$:

$$\int_{S(t)} \varrho\,\boldsymbol{h}\cdot\big(\boldsymbol{a}+[\boldsymbol{b}](\boldsymbol{x}-\boldsymbol{x}_c)\big)\,d\boldsymbol{x} = 0 \quad \text{for all} \quad \boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^3. \tag{3.7}$$

However, we will meet some difficulties when doing so. We say that $\boldsymbol{h}_* \in \mathcal{R}_{v*}^{\perp}$, if $\boldsymbol{h}_*$ satisfies the following relation:

$$\int_{S_*} \varrho_* \boldsymbol{h}_* \cdot \big(\boldsymbol{a} + [\boldsymbol{b}]\boldsymbol{R}(\boldsymbol{\xi} + \boldsymbol{\eta})\big)\, d\boldsymbol{\xi} = 0 \quad \text{for all} \quad \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$$

that is obtained by passage to the Lagrangian coordinates in (3.7). The orthogonal in $L^2(S_*)$ projector onto $\mathcal{R}_{v*}^{\perp}$ has a sufficiently complicated structure and we get a complicated equation after substituting the test functions from $\mathcal{R}_{v*}^{\perp}$ into (3.6). For this reason, we apply another approach.

Let us substitute $\boldsymbol{\phi}(\boldsymbol{\xi}, t) = \boldsymbol{x}_c(t) + \boldsymbol{R}(t)\big(\boldsymbol{\xi} + \boldsymbol{\eta}(\boldsymbol{\xi}, t)\big)$ into (3.6). Since the function $\boldsymbol{h}$ in (3.3) is arbitrary, we can use an arbitrary function $\boldsymbol{\zeta}$ instead of $\boldsymbol{h}_*$ in (3.6). As a result, we obtain the following identity:

$$\int_{t_*}^{T}\int_{S_*} \Big(\varrho_* \big(\dot{\boldsymbol{x}}_c + \dot{\boldsymbol{R}}(\boldsymbol{\xi}+\boldsymbol{\eta}) + \boldsymbol{R}\partial_t\boldsymbol{\eta}\big)\cdot\partial_t\boldsymbol{\zeta} - \big(\boldsymbol{R}(\boldsymbol{I}+\nabla_\xi\boldsymbol{\eta})\boldsymbol{T}^{(2)}\big) : \nabla_\xi\boldsymbol{\zeta} + \varrho_*\boldsymbol{f}_*\cdot\boldsymbol{\zeta}\Big)\, d\boldsymbol{\xi}dt$$
$$= -\int_{t_*}^{T}\int_{\partial S_*} \boldsymbol{g}_*\cdot\boldsymbol{\zeta}\, ds_\xi dt \tag{3.8}$$

for an arbitrary function $\boldsymbol{\zeta}$ such that $\boldsymbol{\zeta}|_{t=t_*} = \boldsymbol{\zeta}|_{t=T} = \boldsymbol{0}$. We forget for a moment that the functions $\boldsymbol{x}_c$, $\boldsymbol{R}$, and $\boldsymbol{\eta}$ must satisfy the variational principle (2.3) (or, equivalently, the conditions of Proposition 2.2) and suppose that the functions $\boldsymbol{x}_c$, $\boldsymbol{\omega}$, and $\boldsymbol{R}$ in (3.8) are determined from equations (3.4), (3.5), and (2.13). Let us rewrite equations (3.4) and (3.5) in Lagrangian variables:

$$m\,\ddot{\boldsymbol{x}}_c = \int_{\partial S_*} \boldsymbol{g}_*\, ds_\xi + \int_{S_*} \varrho_*\,\boldsymbol{f}_*\, d\boldsymbol{\xi}, \tag{3.9}$$

$$\frac{d}{dt}\big(\boldsymbol{J}_c\,\boldsymbol{\omega}\big) = \int_{\partial S_*} \boldsymbol{R}(\boldsymbol{\xi}+\boldsymbol{\eta}) \times \boldsymbol{g}_*\, ds_\xi + \int_{S_*} \varrho_*\,\boldsymbol{R}(\boldsymbol{\xi}+\boldsymbol{\eta}) \times \boldsymbol{f}_*\, d\boldsymbol{\xi}, \tag{3.10}$$

$$\boldsymbol{J}_c = \int_{S_*} \varrho_*\big(\boldsymbol{I}|\boldsymbol{\xi}+\boldsymbol{\eta}|^2 - \boldsymbol{R}(\boldsymbol{\xi}+\boldsymbol{\eta}) \otimes \boldsymbol{R}(\boldsymbol{\xi}+\boldsymbol{\eta})\big)\, d\boldsymbol{\xi}.$$

One can put the question, are equations (3.8), (3.9), and (3.10) independent? It seems that if we take the test function $\boldsymbol{\zeta}$ in (3.8) of the form $\boldsymbol{a}(t) + [\boldsymbol{b}(t)]\boldsymbol{R}(\boldsymbol{\xi}+\boldsymbol{\eta})$, then, exactly as it was done when deriving equation (3.4) and (3.5), we obtain (3.9) and (3.10). However, we have obtained equations (3.4) and (3.5) with the help of relations (2.7), (2.11), and (2.8) that are equivalent to (2.17) and (2.18). So, we will suppose that $\boldsymbol{x}_c$, $\boldsymbol{\omega}$, $\boldsymbol{R}$, and $\boldsymbol{\eta}$ satisfy (3.8), (3.9), (3.10), and (2.13) only. The next proposition shows that relations (2.17) and (2.18) follow from these equations.

**Proposition 3.1.** *Suppose that functions* $\boldsymbol{x}_c : [t_*, \infty) \to \mathbb{R}^3$, $\boldsymbol{\omega} : [t_*, \infty) \to \mathbb{R}^3$, $\boldsymbol{\eta} : S_* \times [t_*, \infty) \to \mathbb{R}^3$, *and* $\boldsymbol{R} : [t_*, \infty) \to SO(3)$ *satisfy the following conditions:*

(a) $\boldsymbol{x}_c(t_*) = \boldsymbol{0}$, $\boldsymbol{R}(t_*) = \boldsymbol{I}$, *and* $\boldsymbol{\eta}(\boldsymbol{\xi}, t_*) = \boldsymbol{0}$ *for all* $\boldsymbol{\xi} \in S_*$;

(b) $\dot{\boldsymbol{R}}(t) = [\boldsymbol{\omega}(t)]\boldsymbol{R}(t)$ *for all* $t \geqslant t_*$;

(c) *for every* $t \geqslant t_*$ *the mapping* $\boldsymbol{\phi}(\cdot, t) : S_* \to \mathbb{R}^3$ *defined as* $\boldsymbol{\phi}(\boldsymbol{\xi}, t) = \boldsymbol{x}_c(t) + \boldsymbol{R}(t)(\boldsymbol{\xi} + \boldsymbol{\eta}(\boldsymbol{\xi}, t))$ *is invertible;*

(d) *equations* (3.8), (3.9), *and* (3.10) *are satisfied;*
(e) *conditions* (2.17) *and* (2.18) *are satisfied for* $t = t_*$.

*Then conditions* (2.17) *and* (2.18) *are satisfied for all* $t \geqslant t_*$.

*Proof.* At first we prove (2.17). Let us take an (independent of $\boldsymbol{\xi}$) function $\boldsymbol{a} : [t_*, T] \to \mathbb{R}^3$ such that $\boldsymbol{a}(t_*) = \boldsymbol{a}(T) = \mathbf{0}$ as the test function in (3.8). Then, due to (3.9), it is not difficult to find that

$$\int_{t_*}^{T} \dot{\boldsymbol{a}} \cdot \frac{d}{dt} \int_{S_*} \varrho_* \boldsymbol{R}\boldsymbol{\eta}\, d\boldsymbol{\xi}\, dt = 0.$$

Since $\int_{S_*} \varrho_* \boldsymbol{R}\boldsymbol{\eta}\, d\boldsymbol{\xi} = 0$ and $\int_{S_*} \varrho_* \partial_t(\boldsymbol{R}\boldsymbol{\eta})\, d\boldsymbol{\xi} = 0$ at the time $t = t_*$, this equation implies that $\int_{S_*} \varrho_* \boldsymbol{R}\boldsymbol{\eta}\, d\boldsymbol{\xi} = 0$ and, as a consequence, that $\int_{S_*} \varrho_* \boldsymbol{\eta}\, d\boldsymbol{\xi} = 0$ for all $t > t_*$. Differentiation of the last equality with respect to $t$ gives (2.17).

Let us prove (2.18). If we take $\boldsymbol{\zeta} = [\boldsymbol{b}]\boldsymbol{R}(\boldsymbol{\xi} + \boldsymbol{\eta})$ with an arbitrary function $\boldsymbol{b} : [t_*, T] \to \mathbb{R}^3$ such that $\boldsymbol{b}(t_*) = \boldsymbol{b}(T) = \mathbf{0}$ in (3.8), then after a simple calculation we get the equality

$$\int_{t_*}^{T} \dot{\boldsymbol{b}} \cdot \int_{S_*} \varrho_* \boldsymbol{R}(\boldsymbol{\xi} + \boldsymbol{\eta}) \times \boldsymbol{R}\, \partial_t \boldsymbol{\eta}\, d\boldsymbol{\xi}\, dt = 0.$$

Since $\int_{S_*} \varrho_* \boldsymbol{R}(\boldsymbol{\xi} + \boldsymbol{\eta}) \times \boldsymbol{R}\, \partial_t \boldsymbol{\eta}\, d\boldsymbol{\xi} = \int_{S_*} \varrho_* (\boldsymbol{\xi} + \boldsymbol{\eta}) \times \partial_t \boldsymbol{\eta}\, d\boldsymbol{\xi} = 0$ at $t = t_*$, we obtain that

$$\int_{S_*} \varrho_* \boldsymbol{R}(\boldsymbol{\xi} + \boldsymbol{\eta}) \times \boldsymbol{R}\, \partial_t \boldsymbol{\eta}\, d\boldsymbol{\xi} = 0 \quad \text{for all } t > t_*.$$

Multiplication of this equality by an arbitrary vector $\boldsymbol{b}$ gives (2.19) which is equivalent to (2.18) (see Proposition 2.1). The proposition is proved. □

Proposition 3.1 implies that if we find functions $\boldsymbol{x}_c$, $\boldsymbol{\omega}$, $\boldsymbol{R}$, and $\boldsymbol{\eta}$ that satisfy conditions (a)–(e), then we completely determine the motion $\boldsymbol{\phi}$ of the elastic body by (2.15). Moreover, as it follows from Proposition 2.2, the variational principle (2.3) holds true. Supplementing (a)–(e) with a constitutive equation for $\boldsymbol{T}^{(2)}$, we obtain a closed model of the motion of the elastic body.

Let us make a remark concerning this model. Condition (c) should be satisfied if we want to obtain a mechanically acceptable solution. Since the mapping $\boldsymbol{\phi}(\cdot, t)$ is identical at $t = t_*$, it will be invertible for $t$ sufficiently closed to $t_*$. Thus, we should consider the problem on a sufficiently short time interval. Notice that such a restriction must be met in all non-steady elasticity problems. However, we can consider a mathematical problem of determining the functions $\boldsymbol{x}_c$, $\boldsymbol{\omega}$, $\boldsymbol{R}$, and $\boldsymbol{\eta}$ that satisfy conditions (a), (b), (d), and (e). In this case, we do not need any external restrictions on the time interval, where we solve the problem. Notice that in [1], besides condition (c) there appears one more restriction that was discussed at the end of Section 1.

## 4. Model with small deformations

The model derived in the previous section is general and can be used for describing the motion of any elastic body. However, it is difficult to investigate even in the case of simple constitutive equations. In this section, we simplify this model assuming that the deformations are small and $t_*$ is the time when the body was in its reference (natural) configuration that is characterized by the absence of internal stresses.

Let $\boldsymbol{u}(\boldsymbol{\xi},t) = \boldsymbol{\phi}(\boldsymbol{\xi},t) - \boldsymbol{\xi}$ be the field of displacements and $\boldsymbol{E}(\boldsymbol{u})$ be the Green–Saint-Venant deformation tensor:

$$2\,\boldsymbol{E}(\boldsymbol{u}) = \nabla_\xi \boldsymbol{u} + \nabla_\xi \boldsymbol{u}^T + \nabla_\xi \boldsymbol{u}^T \nabla_\xi \boldsymbol{u}.$$

It is not difficult to see that

$$2\,\boldsymbol{E}(\boldsymbol{u}) = 2\,\boldsymbol{E}(\boldsymbol{\eta}) = \nabla_\xi \boldsymbol{\eta} + \nabla_\xi \boldsymbol{\eta}^T + \nabla_\xi \boldsymbol{\eta}^T \nabla_\xi \boldsymbol{\eta}.$$

We suppose that the body is of a Saint-Venant–Kirchhoff material, that is the second Piola–Kirchhoff stress tensor is a linear function of $\boldsymbol{E}(\boldsymbol{u})$:

$$\boldsymbol{T}^{(2)} = \lambda\,\boldsymbol{I}\,\mathrm{tr}\boldsymbol{E} + 2\mu\,\boldsymbol{E}, \tag{4.1}$$

where $\lambda$ and $\mu$ are positive constants called Lamé coefficients. Suppose also that the deformations of the body are small and we can neglect in the equations the terms of the order $|\nabla_\xi \boldsymbol{\eta}|^2$. Then after substitution of (4.1) into (3.8), we obtain:

$$\int_{t_*}^T \int_{S_*} \Big( \varrho_*\,\partial_t\big(\dot{\boldsymbol{x}}_c + \dot{\boldsymbol{R}}(\boldsymbol{\xi}+\boldsymbol{\eta}) + \boldsymbol{R}\,\partial_t\boldsymbol{\eta}\big)\cdot\boldsymbol{\zeta} + \big(\boldsymbol{R}\,\boldsymbol{\Sigma}(\boldsymbol{\eta})\big) : \nabla_\xi\boldsymbol{\zeta} - \varrho_*\boldsymbol{f}_*\cdot\boldsymbol{\zeta}\Big)\,d\boldsymbol{\xi}dt \\ = \int_{t_*}^T \int_{\partial S_*} \boldsymbol{g}_*\cdot\boldsymbol{\zeta}\,ds_\xi dt \tag{4.2}$$

where $\boldsymbol{\Sigma}(\boldsymbol{\eta}) = \lambda\,\boldsymbol{I}\,\mathrm{div}_{\,\xi}\boldsymbol{\eta} + \mu(\nabla_\xi\boldsymbol{\eta} + \nabla_\xi\boldsymbol{\eta}^T)$. However, this equation is not satisfactory since we cannot prove a statement similar to Proposition 3.1 with equation (4.2) instead of (3.8). That is the original variational principle (2.3) will be broken. Of course, we can also consider the problem with equation (4.2), but it will be another problem and it will be not so easy for investigation. There will appear difficulties even with obtaining the energy estimate. For this reason, we choose another way. Namely, at first we rewrite equation (3.8) in an equivalent form and then linearize it. We will need an auxiliary result.

Let us decompose the space $L^2(S_*)$ at every time moment $t$ into the orthogonal sum of the subspaces $\mathcal{R}_{v*}(t)$ and $\mathcal{R}_{v*}^{\perp}(t)$, where $\mathcal{R}_{v*}(t) = \big\{\boldsymbol{a} + [\boldsymbol{b}]\boldsymbol{R}(t)\big(\boldsymbol{\xi} + \boldsymbol{\eta}(\boldsymbol{\xi},t)\big) \mid \boldsymbol{a}\in\mathbb{R}^3,\ \boldsymbol{b}\in\mathbb{R}^3\big\}$. The projectors onto these subspaces will be denoted by $\boldsymbol{P}_t$ and $\boldsymbol{Q}_t$, respectively. Sometimes, the subscript $t$ will be omitted. It is not difficult to verify that

$$(\boldsymbol{P}_t\boldsymbol{\zeta})(\boldsymbol{\xi},t) = \boldsymbol{a} + [\boldsymbol{b}(t)]\boldsymbol{R}(t)\big(\boldsymbol{\xi} + \boldsymbol{\eta}(\boldsymbol{\xi},t)\big), \qquad \boldsymbol{\zeta}\in L^2(S_*),$$

where

$$\boldsymbol{a} = m^{-1}\int_{S_*}\varrho_*\boldsymbol{\zeta}\,d\boldsymbol{\xi}, \qquad \boldsymbol{b} = \boldsymbol{J}_c^{-1}\int_{S_*}\varrho_*\big(\boldsymbol{R}(\boldsymbol{\xi}+\boldsymbol{\eta})\big)\times\boldsymbol{\zeta}\,d\boldsymbol{\xi}.$$

Notice that $\boldsymbol{R}\,\partial_t\boldsymbol{\eta} \in \mathcal{R}_{v*}^{\perp}$ (i.e., $\boldsymbol{Q}(\boldsymbol{R}\,\partial_t\boldsymbol{\eta}) = \boldsymbol{R}\,\partial_t\boldsymbol{\eta}$) due to Proposition 2.1.

**Lemma 4.1.** *Let $\boldsymbol{S}$ be a symmetric tensor field in $S_*$. Then*

$$\int_{S_*} \boldsymbol{R}(\boldsymbol{I}+\nabla\boldsymbol{\eta})\boldsymbol{S} : \nabla\boldsymbol{\zeta}\,d\boldsymbol{\xi} = \int_{S_*} \boldsymbol{R}(\boldsymbol{I}+\nabla\boldsymbol{\eta})\boldsymbol{S} : \nabla\boldsymbol{Q}(\boldsymbol{\zeta})\,d\boldsymbol{\xi}.$$

*Proof.* Since $\boldsymbol{\zeta} = \boldsymbol{P}(\boldsymbol{\zeta}) + \boldsymbol{Q}(\boldsymbol{\zeta})$, it is enough to prove that $\int_{S_*} \boldsymbol{R}(\boldsymbol{I}+\nabla\boldsymbol{\eta})\boldsymbol{S} : \nabla\boldsymbol{P}(\boldsymbol{\zeta})\,d\boldsymbol{\xi} = 0$. For arbitrary $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ we have:

$$\int_{S_*} \boldsymbol{R}(\boldsymbol{I}+\nabla\boldsymbol{\eta})\boldsymbol{S} : \nabla\big(\boldsymbol{a} + [\boldsymbol{b}]\boldsymbol{R}(\boldsymbol{\xi}+\boldsymbol{\eta})\big)\,d\boldsymbol{\xi}$$
$$= \int_{S_*} \boldsymbol{R}(\boldsymbol{I}+\nabla\boldsymbol{\eta})\boldsymbol{S} : [\boldsymbol{b}]\boldsymbol{R}(\boldsymbol{I}+\nabla\boldsymbol{\eta})\,d\boldsymbol{\xi} = \int_{S_*} \boldsymbol{S} : \boldsymbol{B}\,d\boldsymbol{\xi},$$

where $\boldsymbol{B} = (\boldsymbol{I}+\nabla\boldsymbol{\eta})^T\boldsymbol{R}^T[\boldsymbol{b}]\boldsymbol{R}(\boldsymbol{I}+\nabla\boldsymbol{\eta})$. As the matrix $\boldsymbol{B}$ is skew-symmetric, $\boldsymbol{S} : \boldsymbol{B} = 0$. The lemma is proved. □

By making use of the lemma above, we can rewrite (3.8) in the following form:

$$\int_{t_*}^{T}\int_{S_*} \Big(\varrho_*\partial_t\big(\dot{\boldsymbol{x}}_c + \dot{\boldsymbol{R}}(\boldsymbol{\xi}+\boldsymbol{\eta}) + \boldsymbol{R}\partial_t\boldsymbol{\eta}\big)\cdot\boldsymbol{\zeta} + \boldsymbol{R}(\boldsymbol{I}+\nabla_\xi\boldsymbol{\eta})\boldsymbol{T}^{(2)} : \nabla_\xi\boldsymbol{Q}(\boldsymbol{\zeta}) - \varrho_*\boldsymbol{f}_*\cdot\boldsymbol{\zeta}\Big)\,d\boldsymbol{\xi}dt$$
$$= \int_{t_*}^{T}\int_{\partial S_*} \boldsymbol{g}_*\cdot\boldsymbol{\zeta}\,ds_\xi dt.$$

Now, suggesting that the deformations are small, substitute (4.1) into this identity and neglect the terms of the order $|\nabla\boldsymbol{\eta}|^2$. Notice that in general $\boldsymbol{Q}(\boldsymbol{\zeta})$ is not small and can be any vector field from $\mathcal{R}_{v*}^{\perp}$. As a result, we obtain the integral identity

$$\int_{t_*}^{T}\int_{S_*} \Big(\varrho_*\,\partial_t\big(\dot{\boldsymbol{x}}_c + \dot{\boldsymbol{R}}(\boldsymbol{\xi}+\boldsymbol{\eta}) + \boldsymbol{R}\,\partial_t\boldsymbol{\eta}\big)\cdot\boldsymbol{\zeta} + \big(\boldsymbol{R}\,\boldsymbol{\Sigma}(\boldsymbol{\eta})\big) : \nabla_\xi\boldsymbol{Q}(\boldsymbol{\zeta}) - \varrho_*\boldsymbol{f}_*\cdot\boldsymbol{\zeta}\Big)\,d\boldsymbol{\xi}dt$$
$$= \int_{t_*}^{T}\int_{\partial S_*} \boldsymbol{g}_*\cdot\boldsymbol{\zeta}\,ds_\xi dt, \tag{4.3}$$

where $\boldsymbol{\Sigma}(\boldsymbol{\eta}) = \lambda\,\boldsymbol{I}\,\mathrm{div}_{\,\xi}\boldsymbol{\eta} + 2\mu\,\boldsymbol{\varepsilon}(\boldsymbol{\eta})$, $\boldsymbol{\varepsilon}(\boldsymbol{\eta}) = (\nabla_\xi\boldsymbol{\eta} + \nabla_\xi\boldsymbol{\eta}^T)/2$. Compared with (4.2), equation (4.3) has several advantages and the principal one is that the following statement holds.

**Proposition 4.2.** *The assertion of Proposition* 3.1 *holds true with condition* (d) *replaced by the following one:*

(d′) *equations* (4.3), (3.9), *and* (3.10) *are satisfied.*

The proof is absolutely similar to that of Proposition 3.1.

## 5. The energy estimate

In this section, we obtain an energy equality for the linearized problem. Let us take $\boldsymbol{\zeta} = \boldsymbol{R}\,\partial_t\boldsymbol{\eta}$ in equation (4.3). Then, as it follows from Proposition 2.1, $\boldsymbol{Q}(\boldsymbol{\zeta}) = \boldsymbol{R}\,\partial_t\boldsymbol{\eta}$ and $\int_{S_*} \varrho_* \ddot{\boldsymbol{x}}_c \cdot \boldsymbol{R}\,\partial_t\boldsymbol{\eta}\,d\boldsymbol{\xi} = 0$. Therefore,

$$\int_{t_*}^{T}\int_{S_*} \Big(\varrho_*\,\ddot{\boldsymbol{R}}(\boldsymbol{\xi}+\boldsymbol{\eta})\cdot\boldsymbol{R}\,\partial_t\boldsymbol{\eta} + \varrho_*\,\dot{\boldsymbol{R}}\,\partial_t\boldsymbol{\eta}\cdot\boldsymbol{R}\,\partial_t\boldsymbol{\eta} + \varrho_*\,\partial_t(\boldsymbol{R}\,\partial_t\boldsymbol{\eta})\cdot\boldsymbol{R}\,\partial_t\boldsymbol{\eta}$$
$$+\,\boldsymbol{\Sigma}(\boldsymbol{\eta}) : \partial_t\nabla\boldsymbol{\eta} - \varrho_*\boldsymbol{f}_*\cdot\boldsymbol{R}\,\partial_t\boldsymbol{\eta}\Big)\,d\boldsymbol{\xi}dt = \int_{t_*}^{T}\int_{\partial S_*} \boldsymbol{g}_*\cdot\boldsymbol{R}\,\partial_t\boldsymbol{\eta}\,ds_\xi dt. \tag{5.1}$$

Notice that

$$\dot{\boldsymbol{R}}\,\partial_t\boldsymbol{\eta}\cdot\boldsymbol{R}\,\partial_t\boldsymbol{\eta} = \dot{\boldsymbol{R}}\boldsymbol{R}^T\boldsymbol{R}\,\partial_t\boldsymbol{\eta}\cdot\boldsymbol{R}\,\partial_t\boldsymbol{\eta} = [\boldsymbol{\omega}]\boldsymbol{R}\,\partial_t\boldsymbol{\eta}\cdot\boldsymbol{R}\,\partial_t\boldsymbol{\eta} = 0,$$

$$\partial_t(\boldsymbol{R}\,\partial_t\boldsymbol{\eta})\cdot\boldsymbol{R}\,\partial_t\boldsymbol{\eta} = \frac{1}{2}\partial_t|\partial_t\boldsymbol{\eta}|^2,$$

$$\boldsymbol{\Sigma}(\boldsymbol{\eta}) : \partial_t\nabla\boldsymbol{\eta} = \partial_t\Big(\frac{\lambda}{2}\,|\mathrm{div}\,\boldsymbol{\eta}|^2 + \mu\,|\boldsymbol{\varepsilon}(\boldsymbol{\eta})|^2\Big),$$

where $|\boldsymbol{\varepsilon}(\boldsymbol{\eta})|^2 = \boldsymbol{\varepsilon}(\boldsymbol{\eta}) : \boldsymbol{\varepsilon}(\boldsymbol{\eta})$. Let us investigate the first term on the left-hand side of equality (5.1).

Since $\ddot{\boldsymbol{R}} = [\dot{\boldsymbol{\omega}}]\boldsymbol{R} + [\boldsymbol{\omega}]\dot{\boldsymbol{R}} = [\dot{\boldsymbol{\omega}}]\boldsymbol{R} + [\boldsymbol{\omega}][\boldsymbol{\omega}]\boldsymbol{R}$,

$$\int_{S_*} \varrho_*\ddot{\boldsymbol{R}}(\boldsymbol{\xi}+\boldsymbol{\eta})\cdot\boldsymbol{R}\,\partial_t\boldsymbol{\eta}\,d\boldsymbol{\xi}$$
$$= \int_{S_*} \varrho_*[\dot{\boldsymbol{\omega}}]\boldsymbol{R}(\boldsymbol{\xi}+\boldsymbol{\eta})\cdot\boldsymbol{R}\,\partial_t\boldsymbol{\eta}\,d\boldsymbol{\xi} + \int_{S_*} \varrho_*[\boldsymbol{\omega}][\boldsymbol{\omega}]\boldsymbol{R}(\boldsymbol{\xi}+\boldsymbol{\eta})\cdot\boldsymbol{R}\,\partial_t\boldsymbol{\eta}\,d\boldsymbol{\xi}.$$

The first term on the right-hand side of this equality is equal to zero due to Proposition 2.1; we transform the second one as follows:

$$\int_{S_*} \varrho_*[\boldsymbol{\omega}][\boldsymbol{\omega}]\boldsymbol{R}(\boldsymbol{\xi}+\boldsymbol{\eta})\cdot\boldsymbol{R}\,\partial_t\boldsymbol{\eta}\,d\boldsymbol{\xi} = \int_{S_*} \varrho_*\boldsymbol{\omega}\times\big(\boldsymbol{\omega}\times\boldsymbol{R}(\boldsymbol{\xi}+\boldsymbol{\eta})\big)\cdot\boldsymbol{R}\,\partial_t\boldsymbol{\eta}\,d\boldsymbol{\xi}$$
$$= \int_{S_*} \varrho_*\Big(\boldsymbol{\omega}\big(\boldsymbol{\omega}\cdot\boldsymbol{R}(\boldsymbol{\xi}+\boldsymbol{\eta})\big) - |\boldsymbol{\omega}|^2\boldsymbol{R}(\boldsymbol{\xi}+\boldsymbol{\eta})\Big)\cdot\boldsymbol{R}\,\partial_t(\boldsymbol{\xi}+\boldsymbol{\eta})\,d\boldsymbol{\xi}$$
$$= \int_{S_*} \varrho_*\boldsymbol{\omega}\cdot\Big(\boldsymbol{R}\,(\boldsymbol{\xi}+\boldsymbol{\eta})\otimes\boldsymbol{R}\,\partial_t(\boldsymbol{\xi}+\boldsymbol{\eta}) - \boldsymbol{I}\big(\boldsymbol{R}(\boldsymbol{\xi}+\boldsymbol{\eta})\cdot\boldsymbol{R}\,\partial_t(\boldsymbol{\xi}+\boldsymbol{\eta})\big)\Big)\boldsymbol{\omega}\,d\boldsymbol{\xi}.$$

Notice that

$$\boldsymbol{\omega}\cdot\dot{\boldsymbol{J}}_c\,\boldsymbol{\omega} = 2\int_{S_*} \varrho_*\boldsymbol{\omega}\cdot\Big(\boldsymbol{I}\big(\boldsymbol{R}(\boldsymbol{\xi}+\boldsymbol{\eta})\cdot\boldsymbol{R}\,\partial_t(\boldsymbol{\xi}+\boldsymbol{\eta})\big) - \boldsymbol{R}\,(\boldsymbol{\xi}+\boldsymbol{\eta})\otimes\boldsymbol{R}\,\partial_t(\boldsymbol{\xi}+\boldsymbol{\eta})\Big)\boldsymbol{\omega}\,d\boldsymbol{\xi}.$$

Therefore,

$$\int_{S_*} \varrho_*\ddot{\boldsymbol{R}}(\boldsymbol{\xi}+\boldsymbol{\eta})\cdot\boldsymbol{R}\,\partial_t\boldsymbol{\eta}\,d\boldsymbol{\xi} = -\frac{1}{2}\,\boldsymbol{\omega}\cdot\dot{\boldsymbol{J}}_c\,\boldsymbol{\omega}.$$

Thus, (5.1) is equivalent to the following equality:

$$\int_{t_*}^{T}\int_{S_*}\Big(\frac{\varrho_*}{2}\,\partial_t|\partial_t\boldsymbol{\eta}|^2+\partial_t\big(\frac{\lambda}{2}\,|\mathrm{div}\,\boldsymbol{\eta}|^2+\mu\,|\boldsymbol{\varepsilon}(\boldsymbol{\eta})|^2\big)-\varrho_*\boldsymbol{f}_*\cdot\boldsymbol{R}\,\partial_t\boldsymbol{\eta}\Big)\,d\boldsymbol{\xi}dt-\frac{1}{2}\,\boldsymbol{\omega}\cdot\dot{\boldsymbol{J}}_c\,\boldsymbol{\omega}$$
$$=\int_{t_*}^{T}\int_{\partial S_*}\boldsymbol{g}_*\cdot\boldsymbol{R}\,\partial_t\boldsymbol{\eta}\,ds_\xi dt. \quad (5.2)$$

Since $\boldsymbol{J}_c$ is a symmetric matrix,

$$\frac{d}{dt}(\boldsymbol{\omega}\cdot\boldsymbol{J}_c\,\boldsymbol{\omega})=2\,\boldsymbol{\omega}\cdot\frac{d}{dt}(\boldsymbol{J}_c\,\boldsymbol{\omega})-\boldsymbol{\omega}\cdot\dot{\boldsymbol{J}}_c\,\boldsymbol{\omega}.$$

Hence, by multiplying (3.10) by $\boldsymbol{\omega}$, we obtain

$$\frac{1}{2}\,\frac{d}{dt}(\boldsymbol{\omega}\cdot\boldsymbol{J}_c\,\boldsymbol{\omega})+\frac{1}{2}\,\boldsymbol{\omega}\cdot\dot{\boldsymbol{J}}_c\,\boldsymbol{\omega}=\int_{\partial S_*}\boldsymbol{g}_*\cdot[\boldsymbol{\omega}]\boldsymbol{R}(\boldsymbol{\xi}+\boldsymbol{\eta})\,ds_\xi+\int_{S_*}\varrho_*\boldsymbol{f}_*\cdot[\boldsymbol{\omega}]\boldsymbol{R}(\boldsymbol{\xi}+\boldsymbol{\eta})\,d\boldsymbol{\xi}. \quad (5.3)$$

At the same time, multiplication of (3.9) by $\dot{\boldsymbol{x}}_c$ gives

$$\frac{m}{2}\,\frac{d}{dt}|\dot{\boldsymbol{x}}_c|^2=\int_{\partial S_*}\boldsymbol{g}_*\cdot\dot{\boldsymbol{x}}_c\,ds_\xi+\int_{S_*}\varrho_*\boldsymbol{f}_*\cdot\dot{\boldsymbol{x}}_c\,d\boldsymbol{\xi}. \quad (5.4)$$

Identities (5.2), (5.3), and (5.4) imply the following energy equality:

$$\int_{t_*}^{T}\frac{1}{2}\,\frac{d}{dt}\Big(\int_{S_*}\varrho_*|\partial_t\boldsymbol{\eta}|^2\,d\boldsymbol{\xi}+m|\dot{\boldsymbol{x}}_c|^2+\boldsymbol{\omega}\cdot\boldsymbol{J}_c\,\boldsymbol{\omega}+\int_{S_*}\big(\lambda|\mathrm{div}\,\boldsymbol{\eta}|^2+2\mu|\boldsymbol{\varepsilon}(\boldsymbol{\eta})|^2\big)\,d\boldsymbol{\xi}\Big)\,dt$$
$$=\int_{t_*}^{T}\int_{S_*}\varrho_*\boldsymbol{f}_*\cdot\big(\dot{\boldsymbol{x}}_c+\dot{\boldsymbol{R}}(\boldsymbol{\xi}+\boldsymbol{\eta})+\boldsymbol{R}\,\partial_t\boldsymbol{\eta}\big)\,d\boldsymbol{\xi}dt$$
$$+\int_{t_*}^{T}\int_{\partial S_*}\boldsymbol{g}_*\cdot\big(\dot{\boldsymbol{x}}_c+\dot{\boldsymbol{R}}(\boldsymbol{\xi}+\boldsymbol{\eta})+\boldsymbol{R}\,\partial_t\boldsymbol{\eta}\big)\,ds_\xi dt.$$

Here, $T$ can be an arbitrary number greater than $t_*$. So that, for any $T$ and $t_0$ such that $T>t_0>t_*$, we find:

$$\int_{S_*}\varrho_*|\partial_t\boldsymbol{\eta}(T)|^2\,d\boldsymbol{\xi}+m|\dot{\boldsymbol{x}}_c(T)|^2+\boldsymbol{\omega}(T)\cdot\boldsymbol{J}_c(T)\,\boldsymbol{\omega}(T)$$
$$+\int_{S_*}\big(\lambda|\mathrm{div}\,\boldsymbol{\eta}(T)|^2+2\mu|\boldsymbol{\varepsilon}(\boldsymbol{\eta}(T))|^2\big)\,d\boldsymbol{\xi}$$
$$=\int_{S_*}\varrho_*|\partial_t\boldsymbol{\eta}(t_0)|^2\,d\boldsymbol{\xi}+m|\dot{\boldsymbol{x}}_c(t_0)|^2+\boldsymbol{\omega}(t_0)\cdot\boldsymbol{J}_c(t_0)\,\boldsymbol{\omega}(t_0)$$
$$+\int_{S_*}\big(\lambda|\mathrm{div}\,\boldsymbol{\eta}(t_0)|^2+2\mu|\boldsymbol{\varepsilon}(\boldsymbol{\eta}(t_0))|^2\big)\,d\boldsymbol{\xi}$$
$$+2\int_{t_0}^{T}\int_{S_*}\varrho_*\boldsymbol{f}_*\cdot\big(\dot{\boldsymbol{x}}_c+\dot{\boldsymbol{R}}(\boldsymbol{\xi}+\boldsymbol{\eta})+\boldsymbol{R}\,\partial_t\boldsymbol{\eta}\big)\,d\boldsymbol{\xi}dt$$
$$+2\int_{t_0}^{T}\int_{\partial S_*}\boldsymbol{g}_*\cdot\big(\dot{\boldsymbol{x}}_c+\dot{\boldsymbol{R}}(\boldsymbol{\xi}+\boldsymbol{\eta})+\boldsymbol{R}\,\partial_t\boldsymbol{\eta}\big)\,ds_\xi dt. \quad (5.5)$$

By making use of Proposition 2.1, it is not difficult to show that

$$\int_{S_*} \varrho_* |\partial_t \boldsymbol{\eta}|^2 \, d\boldsymbol{\xi} + m|\dot{\boldsymbol{x}}_c|^2 + \boldsymbol{\omega} \cdot \boldsymbol{J}_c \boldsymbol{\omega} = \int_{S_*} \varrho_* |\partial_t \boldsymbol{\phi}|^2 \, d\boldsymbol{\xi}.$$

Therefore, equality (5.5) can be also rewritten as

$$\begin{aligned} &\int_{S_*} \varrho_* |\partial_t \boldsymbol{\phi}(T)|^2 \, d\boldsymbol{\xi} + \int_{S_*} \big(\lambda |\mathrm{div}\, \boldsymbol{\eta}(T)|^2 + 2\mu |\boldsymbol{\varepsilon}(\boldsymbol{\eta}(T))|^2\big) \, d\boldsymbol{\xi} \\ &\quad = \int_{S_*} \varrho_* |\partial_t \boldsymbol{\phi}(t_0)|^2 \, d\boldsymbol{\xi} + \int_{S_*} \big(\lambda |\mathrm{div}\, \boldsymbol{\eta}(t_0)|^2 + 2\mu |\boldsymbol{\varepsilon}(\boldsymbol{\eta}(t_0))|^2\big) \, d\boldsymbol{\xi} \\ &\qquad + 2\int_{t_0}^{T} \int_{S_*} \varrho_* \boldsymbol{f}_* \cdot \partial_t \boldsymbol{\phi} \, d\boldsymbol{\xi} dt + 2\int_{t_0}^{T} \int_{\partial S_*} \boldsymbol{g}_* \cdot \partial_t \boldsymbol{\phi} \, ds_\xi dt. \end{aligned}$$

## 6. The final formulation of the problem

We have suggested that the body is in its natural state at the time $t_*$, i.e., $S_*$ is the reference configuration of the body. However, the problem can be considered on an arbitrary time interval $[t_0, T]$ with $t_0 \geqslant t_*$. At the time moment $t_0$, we have to prescribe $\boldsymbol{x}_c$ $\dot{\boldsymbol{x}}_c$, $\boldsymbol{R}$, $\boldsymbol{\omega}$, $\boldsymbol{\eta}$, and $\partial_t \boldsymbol{\eta}$. From a mechanical point of view, the values of these functions at $t = t_0$ cannot be arbitrary but must be found by solving the problem (a)–(e) on the interval $[t_*, t_0]$ with some external forces $\boldsymbol{g}_*$ and $\boldsymbol{f}_*$. The point is that, generally speaking, not every state of the body can be achieved from the reference configuration. On the other hand, if we consider a mathematical problem, then we can take any initial values of these functions.

Problem L. Suppose that the body in its reference configuration occupies a domain $S_* \subset \mathbb{R}^3$ at the time moment $t_*$. Let $\varrho_* : S_* \to (0, \infty)$ be its density distribution at this moment and $T$ and $t_0$ be real numbers such that $T > t_0 \geqslant t_*$. It is necessary to find functions $\boldsymbol{x}_c : [t_0, T] \to \mathbb{R}^3$, $\boldsymbol{\omega} : [t_0, T] \to \mathbb{R}^3$, $\boldsymbol{R} : [t_0, T] \to SO(3)$, and $\boldsymbol{\eta} : S_* \times [t_0, T] \to \mathbb{R}^3$ that satisfy the following conditions:

(i) $\boldsymbol{x}_c(t_0) = \overline{\boldsymbol{x}}_c$, $\dot{\boldsymbol{x}}_c(t_0) = \overline{\overline{\boldsymbol{x}}}_c$, $\boldsymbol{R}(t_0) = \overline{\boldsymbol{R}}$, $\boldsymbol{\omega}(t_0) = \overline{\boldsymbol{\omega}}$, $\boldsymbol{\eta}(t_0) = \overline{\boldsymbol{\eta}}$, $\partial_t \boldsymbol{\eta}(t_0) = \overline{\overline{\boldsymbol{\eta}}}$, where $\overline{\boldsymbol{x}}_c$, $\overline{\overline{\boldsymbol{x}}}_c$, $\overline{\boldsymbol{\omega}}$ are prescribed vectors, $\overline{\boldsymbol{R}}$ is a prescribed matrix from $SO(3)$, and $\overline{\boldsymbol{\eta}}, \overline{\overline{\boldsymbol{\eta}}} : S_* \to \mathbb{R}^3$ are prescribed functions such that

$$\int_{S_*} \varrho_*(\boldsymbol{\xi}) \overline{\boldsymbol{\eta}}(\boldsymbol{\xi}) \, d\boldsymbol{\xi} = \boldsymbol{0}, \quad \int_{S_*} \varrho_*(\boldsymbol{\xi}) \overline{\overline{\boldsymbol{\eta}}}(\boldsymbol{\xi}) \, d\boldsymbol{\xi} = \boldsymbol{0},$$

$$\int_{S_*} \varrho_*(\boldsymbol{\xi}) \big(\boldsymbol{\xi} + \overline{\boldsymbol{\eta}}(\boldsymbol{\xi})\big) \times \overline{\overline{\boldsymbol{\eta}}}(\boldsymbol{\xi}) \, d\boldsymbol{\xi} = \boldsymbol{0}.$$

(ii) The following equations are satisfied on the time interval $[t_0, T]$:

$$\dot{\boldsymbol{R}}(t) = [\boldsymbol{\omega}(t)] \boldsymbol{R}(t),$$

$$m \ddot{\boldsymbol{x}}_c(t) = \int_{\partial S_*} \boldsymbol{g}_*(\boldsymbol{\xi}, t) \, ds_\xi + \int_{S_*} \varrho_*(\boldsymbol{\xi}) \boldsymbol{f}_*(\boldsymbol{\xi}, t) \, d\boldsymbol{\xi},$$

$$\frac{d}{dt}\big(\boldsymbol{J}_c(t)\boldsymbol{\omega}(t)\big) = \int_{\partial S_*} \boldsymbol{R}(t)\big(\boldsymbol{\xi} + \boldsymbol{\eta}(\boldsymbol{\xi}, t)\big) \times \boldsymbol{g}_*(\boldsymbol{\xi}, t)\, ds_\xi + \int_{S_*} \varrho_*(\boldsymbol{\xi})\boldsymbol{R}(t)\big(\boldsymbol{\xi} + \boldsymbol{\eta}(\boldsymbol{\xi}, t)\big) \times \boldsymbol{f}_*(\boldsymbol{\xi}, t)\, d\boldsymbol{\xi},$$

where $m = \int_{S_*} \varrho_*(\boldsymbol{\xi})\, d\boldsymbol{\xi}$ is the mass of the body, $\boldsymbol{g}_* : \partial S_* \times [t_0, T] \to \mathbb{R}^3$, $\boldsymbol{f}_* : S_* \times [t_0, T] \to \mathbb{R}^3$ are prescribed functions of external forces,

$$\boldsymbol{J}_c(t) = \int_{S_*} \varrho_*(\boldsymbol{\xi})\Big(\boldsymbol{I}\,|\boldsymbol{\xi} + \boldsymbol{\eta}(\boldsymbol{\xi}, t)|^2 - \boldsymbol{R}(t)\big(\boldsymbol{\xi} + \boldsymbol{\eta}(\boldsymbol{\xi}, t)\big) \otimes \boldsymbol{R}(t)\big(\boldsymbol{\xi} + \boldsymbol{\eta}(\boldsymbol{\xi}, t)\big)\Big)\, d\boldsymbol{\xi}$$

is the tensor of the inertia moments of the body with respect to its mass center.

(iii) The integral identity

$$\int_{t_0}^{T}\int_{S_*} \Big(\varrho_*\, \partial_t\big(\dot{\boldsymbol{x}}_c + \dot{\boldsymbol{R}}(\boldsymbol{\xi} + \boldsymbol{\eta}) + \boldsymbol{R}\,\partial_t\boldsymbol{\eta}\big) \cdot \boldsymbol{\zeta} + \boldsymbol{R}\,\boldsymbol{\Sigma}(\boldsymbol{\eta}) : \nabla_\xi \boldsymbol{Q}(\boldsymbol{\zeta})\Big)\, d\boldsymbol{\xi} dt = \int_{t_0}^{T}\int_{S_*} \varrho_* \boldsymbol{f}_* \cdot \boldsymbol{\zeta}\, d\boldsymbol{\xi} dt + \int_{t_*}^{T}\int_{\partial S_*} \boldsymbol{g}_* \cdot \boldsymbol{\zeta}\, ds_\xi dt$$

holds true for an arbitrary function $\boldsymbol{\zeta} : S_* \times [t_0, T] \to \mathbb{R}^3$. Here, $\boldsymbol{\Sigma}(\boldsymbol{\eta}) = \lambda\,\boldsymbol{I}\,\mathrm{div}_{\,\xi}\boldsymbol{\eta} + 2\mu\,\boldsymbol{\varepsilon}(\boldsymbol{\eta})$, $\boldsymbol{\varepsilon}(\boldsymbol{\eta}) = (\nabla_\xi\boldsymbol{\eta} + \nabla_\xi\boldsymbol{\eta}^T)/2$, and $\boldsymbol{Q}(\boldsymbol{\zeta})$ is the projection in $L^2(S_*)$ of the function $\boldsymbol{\zeta}$ onto $\mathcal{R}_{v*}^{\perp}$ (see Section 4). •

If we want the solution of the problem to be mechanically appropriate, then we need one more condition:

(iv) The mapping $\boldsymbol{\phi}(\cdot, t) : S_* \to \mathbb{R}^3$ defined as $\boldsymbol{\phi}(\boldsymbol{\xi}, t) = \boldsymbol{x}_c(t) + \boldsymbol{R}(t)\big(\boldsymbol{\xi} + \boldsymbol{\eta}(\boldsymbol{\xi}, t)\big)$ is invertible for all $t \in [t_0, T]$.

**Remark.** As it follows from (i) and Proposition 4.2, conditions (2.17) and (2.18) are satisfied for all $t \geqslant t_0$. Therefore, the solution of Problem L solves also problem (2.3) which was our starting point. •

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